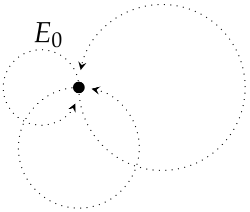
Supersingular Curves You Can Trust

Andrea Basso, Giulio Codogni, Deirdre Connolly, Luca De Feo,
Tako Boris Fouotsa, Guido Maria Lido, Travis Morrison, Lorenz Panny,
Sikhar Patranabis, Benjamin Wesolowski

Lyon, 26 April 2023

Plenty of reasons to distrust a supersingular elliptic curve

- Elliptic curves have a lot of structure, among which the endomorphism ring.



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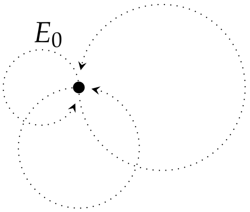

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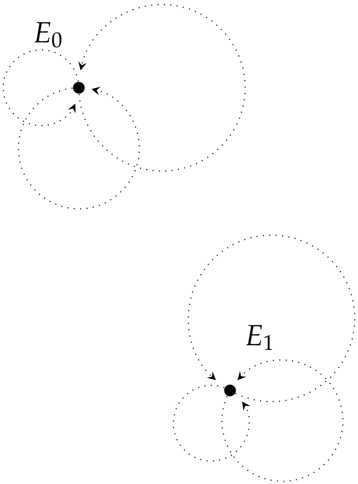

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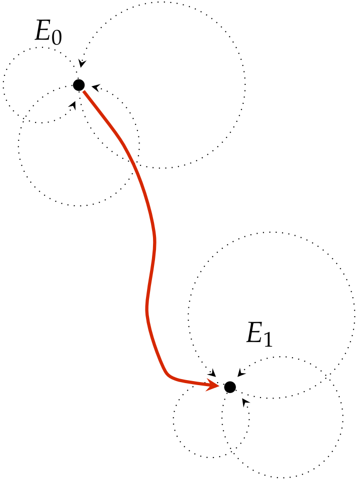

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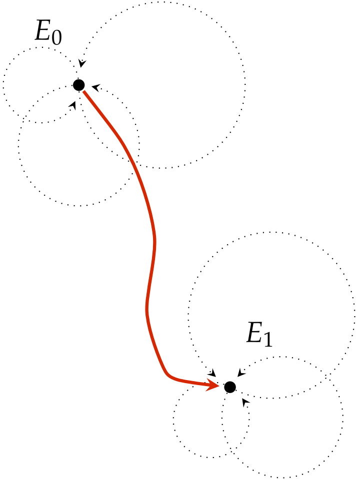
$\Longrightarrow$ End(E) can be used to backdoor several isogeny-based protocols.



A big open problem

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Reality: Less great; next slide.

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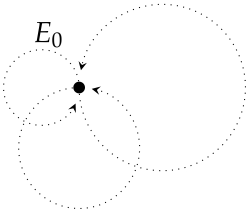

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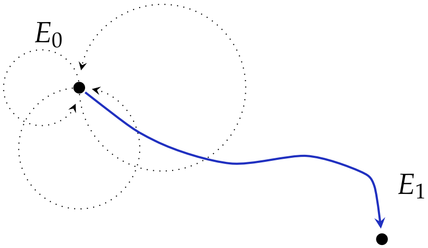

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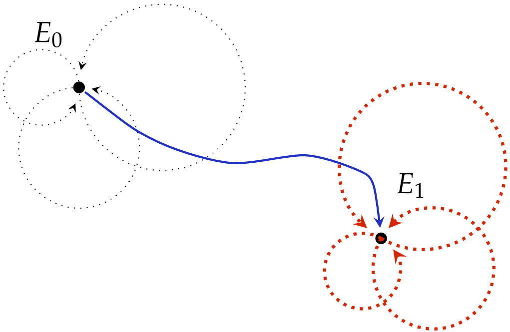

The connecting isogeny is a backdoor to the endomorphism ring.

Folklore workaround: Distributed trusted setup

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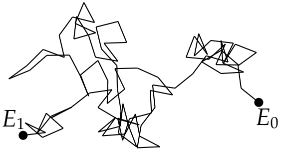

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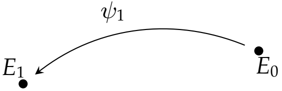

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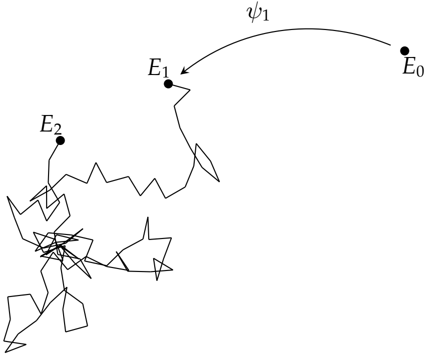

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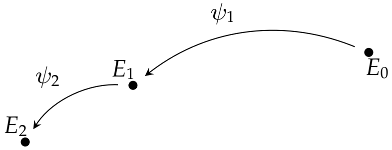

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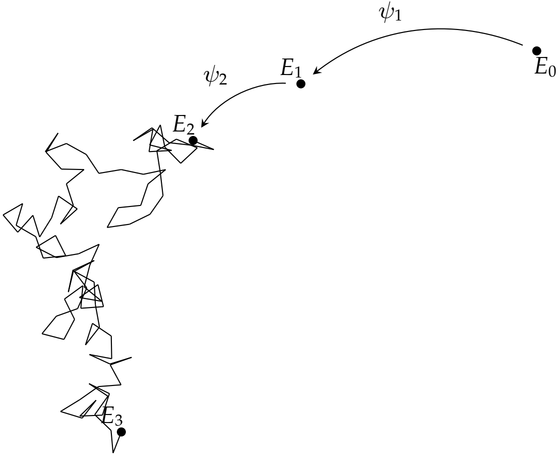

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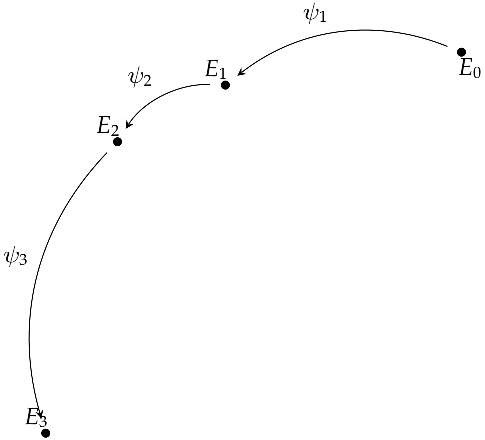

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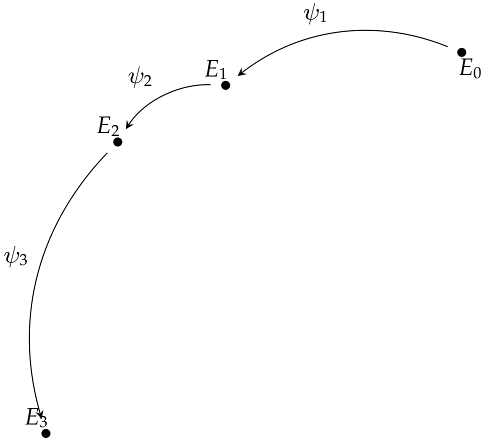

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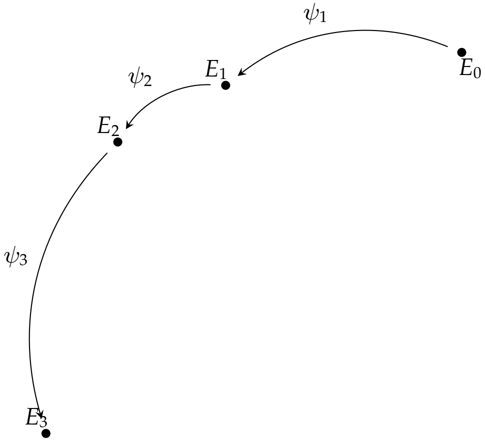

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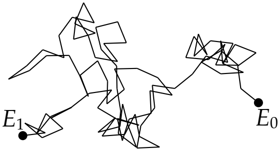

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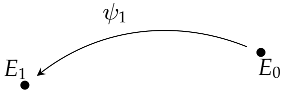

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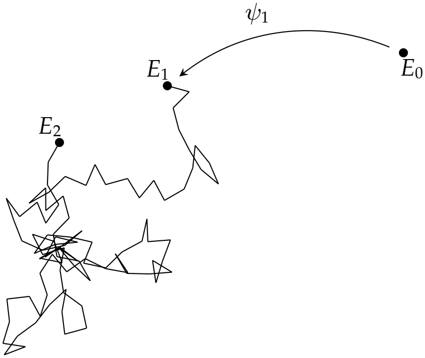

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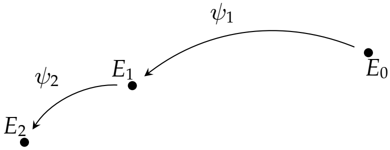

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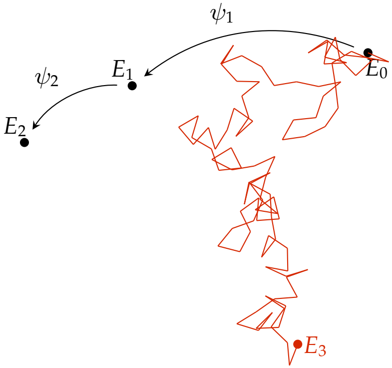

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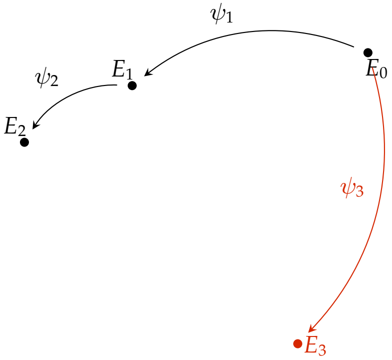

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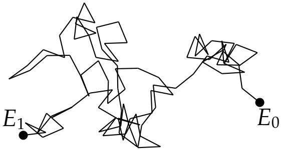

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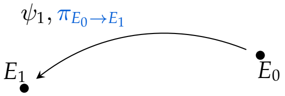

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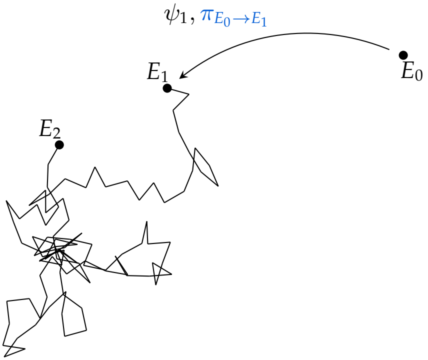

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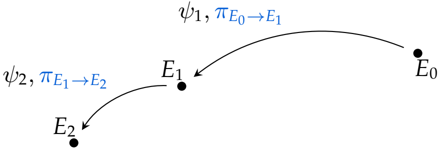

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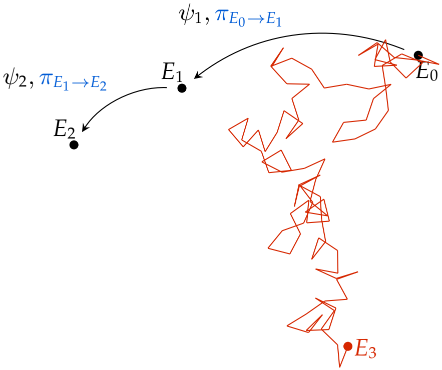

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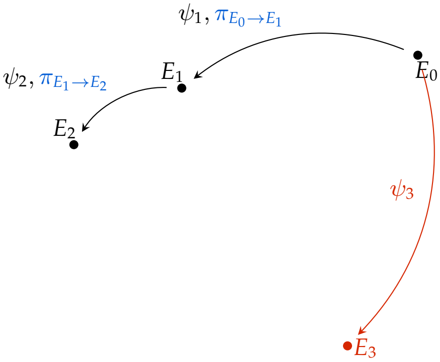

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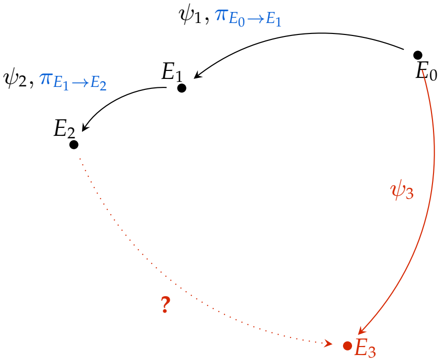

Main result

Assuming $\text{End}(E)$ is hard to compute, the trusted-setup protocol is provably secure in the simplified UC model if the proof of knowledge π is

- Correct for the relation

  $\varphi\colon E_0 \to E_1$ is a cyclic d-isogeny.

- Special-sound for the relation

  $\varphi\colon E_0 \to E_1$ is a cyclic isogeny (not necessarily of degree d).

- Statistically **zero-knowledge**.

  $\Longrightarrow$ Trusted setup is resistant against future cryptanalysis.

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$$E_0 \xrightarrow{\varphi} E_1$$

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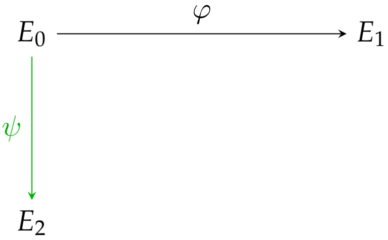

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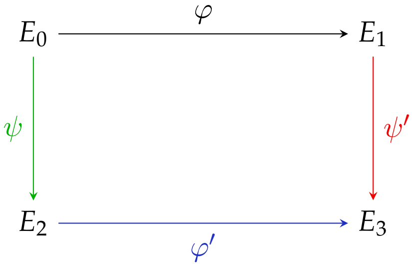

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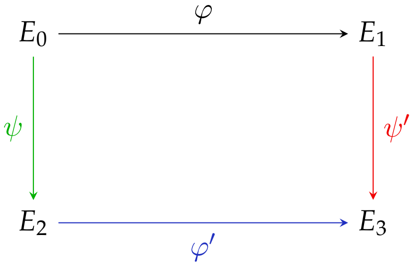

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- No SIDH attacks!!

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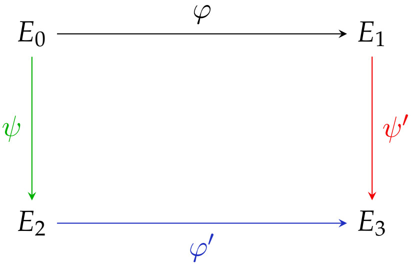

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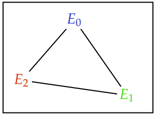
Bad things:

- Isogenies are rational $\Longrightarrow$ short
- Only computational ZK

Achieving statistical zero-knowledge (in theory)

- The supersingular isogeny graph is Ramanujan.
  $\implies$ Random walks quickly converge to $\approx$uniform.



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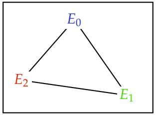

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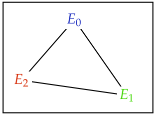

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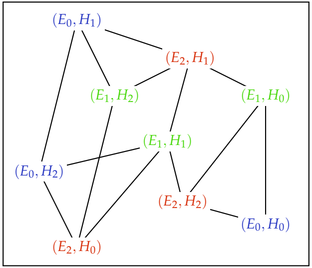

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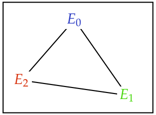

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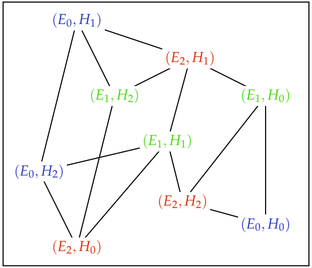
- The graph with level structure is also Ramanujan!
  $\implies$ More information revealed, hence longer walks.



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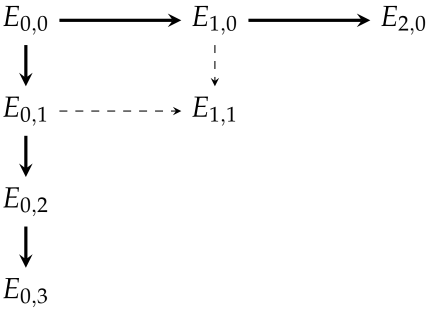

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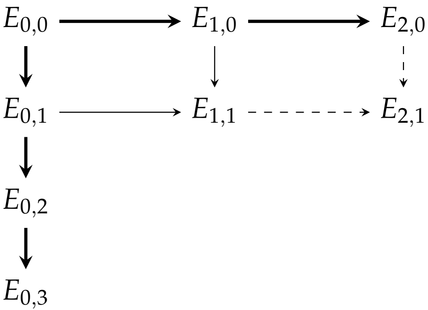

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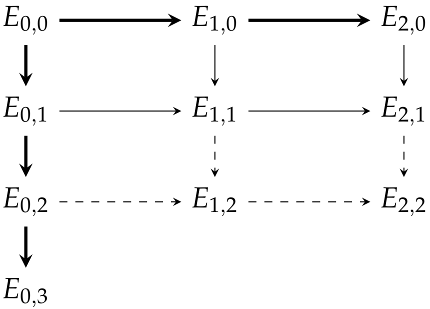

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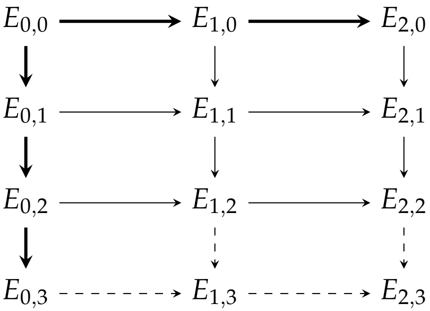

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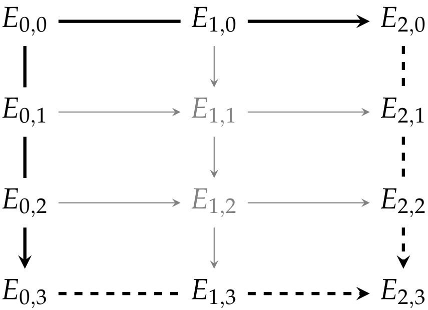

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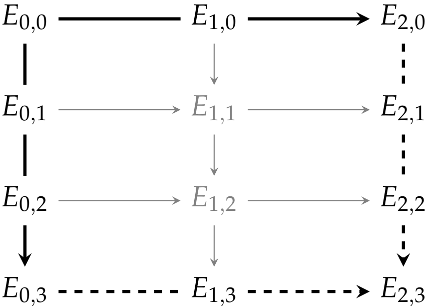

- Gluing $n \times m$ squares with degrees $2^a \times 3^b$: Complexity $nm \cdot \widetilde{O}(a+b)$.

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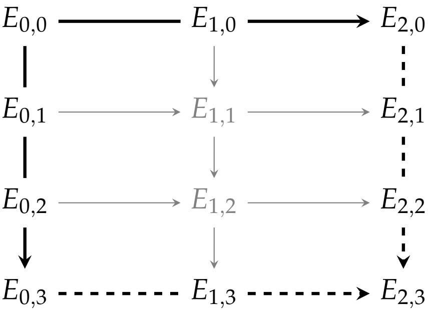
$\Longrightarrow$ Solution: Glue together multiple SIDH squares. "**SIDH ladder**".



- Gluing $n \times m$ squares with degrees $2^a \times 3^b$: Complexity $nm \cdot \widetilde{O}(a+b)$.
- Any base field: Choose $a = b = 1$, potentially going to a degree-$O(1)$ extension.

Performance: Not great, not terrible

	Isogeny Lengths		Proof Size	Running Time	
$\log(p)$	$\rightarrow$	$\downarrow$	(kB)	Prove (s)	Verify (s)
434	705	890	191.19	2.96	0.32
503	774	977	215.75	4.17	0.44
610	1010	1275	404.32	12.12	1.24
751	1280	1616	662.63	26.07	2.89

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- Practical enough for trusted-setup protocols.
- We plan to run a trusted setup ceremony in the real world.

$\Longrightarrow$ Result: the world's first and only **SECUER**s!