ETH zürich

Foundations of Cryptography
foc.ethz.ch

Worst-Case Subexponential Attacks on PRGs of Constant Degree or Constant Locality

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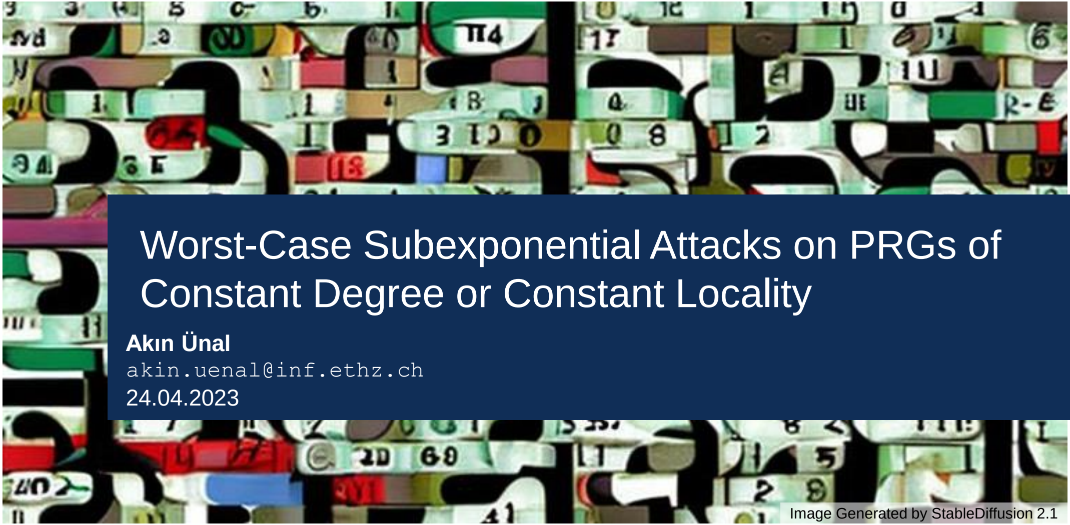

Image Generated by StableDiffusion 2.1

Motivation

Gay-Pass STOC`21
subexp. LWE
+
circular Shielded Randomness
Leakage-security of GSW

Jain-Lin-Sahai EC`22
LPN over $\mathbb{Z}_p$
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Pairings
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Local PRGs $F : \{0,1\}^n \rightarrow \{0,1\}^{n^{1+e}}$

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They need *subexponential security*
i.e. each ppt adversary must have an advantage of
$\leq 2^{-\lambda^c}$ for some $c > 0$.

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Convention

f_i is the function that computes the i-th output value of F.

I.e. $F(x) = \big(f_1(x), \dots, f_m(x)\big)$

Local and Polynomial PRGs

$F: \{0,1\}^n \to \{0,1\}^m$ has **locality** d

iff

each $f_i(x)$ only depends on d bits of $x \in \{0,1\}^n$.

Local and Polynomial PRGs

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$F : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ has **degree** d

iff

each $f_i(X)$ is a polynomial in $\mathbb{Z}_p[X_1, \dots, X_n]$ of total degree d.

Results – Overview

Subexponential Attack on PRGs

$F : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$

of *constant Degree*

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Only high advantage if $p \geq n^c$.

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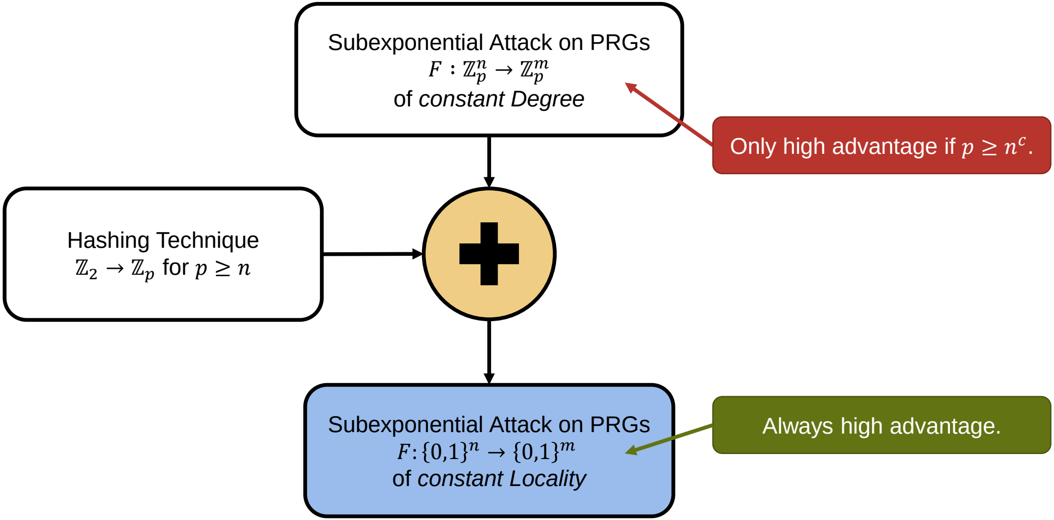

Algebraic Attack

$F : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ PRG of degree d

Each $f_i(X_1, \ldots, X_n)$ is a polynomial in $\mathbb{Z}_p[X_1, \ldots, X_n]$ of degree d

How to distinguish $F(x)$, $x \leftarrow \mathbb{Z}_p^n$, from $y \leftarrow \mathbb{Z}_p^m$?

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First Idea
Assume we have a *linear relationship* between $f_1(X), \ldots, f_m(X)$.

Algebraic Attack: Linear Relationship

$F : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ PRG computed by degree-d polynomials $f_1, \dots, f_m \in \mathbb{Z}_p[X_1, \dots, X_n]$.

Let $v \in \mathbb{Z}_p^m, v \neq 0$, be a linear relationship of $f_1, \dots, f_m$ i.e.

$$v_1 \cdot f_1(X) + \cdots + v_m \cdot f_m(X) = 0$$

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$$\Pr[v_1 \cdot y_1 + \cdots + v_m \cdot y_m = 0] = \frac{1}{p}$$

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This attack has an advantage of $1 - \frac{1}{p}$

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Problem
In general, $f_1, \dots, f_m$ will not be linearly dependent

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An **algebraic relationship** of $f_1(X), \dots, f_m(X) \in \mathbb{Z}_p[X_1, \dots, X_n]$ is a polynomial $h \in \mathbb{Z}_p[Y_1, \dots, Y_m]$ s.t.

1. $h(Y) \neq 0$
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Schwartz-Zippel Lemma

1. How can we compute h?
2. What is an upper bound for $\deg h$?

Bound Degree of Algebraic Relationship h

$$\phi: \mathbb{Z}_p[Y_1, \dots, Y_m] \to \mathbb{Z}_p[X_1, \dots, X_n]$$
$$g(Y_1, \dots, Y_m) \mapsto g\big(f_1(X), \dots, f_m(X)\big)$$

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- $\ker \phi$ contains all algebraic relationships of $f_1, \dots, f_m$

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- $\ker \phi_L$ contains all algebraic relationships of $f_1, \dots, f_m$ of degree $\leq L$
- $\mathbb{Z}_p[Y_1, \dots, Y_m]^{\leq L} = \{g \in \mathbb{Z}_p[Y_1, \dots, Y_m] : \deg g \leq L\}$
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- $\dim \ker \phi_L \geq \dim \mathbb{Z}_p[Y_1, \dots, Y_m]^{\leq L} - \dim \mathbb{Z}_p[X_1, \dots, X_n\,]^{\leq dL}$

Dimension Formula for Linear Maps

Bound Degree of Algebraic Relationship h

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$\Leftrightarrow \dim \ker \phi_L > 0$
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$\Leftarrow L \geq 2^{\frac{d}{d-1}} \cdot n^{1-\frac{e}{d-1}}$

How to Compute h?

We know that $\ker \phi_L$ contains h for $L = \left\lceil 2^{\frac{d}{d-1}} \cdot n^{1-\frac{e}{d-1}} \right\rceil$.

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Compute matrix representation of

$$\phi_L: \mathbb{Z}_p[Y_1, \ldots, Y_m]^{\leq L} \to \mathbb{Z}_p[X_1, \ldots, X_n]^{\leq dL}$$

and solve for $\ker \phi_L$ via Gaussian elimination.

Algebraic Attack: Algorithm

Given PRG $F: \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ consisting of $f_1, \dots, f_m \in \mathbb{Z}_p[X_1, \dots, X_n]$ of degree d, $m \geq n^{1+e}$, and $y \in \mathbb{Z}_p^m$.

Decide if $y = \left(f_1(x), \dots, f_m(x)\right)$ or if $y \leftarrow \mathbb{Z}_p^m$.

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1. Compute $L := \left\lceil 2^{\frac{d}{d-1}} \cdot n^{1-\frac{e}{d-1}} \right\rceil$
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3. If $h(y) = 0$, decide that y is of form $\left(f_1(x), \dots, f_m(x)\right)$
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Good if $p \in \omega\left(n^{1-\frac{e}{d-1}}\right)$.

Bad if $p \in o\left(n^{1-\frac{e}{d-1}}\right)$.

Hashing to Larger Fields

Idea: Convert local PRG $F: \{0,1\}^n \to \{0,1\}^m$ and $y \in \{0,1\}^m$

to a polynomial PRG $G: \{0,1\}^n \to \mathbb{Z}_p^{m'}$ and $y' \in \mathbb{Z}_p^{m'}$

with $p \geq n$ and $m' \approx m$.

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1. Choose prime $p \geq n$
2. Set $m' = \left\lceil \frac{m}{3 \log p} \right\rceil$
3. Draw $A \leftarrow \mathbb{Z}_p^{m' \times m}$
4. Compute $y' \coloneqq A \cdot y$ for $y \in \{0,1\}^m$
5. Compute $G \coloneqq A \cdot F$

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Leftover-Hash-Lemma
y' is close $U(\mathbb{Z}_p^{m'})$ if $y \leftarrow \{0,1\}^m$

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What is the algebraic degree of G?

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4. Compute $y' \coloneqq A \cdot y$ for $y \in \{0,1\}^m$
5. Compute $G \coloneqq A \cdot F$

What is the algebraic degree of G?

$\deg G = \operatorname{loc} F$

Hashing + Algebraic Attack on Binary PRGs

Given a PRG $F : \{0,1\}^n \to \{0,1\}^m$ of locality d and a (pseudo-)random bitstring $y \in \{0,1\}^m$, $m \geq n^{1+e}$.

1. Draw $A \leftarrow \mathbb{Z}_p^{\left\lceil \frac{m}{3 \log p} \right\rceil \times m}$ for prime $p \in [n, 2n]$
2. Compute $G \coloneqq A \cdot F: \{0,1\}^n \to \mathbb{Z}_p^{\left\lceil \frac{m}{3 \log p} \right\rceil}$
3. Compute $y' \coloneqq A \cdot y \in \mathbb{Z}_p^{\left\lceil \frac{m}{3 \log p} \right\rceil}$
4. Compute an algebraic relation $h \in \mathbb{Z}_p[Y]$ for G of degree $O\left((\log n)^{\frac{1}{d-1}} \cdot n^{1-\frac{e}{d-1}}\right)$
5. Output 0 if $h(y') = 0$, otherwise 1

Hashing + Algebraic Attack on Binary PRGs

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Time Complexity:

$$m^{O(\deg h)} = n^{O\left((\log n)^{\frac{1}{d-1}} \cdot n^{1-\frac{e}{d-1}}\right)}$$

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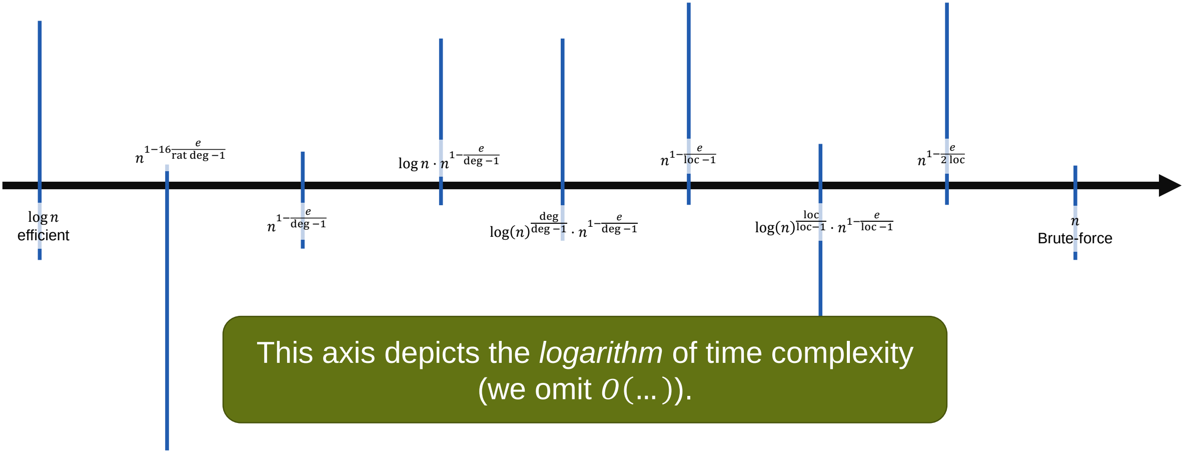
Time Complexity:

$$m^{O(\deg h)} = n^{O\left((\log n)^{\frac{1}{d-1}} \cdot n^{1-\frac{e}{d-1}}\right)}$$

Advantage:

$$1 - O\left(\frac{\deg h}{p}\right) = 1 - O\left(\frac{(\log n)^{\frac{1}{d-1}} \cdot n^{1-\frac{e}{d-1}}}{n}\right) \geq 1 - o(1)$$

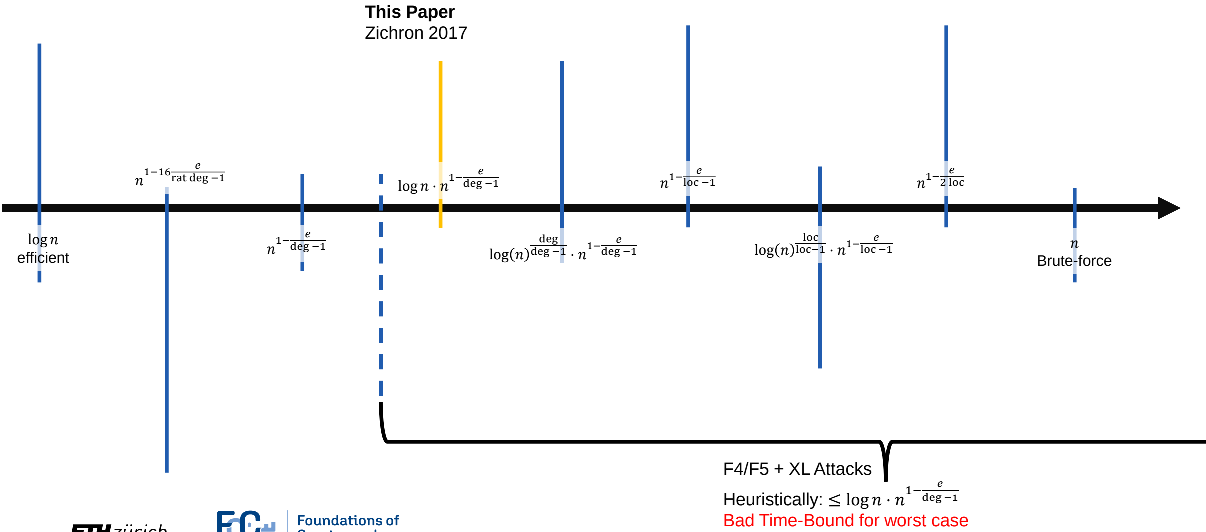
Overview: Attacks on PRGs



This axis depicts the *logarithm* of time complexity (we omit $O(\ldots)$).

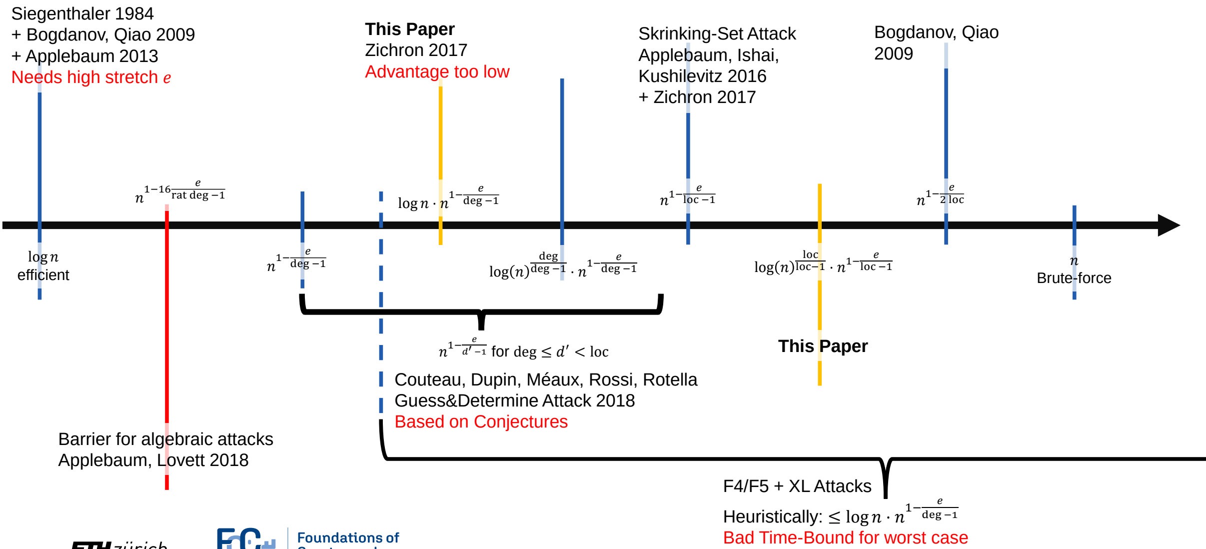
Overview: Attacks on *Constant-Degree* PRGs ($p \geq n^c$)

$$F: \mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^{n^{1+e}}$$



Overview: Attacks on *Local* PRGs

$$F: \{0,1\}^n \to \{0,1\}^{n^{1+e}}$$



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Appendix

How to Bound L

$$\binom{m+L}{L} = \frac{(m+L)\cdots(m+1)}{L\cdots 1} > \frac{(n+dL)\cdots(n+1)}{(dL)\cdots 1} = \binom{n+dL}{dL}$$

$$\Leftrightarrow \quad (m+L)\cdots(m+1)\cdot(dL)\cdots(L+1) > (n+dL)\cdots(n+1)$$

$$\approx \quad m^L \cdot L^{(d-1)L} \geq n^{dL}$$

$$\Leftrightarrow \quad m \cdot L^{d-1} \geq n^d$$

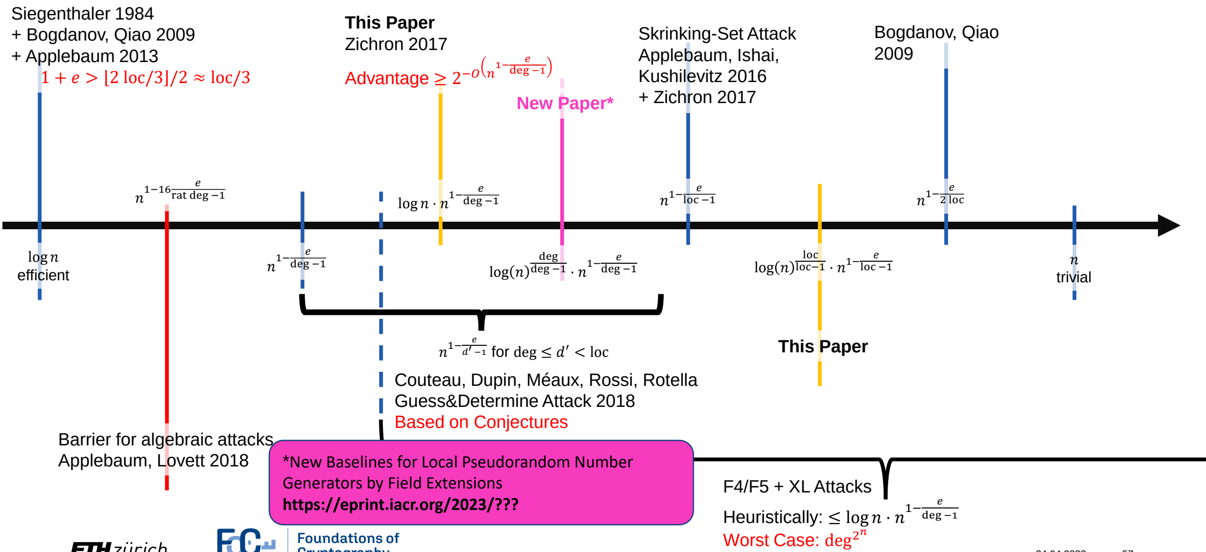
$$\Leftrightarrow \quad L \geq \sqrt[d-1]{n^d/m} \geq n^{1-\frac{e}{d-1}}$$

$(m+L)\cdots(m+1) \approx m^L$
$(dL)\cdots(L+1) \approx L^{(d-1)L}$
$(n+dL)\cdots(n+1) \approx n^{dL}$

Actually $L \geq 2^{\frac{d}{d-1}} \cdot n^{1-\frac{e}{d-1}}$

Overview: Attacks on *Local* PRGs

$$F: \{0,1\}^n \rightarrow \{0,1\}^{n^{1+e}}$$



Hashing Trick: Bad Trade-Off

$F\colon\{0,1\}^n \to \{0,1\}^m$ consists of tri-sum-ands:

$$P \coloneqq (X_1 \wedge X_2) \oplus X_3 \oplus X_4 \oplus X_5 \simeq X_1 \cdot X_2 + X_3 + X_4 + X_5 \bmod 2 \in \mathbb{Z}_2[X_1, \dots, X_5]$$

The same polynomial

$$X_1 \cdot X_2 + X_3 + X_4 + X_5 \bmod p \text{ in } \mathbb{Z}_p[X_1, \dots, X_5]$$

does not compute the same as P over $\{0,1\}^5$:

$$1 \oplus 1 = 1 + 1 = 0 \bmod 2 \qquad\qquad 1 + 1 = 2 \neq 0 \bmod p$$

There is a degree-5 polynomial in $\mathbb{Z}_p[X_1, \dots, X_5]$ that coincides with P on $\{0,1\}^5$:

$$X_1X_2 + X_3 + X_4 + X_5 - X_1X_2X_3 - X_4X_5 - X_1X_2X_4 - X_1X_2X_5 - X_3X_4 - X_3X_5 + X_1X_2X_4X_5 + X_3X_4X_5 + X_1X_2X_3X_4 + X_1X_2X_3X_5 - X_1X_2X_3X_4X_5$$

New Extension Trick

We consider the field extension $GF\left(2^{\lceil \log n \rceil}\right)$ of $\mathbb{Z}_2$.

$$GF\left(2^{\lceil \log n \rceil}\right) \approx \mathbb{Z}_2[\zeta] = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cdot \zeta \oplus \cdots \oplus \mathbb{Z}_2 \cdot \zeta^{\lceil \log n \rceil - 1}$$

We have the bijective map

$$\psi: \{0,1\}^{\lceil \log n \rceil} \to GF\left(2^{\lceil \log n \rceil}\right)$$
$$\left(b_1, \ldots, b_{\lceil \log n \rceil}\right) \mapsto b_1 + b_2 \cdot \zeta + \cdots + b_{\lceil \log n \rceil} \cdot \zeta^{\lceil \log n \rceil - 1}$$

Consider the $m' \times (m' \cdot \lceil \log n \rceil)$-matrix

$$A = I_{m'} \otimes \left(1\ \zeta\ \cdots\ \zeta^{\lceil \log n \rceil - 1}\right) = \begin{pmatrix} 1\ \zeta\ \cdots\ \zeta^{\lceil \log n \rceil - 1} & & \\ & \ddots & \\ & & 1\ \zeta\ \cdots\ \zeta^{\lceil \log n \rceil - 1} \end{pmatrix}$$

For $y \leftarrow \{0,1\}^{m' \cdot \lceil \log n \rceil}$, the vector $A \cdot y$ is a uniformly random element of $GF\left(2^{\lceil \log n \rceil}\right)^{m'}$.

New Extension Trick

We have a natural and homomorphic inclusion of fields $\mathbb{Z}_2 \subset GF\left(2^{\lceil \log n \rceil}\right)$.

This transfers to polynomial rings: $\mathbb{Z}_2[X_1, \dots, X_n] \subset GF\left(2^{\lceil \log n \rceil}\right)[X_1, \dots, X_n]$.

If $f_1, \dots, f_{\lceil \log n \rceil} \in \mathbb{Z}_2[X_1, \dots, X_n]$ are of degree d, then so is

$$f_1(X) + \zeta \cdot f_2(X) + \cdots + \zeta^{\lceil \log n \rceil - 1} \cdot f_{\lceil \log n \rceil}(X)$$

If $F: \{0,1\}^n \to \{0,1\}^{m' \cdot \lceil \log n \rceil}$ is of degree d, then so is $G \coloneqq A \cdot F: \{0,1\}^n \to \{0,1\}^{m'}$.

Degree of new PRG G equals **Degree** of old PRG F.