An efficient key recovery attack on SIDH

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Supersingular Isogeny Diffie–Hellman

- SIDH °2011 (Jao & De Feo)
- Previous security:
  - generic attacks with complexity $\mathcal{O}(\sqrt[4]{p})$ classical and $\mathcal{O}(\sqrt[6]{p})$ quantum (claw finding);
  - attack on unbalanced parameters (de Quehen, Kutas, Leonardi, Martindale, Panny, Petit, Stange);
  - chosen ciphertext attack against static key SIDH (Galbraith, Petit, Shani, Ti).

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- SIKE (Supersingular Isogeny Key Encapsulation) allows long term public keys.
- July 5th 2022: SIKE advances to Round 4 in NIST's post-quantum standardization process.

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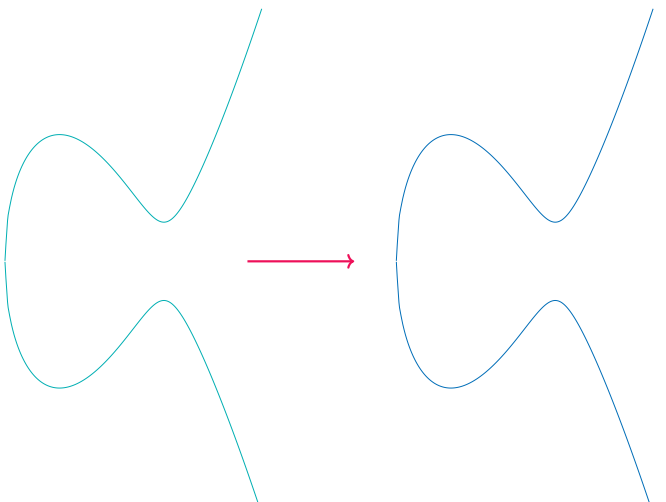
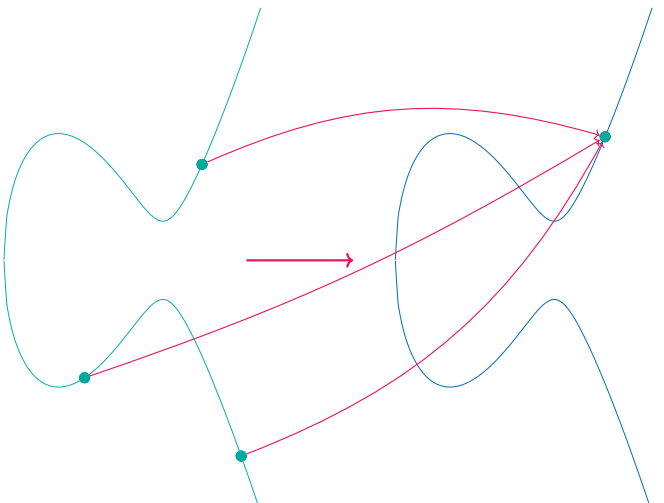
- This work: attack in (heuristic) polynomial time or subexponential time.

Isogenies



Isogenies



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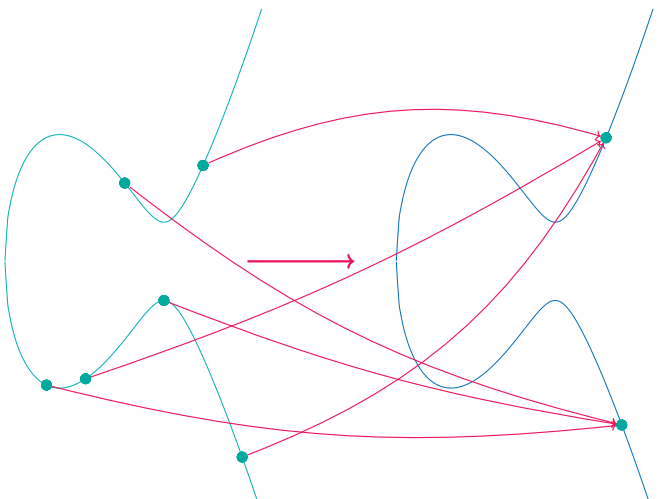

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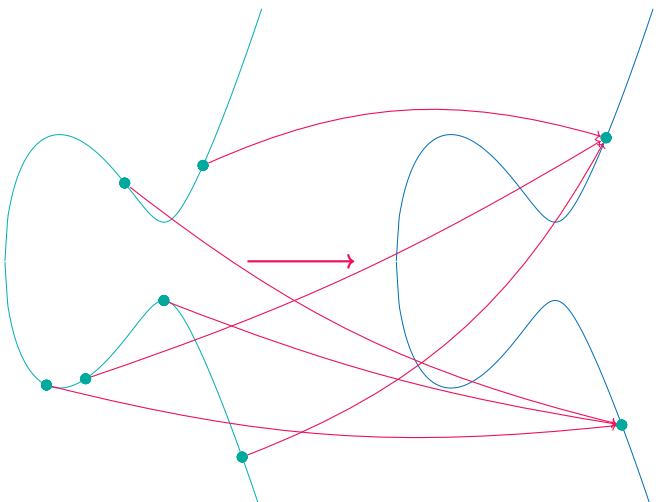
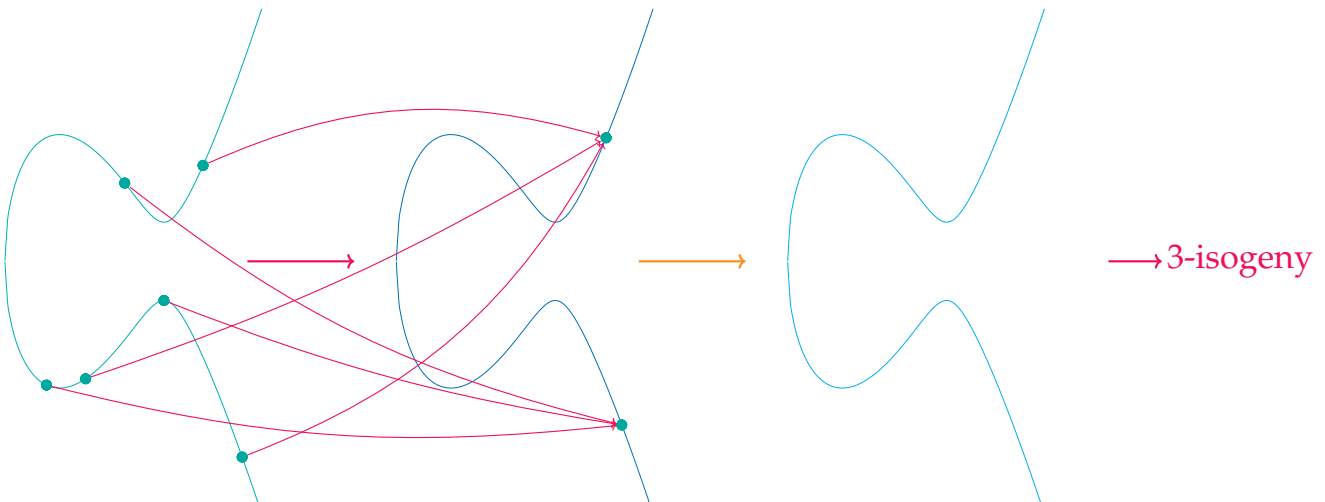


$\longrightarrow$ 3-isogeny

Isogenies



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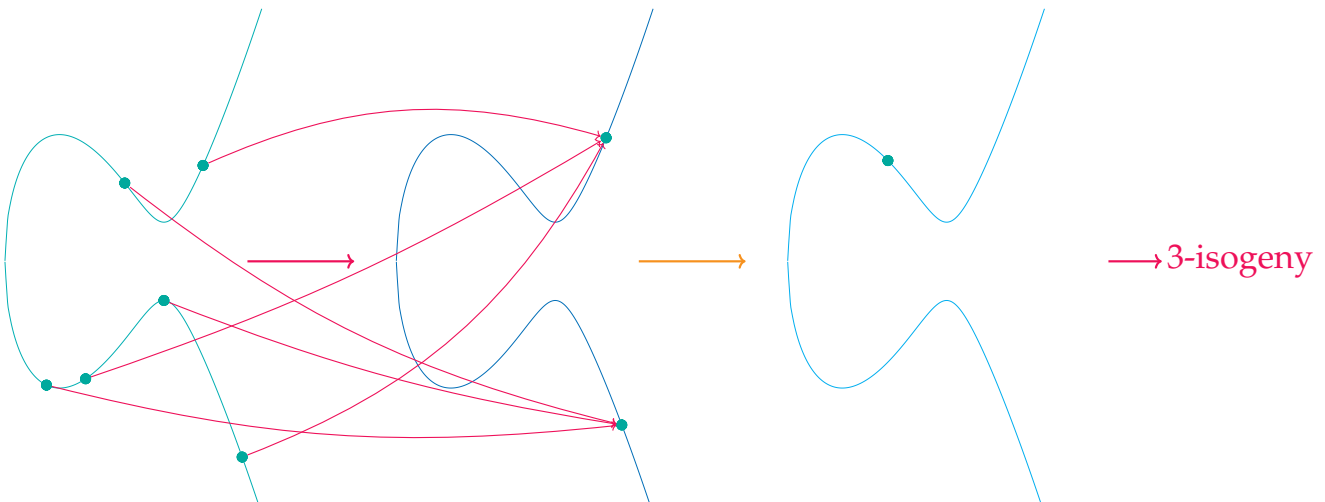

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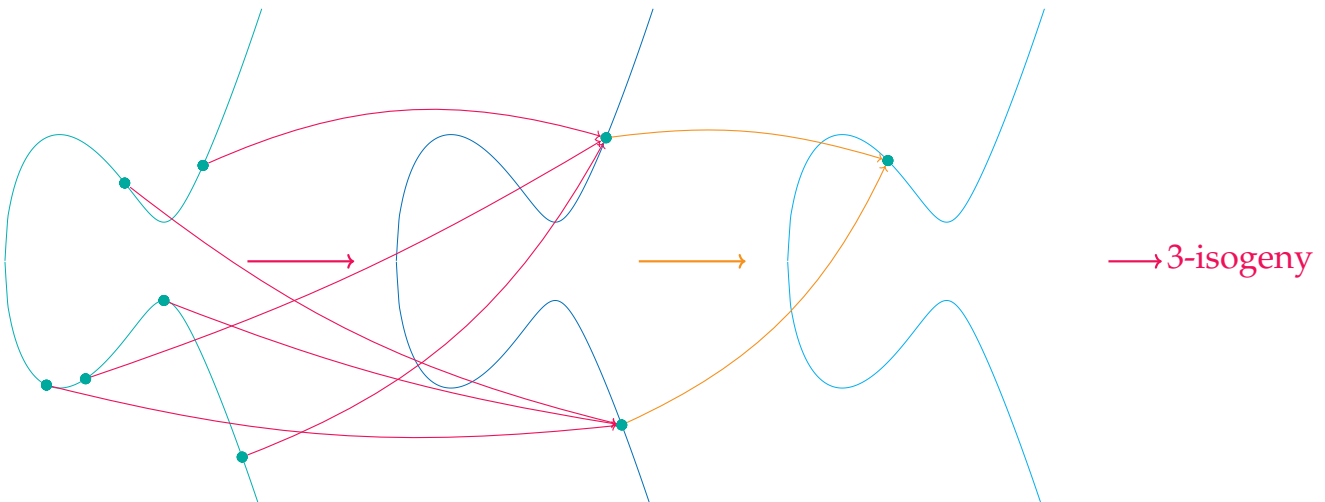

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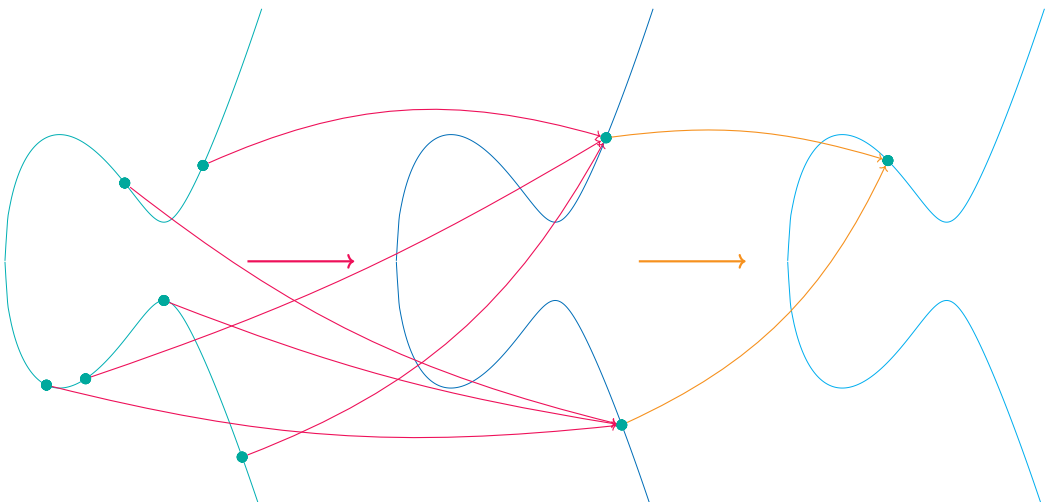
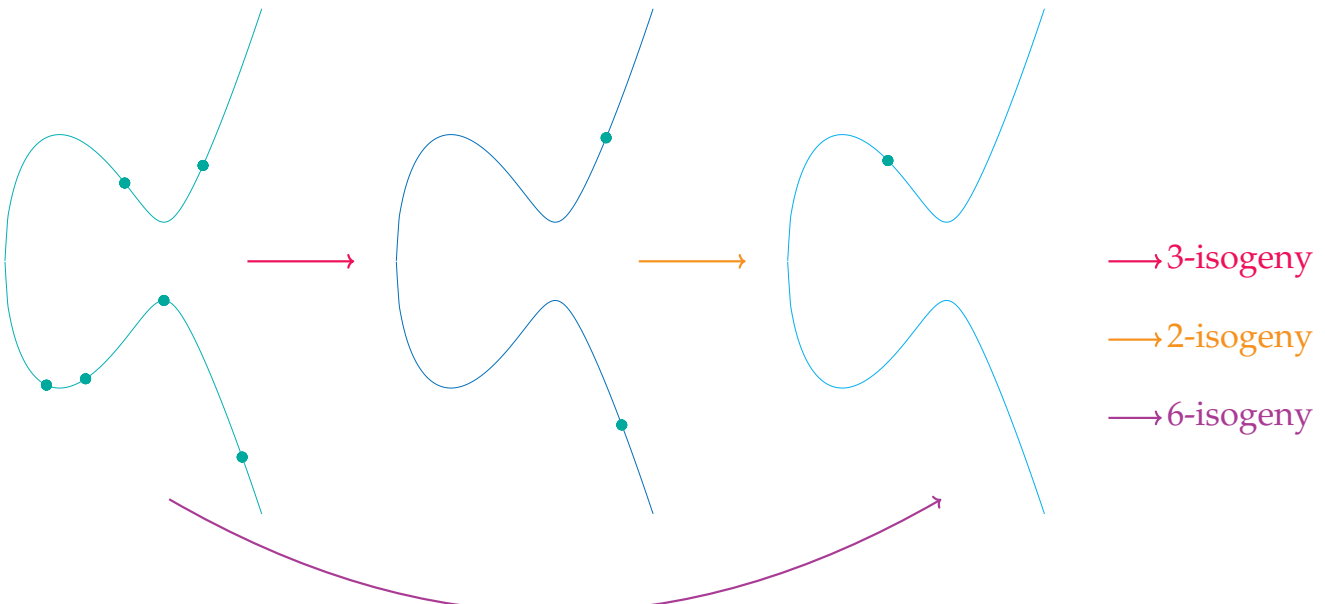


$\longrightarrow$ 3-isogeny

$\longrightarrow$ 2-isogeny

Isogenies



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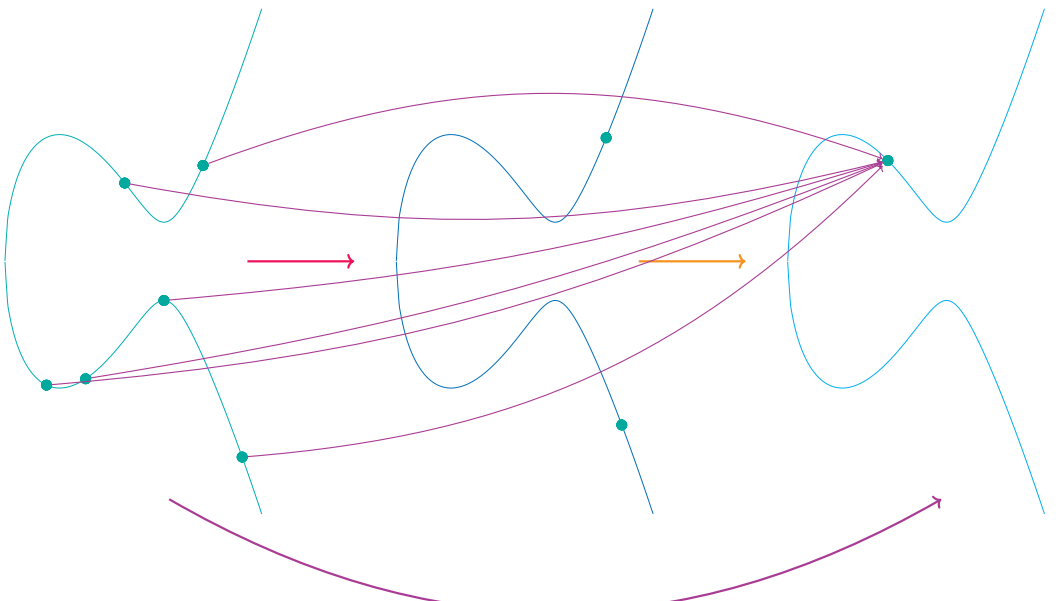
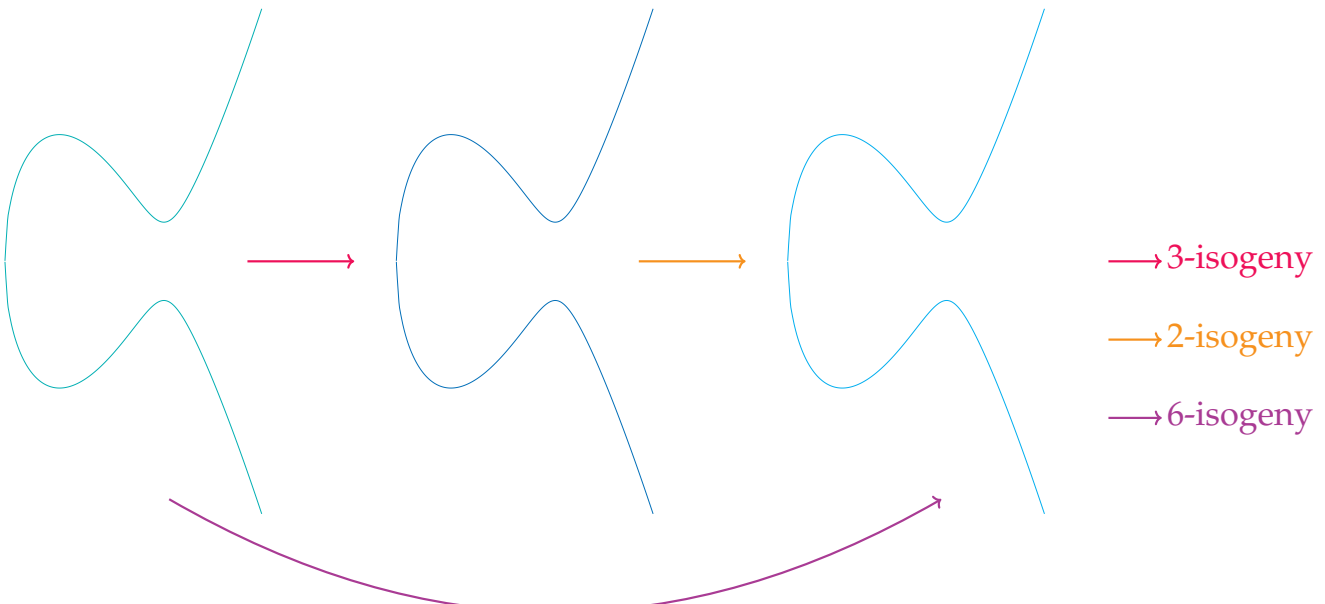


$\longrightarrow$ 3-isogeny

$\longrightarrow$ 2-isogeny

$\longrightarrow$ 6-isogeny

Isogenies



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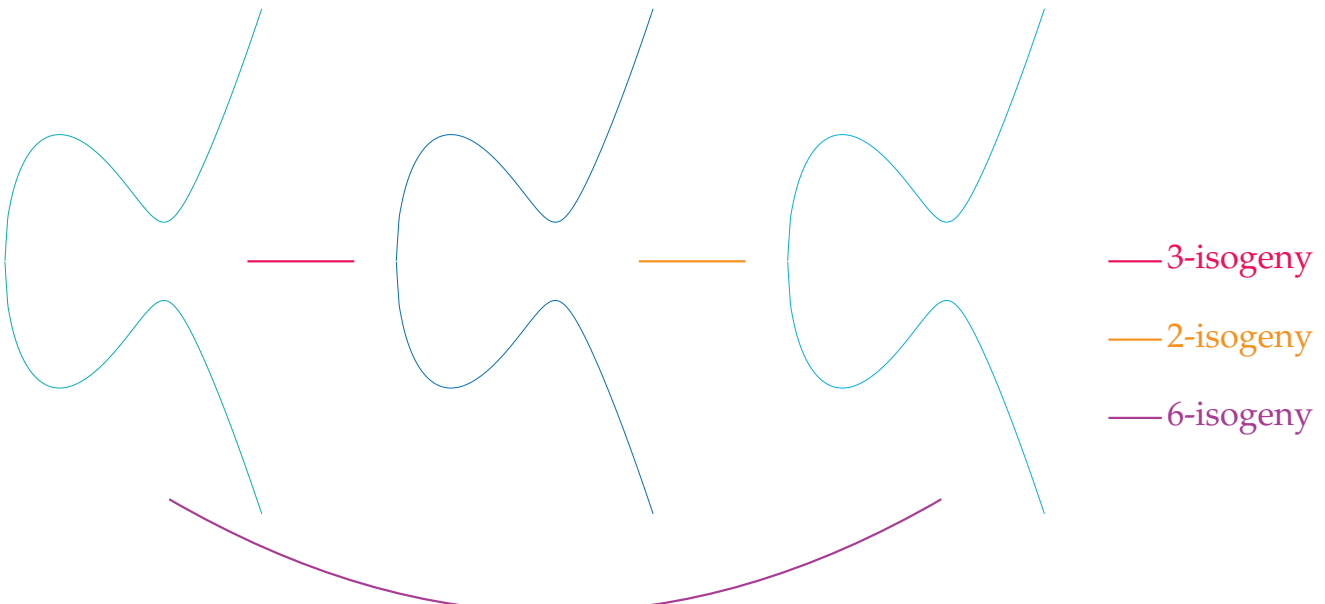

SIDH KEY EXCHANGE

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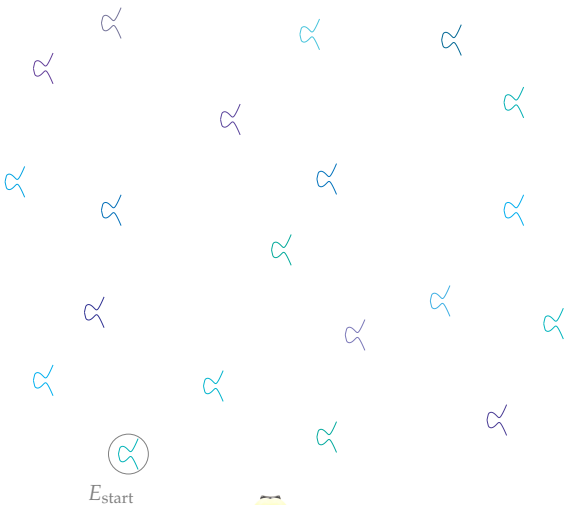

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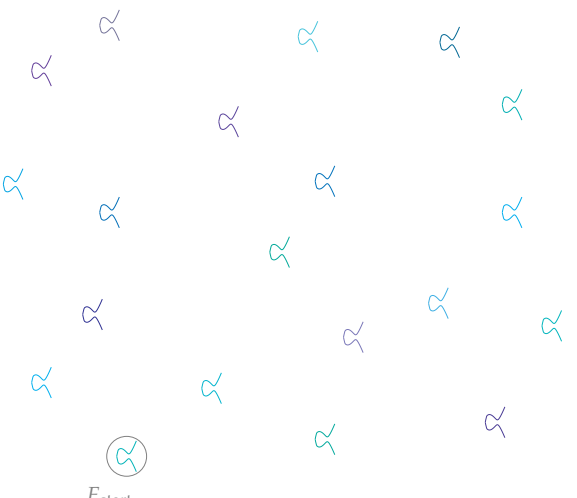

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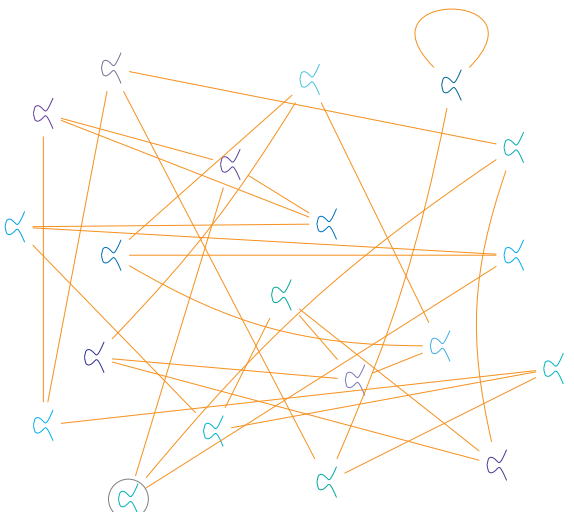

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— 2-isogeny

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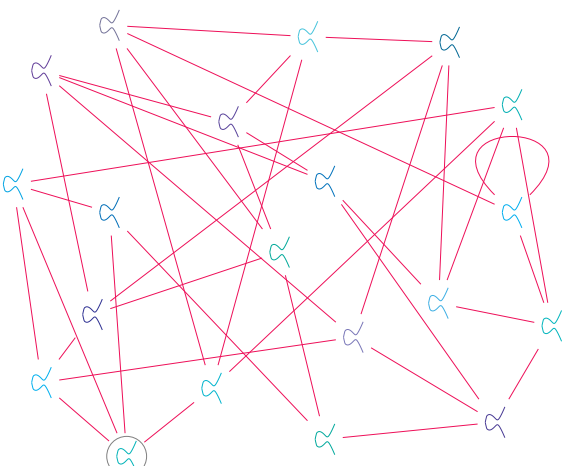

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BOB

— 3-isogeny

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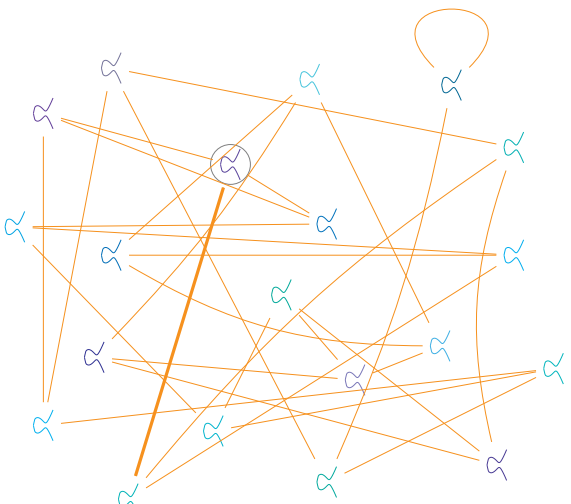
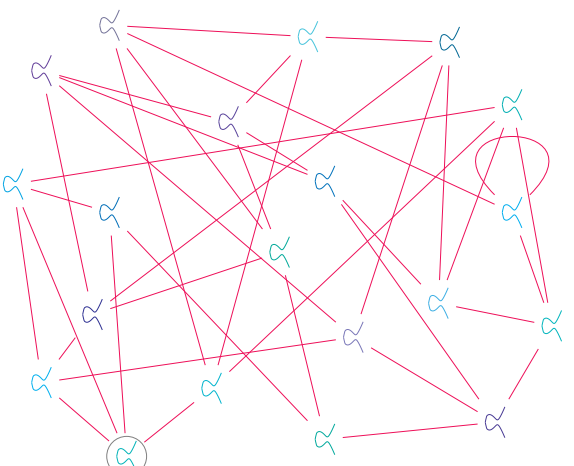


E_{start}

ALICE

2-isogeny

secret path



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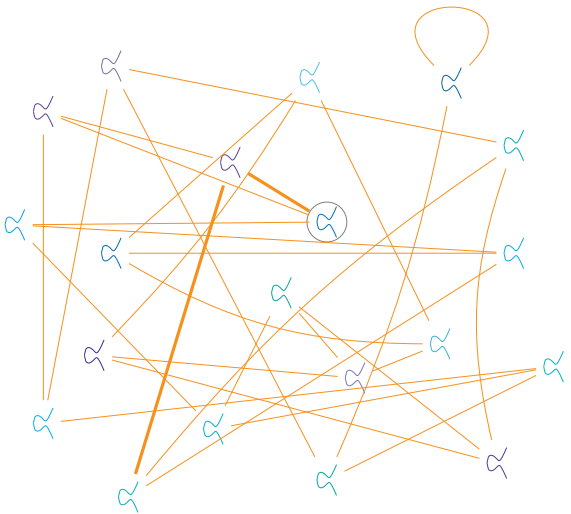

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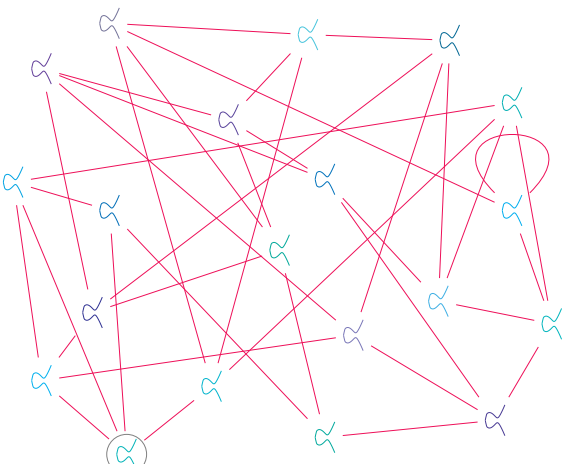

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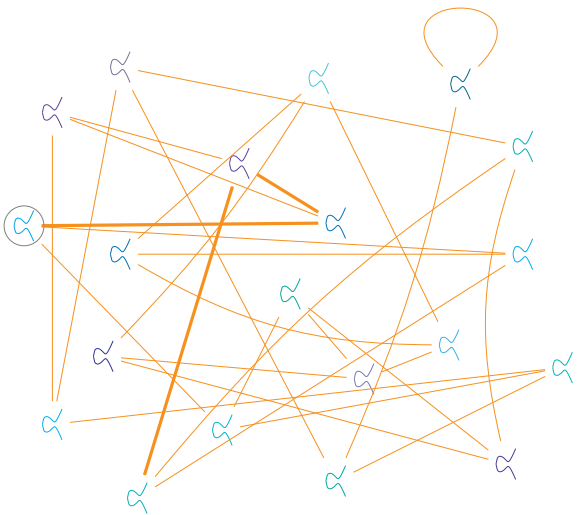

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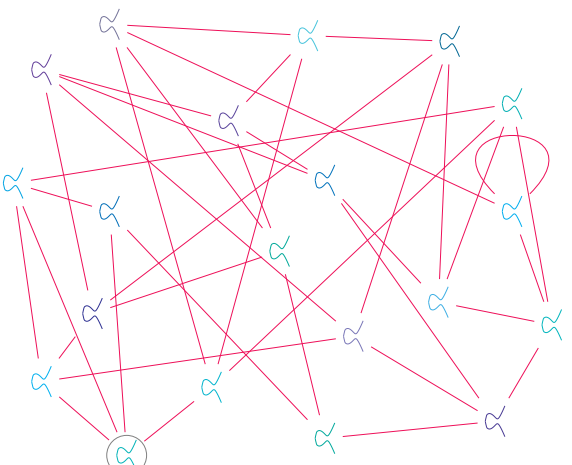

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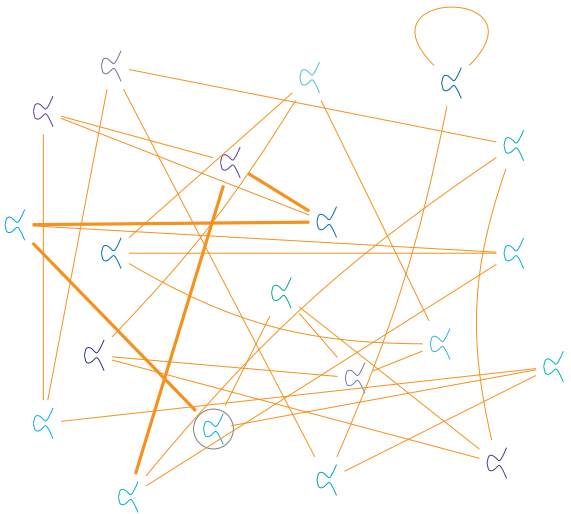

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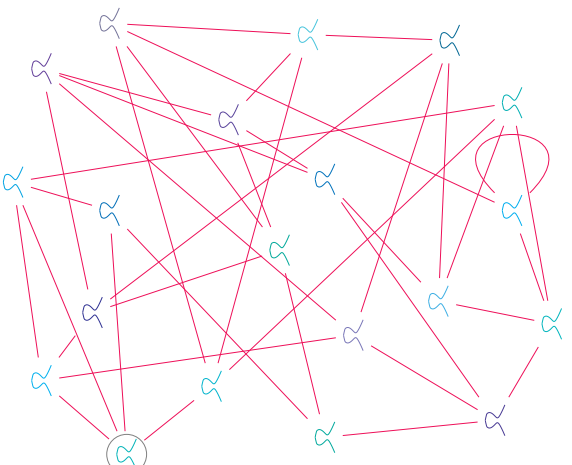

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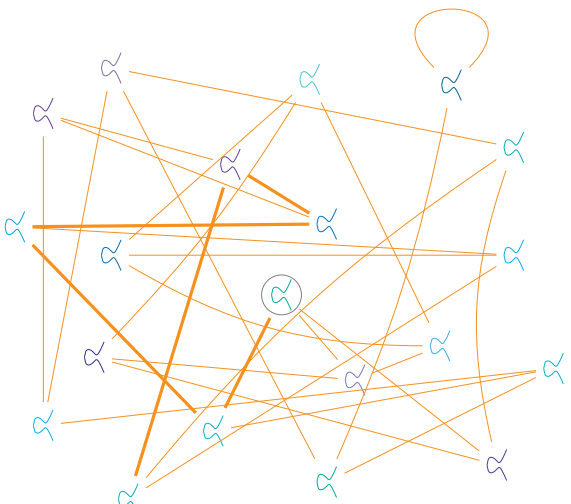

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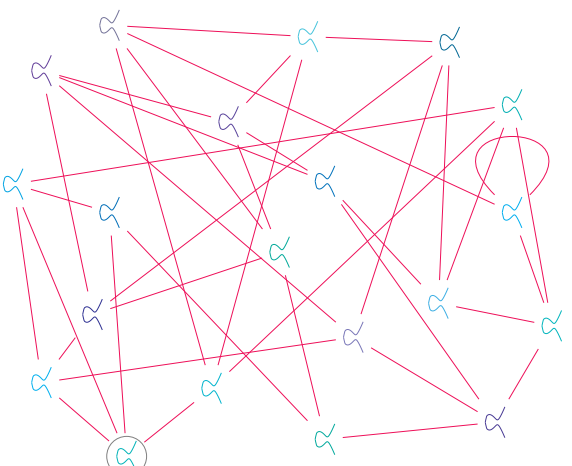

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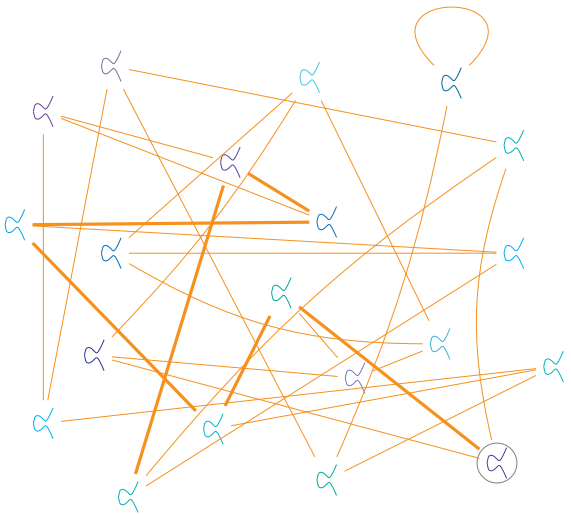

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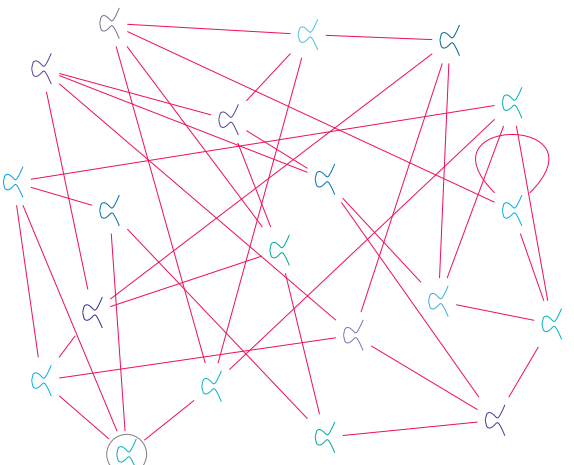

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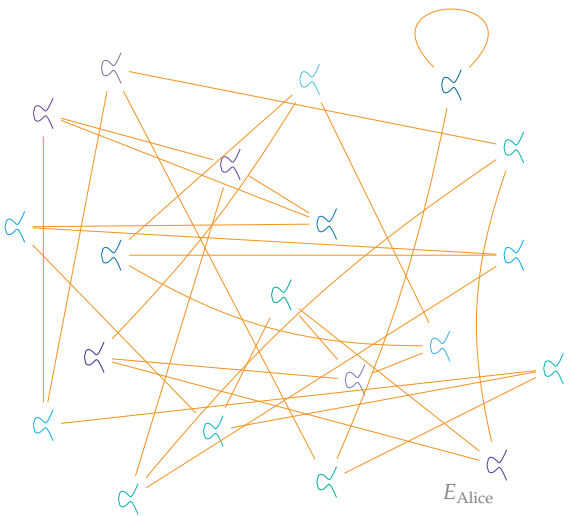

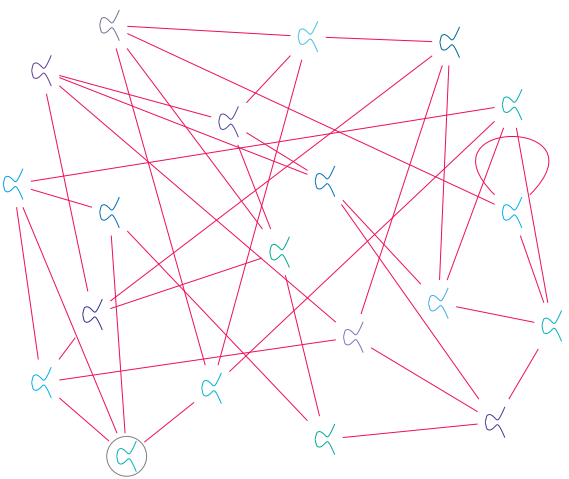

— 2-isogeny
— secret path

— 3-isogeny

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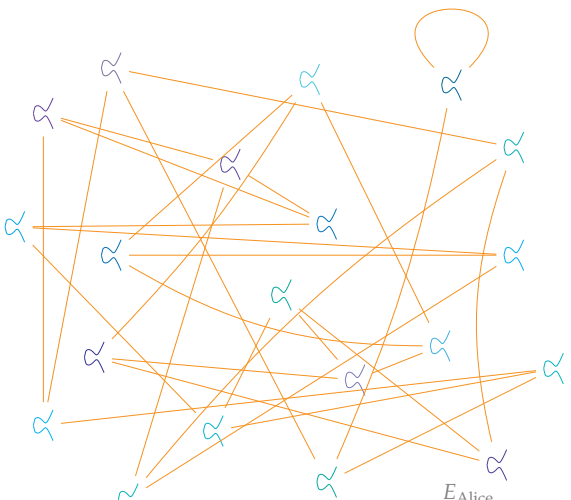

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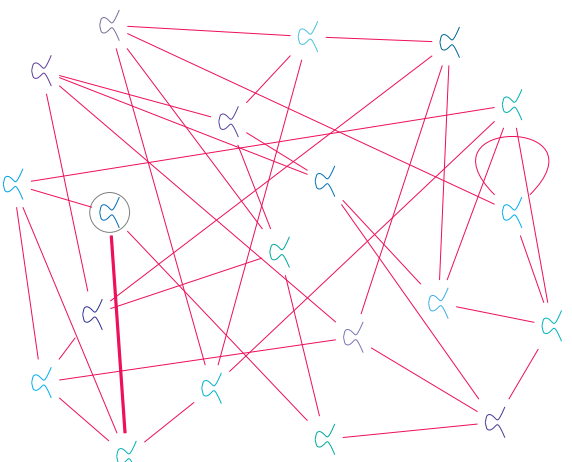

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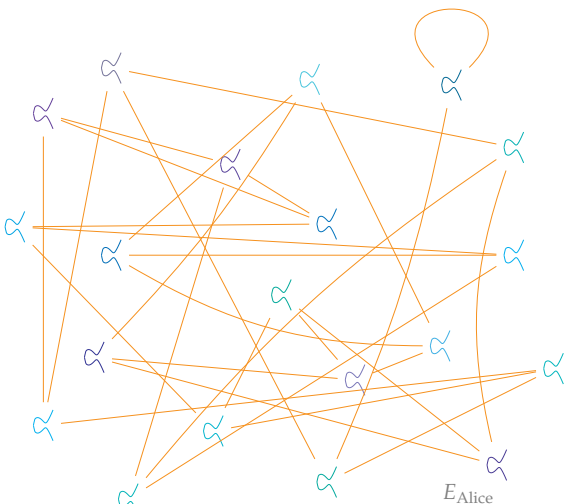

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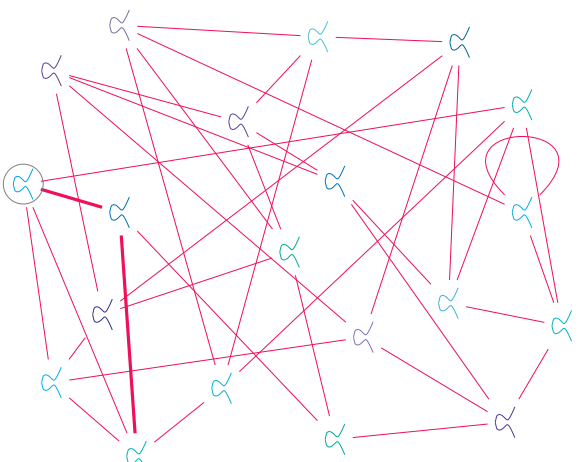
ALICE

2-isogeny

secret path



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BOB

3-isogeny

secret path

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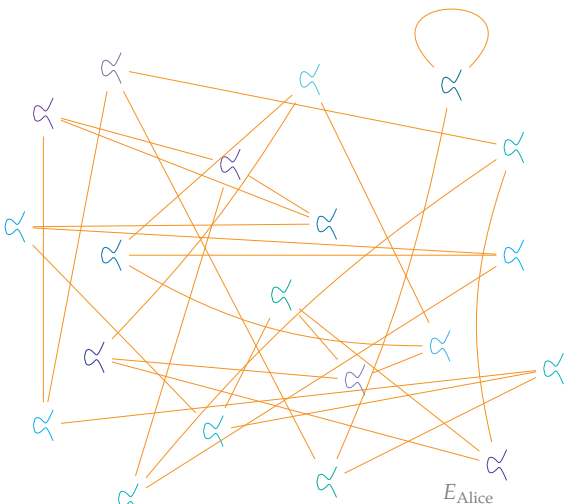

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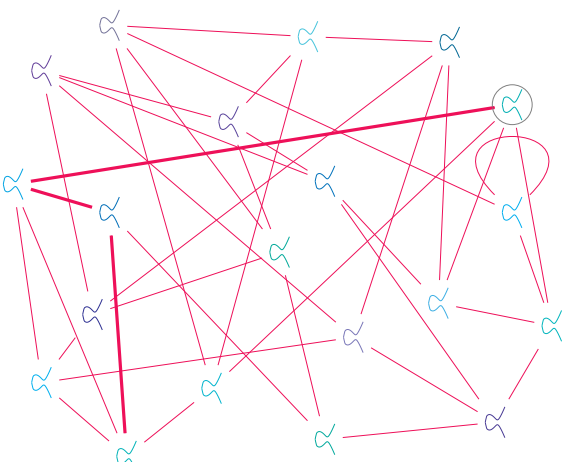

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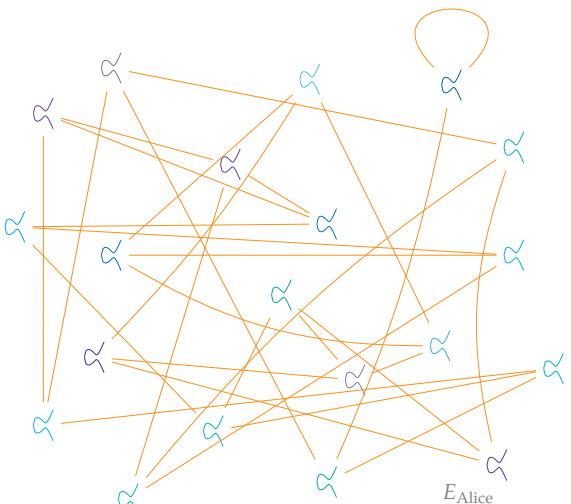

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ALICE

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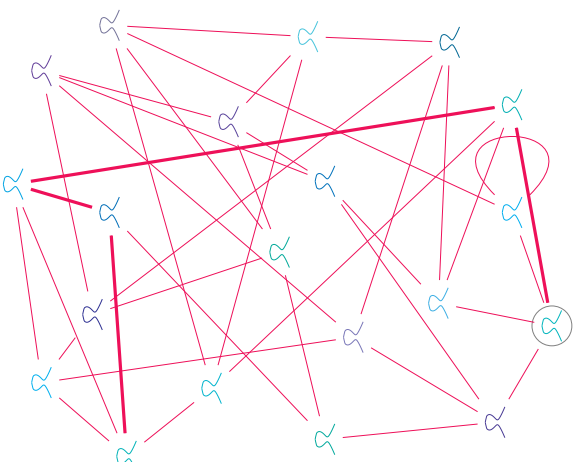

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BOB

3-isogeny

secret path

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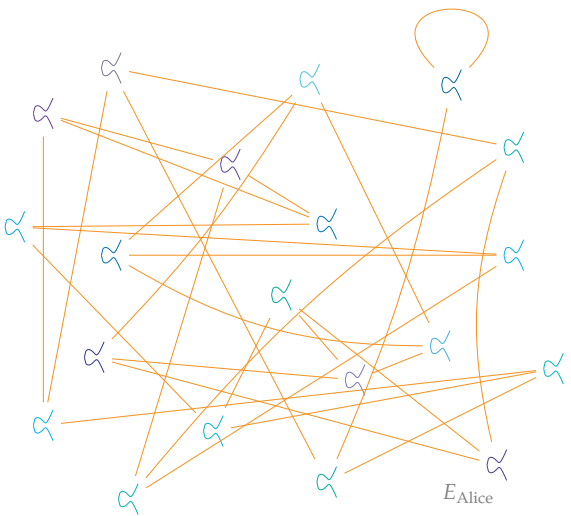

2-isogeny
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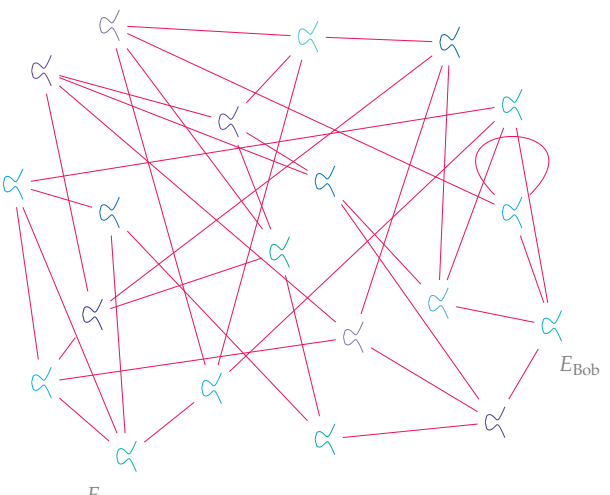

3-isogeny
secret path

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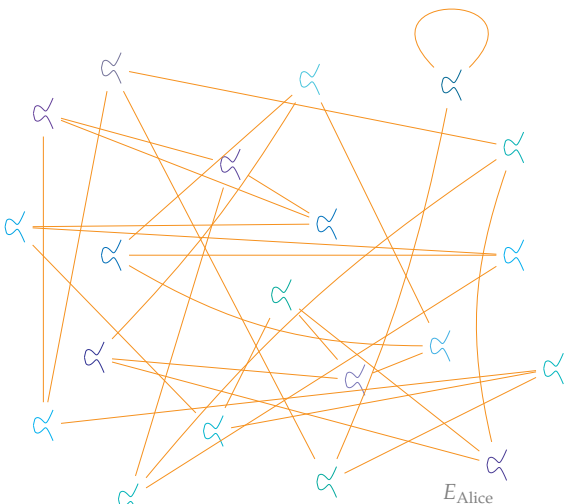
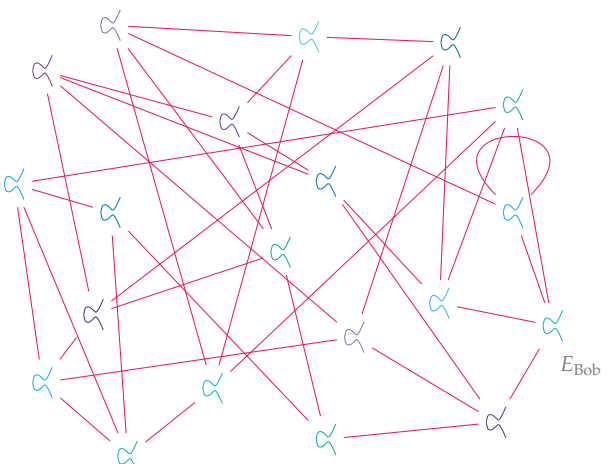




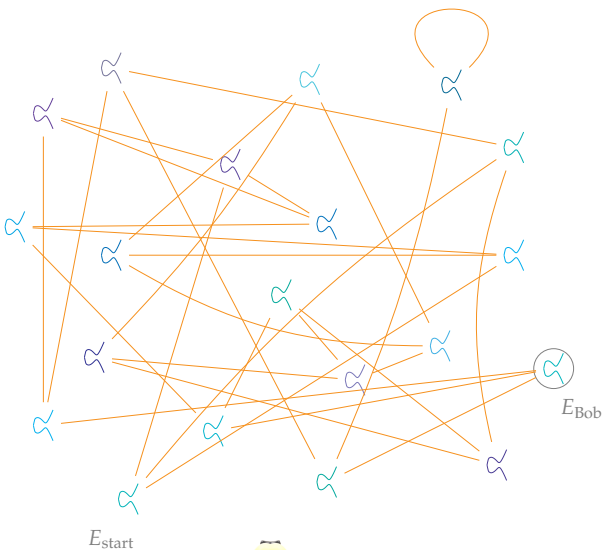
2-isogeny
secret path

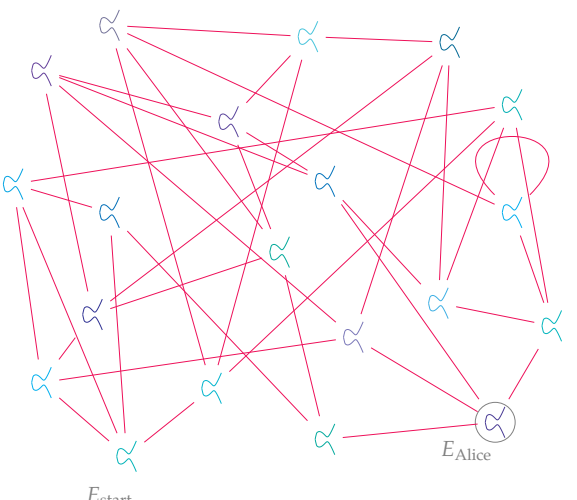
E_{Alice}

E_{Bob}

3-isogeny
secret path

SIDH KEY EXCHANGE





2-isogeny
secret path

3-isogeny
secret path

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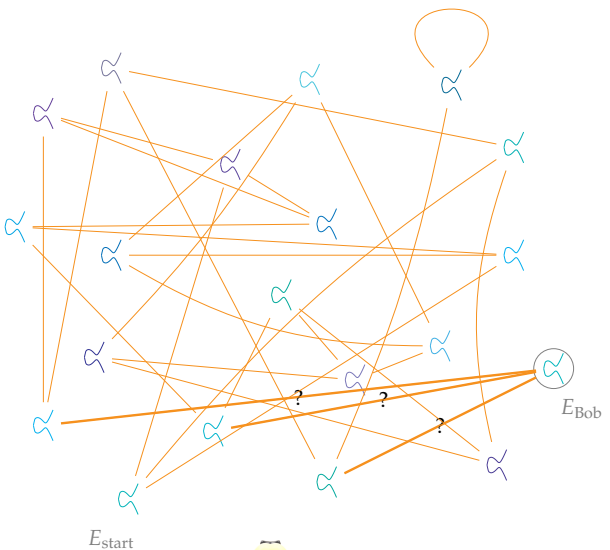

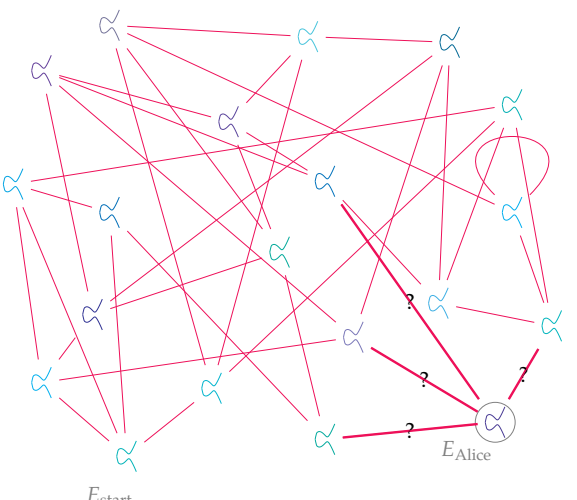

2-isogeny
secret path

3-isogeny
secret path

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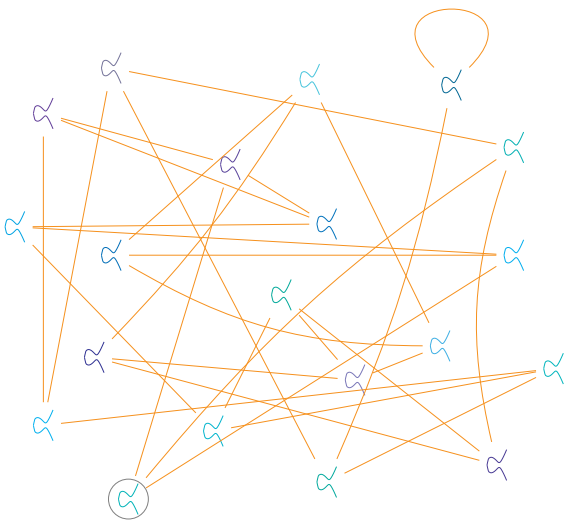

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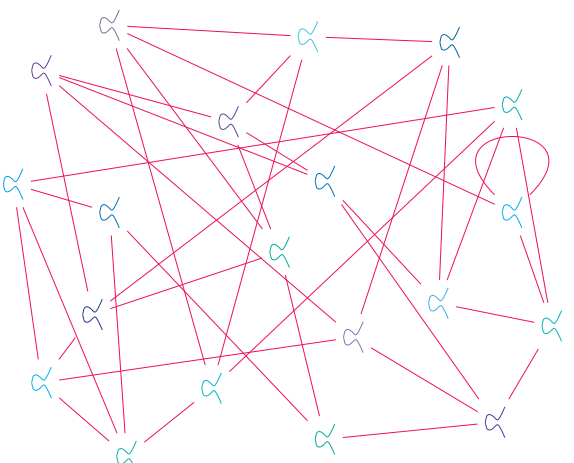

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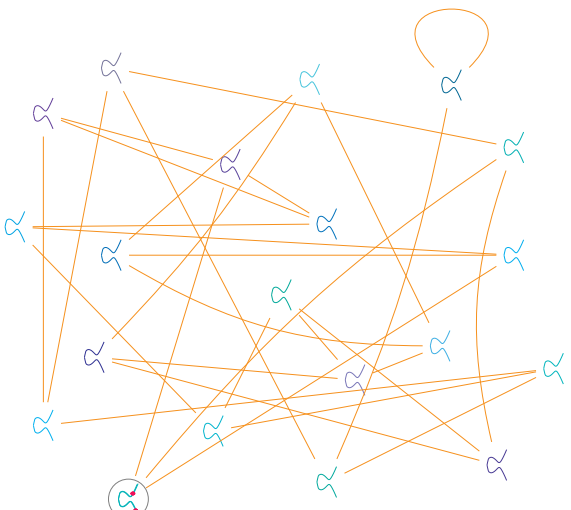

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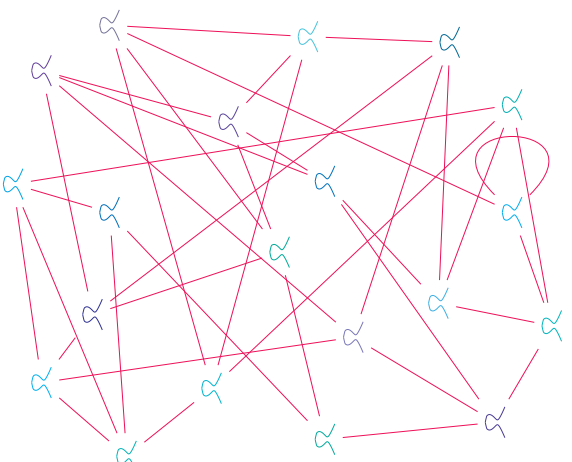

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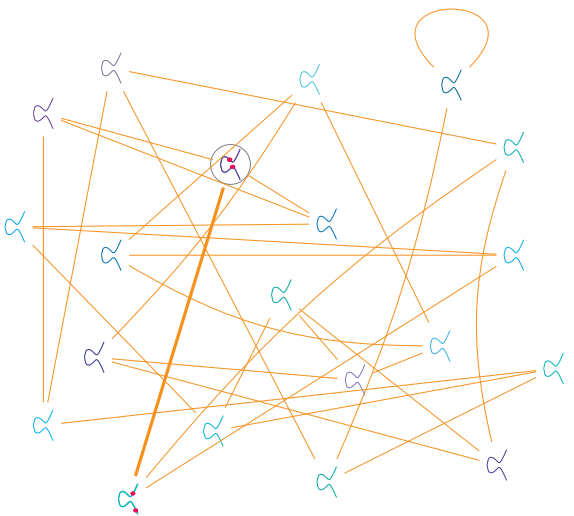
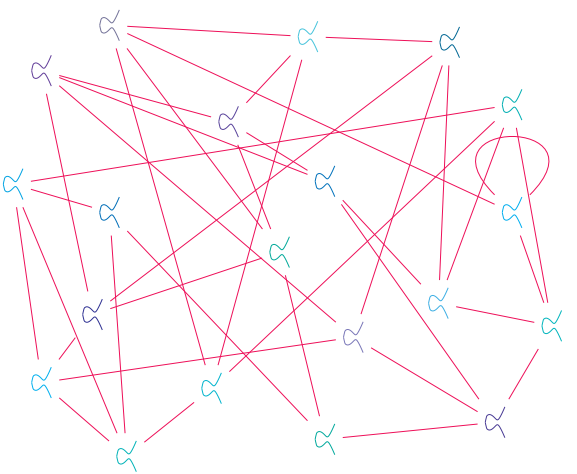
3-isogeny
secret path

SIDH KEY EXCHANGE



E_{start}

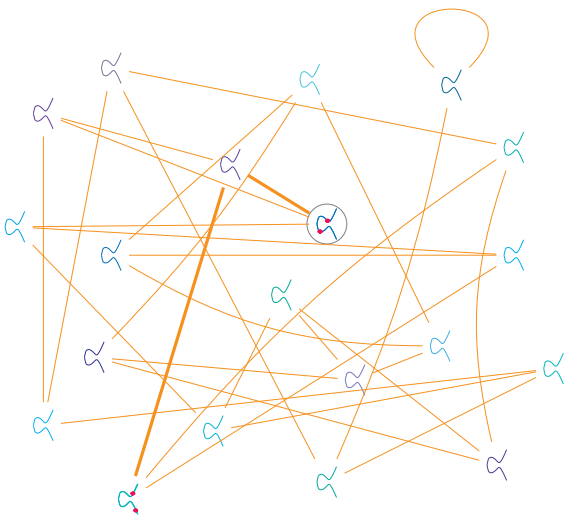
2-isogeny
secret path



E_{start}

3-isogeny
secret path

SIDH KEY EXCHANGE

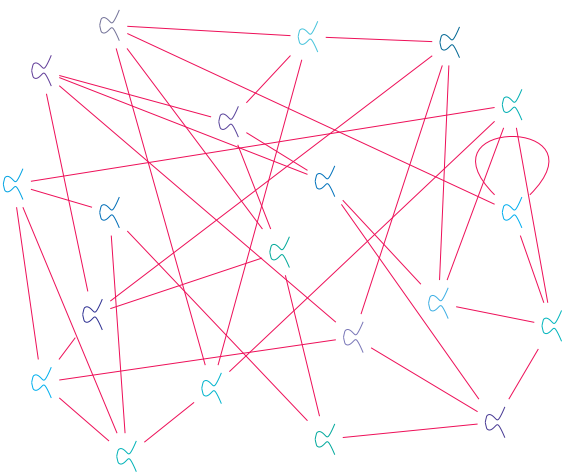


E_{start}

ALICE

2-isogeny

secret path



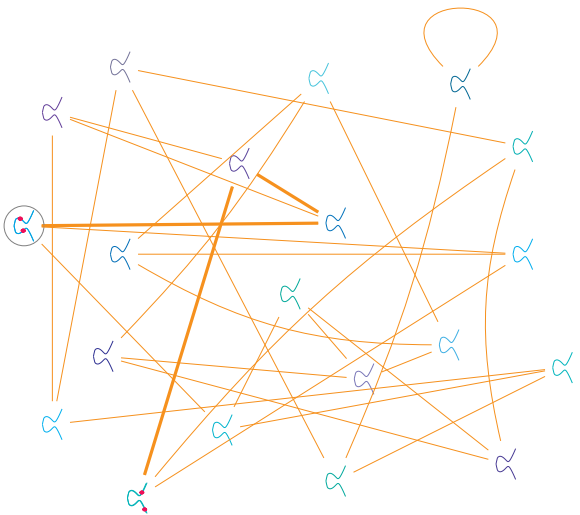
E_{start}

BOB

3-isogeny

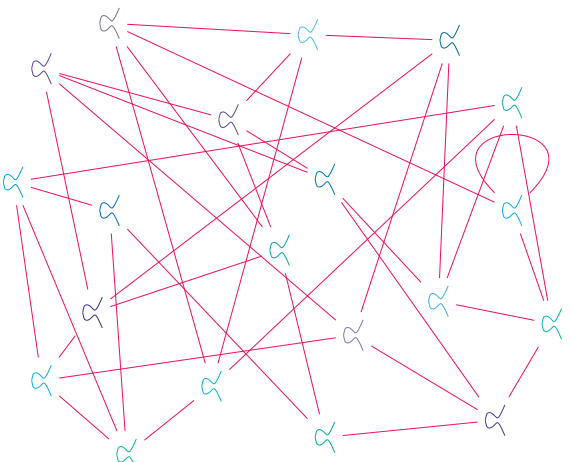
secret path

SIDH KEY EXCHANGE



E_{start}

2-isogeny
secret path



E_{start}

3-isogeny
secret path

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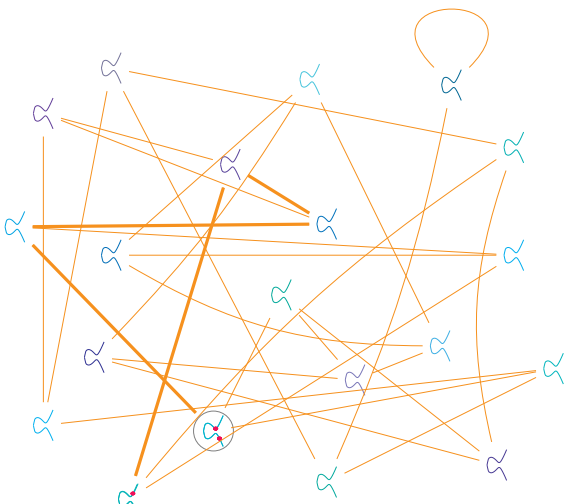

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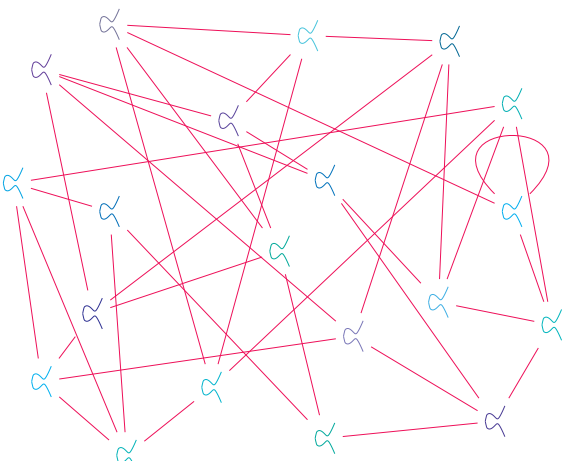
2-isogeny
secret path



E_{start}

3-isogeny
secret path

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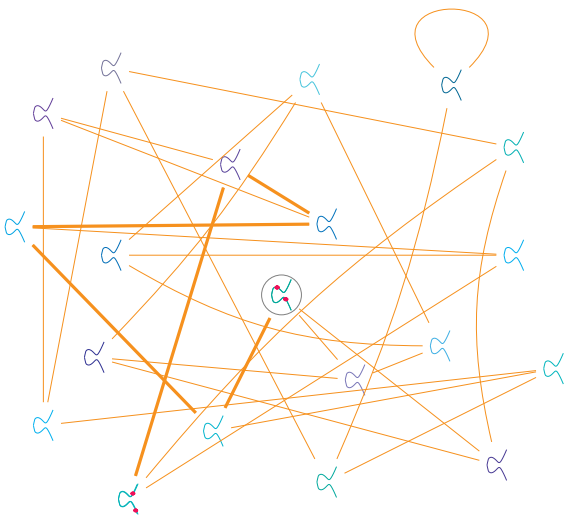
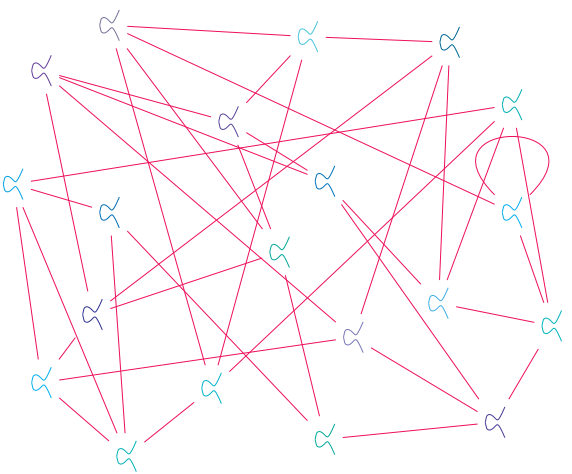


E_{start}

2-isogeny
secret path



E_{start}

3-isogeny
secret path

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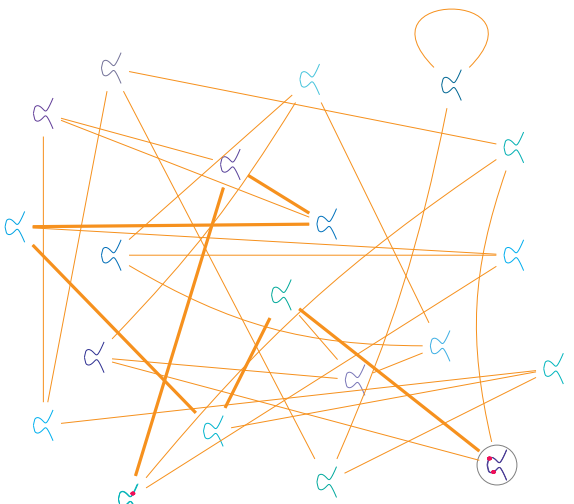
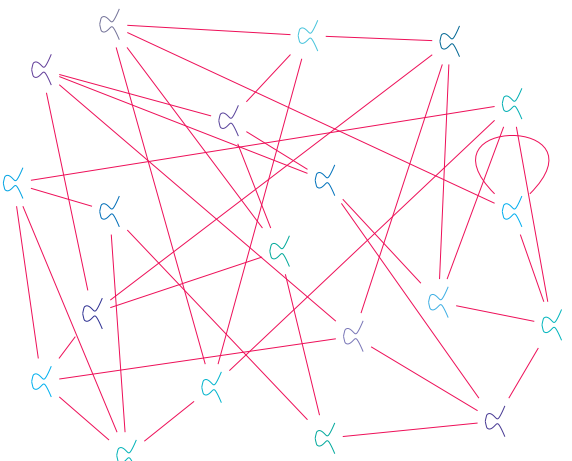


E_{start}

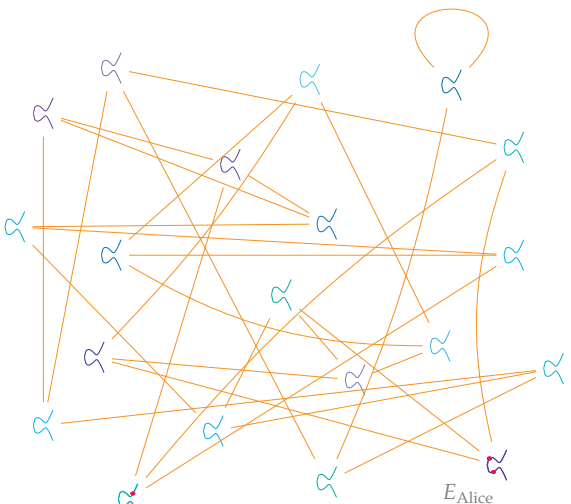
2-isogeny
secret path



E_{start}

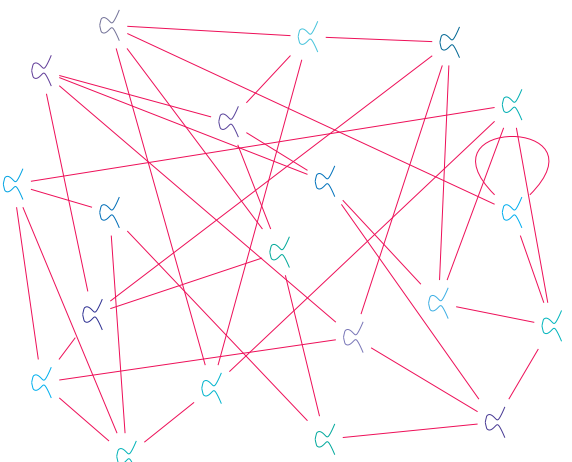
3-isogeny
secret path

SIDH KEY EXCHANGE



E_{start}

2-isogeny
secret path



E_{start}

3-isogeny
secret path

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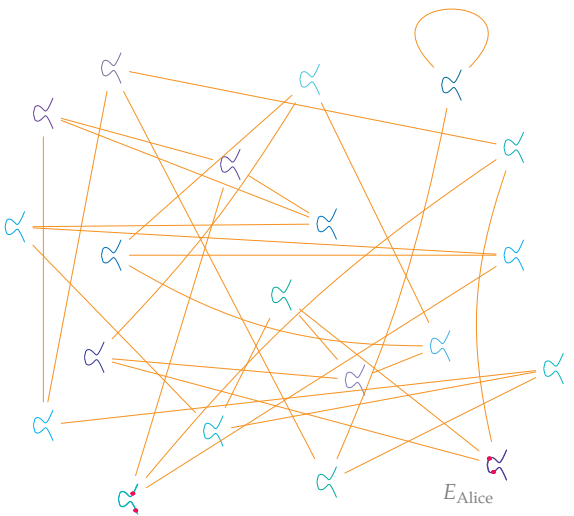
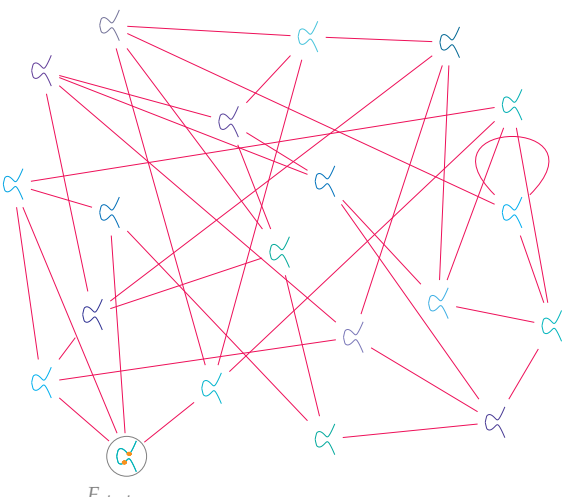


2-isogeny
secret path



3-isogeny
secret path

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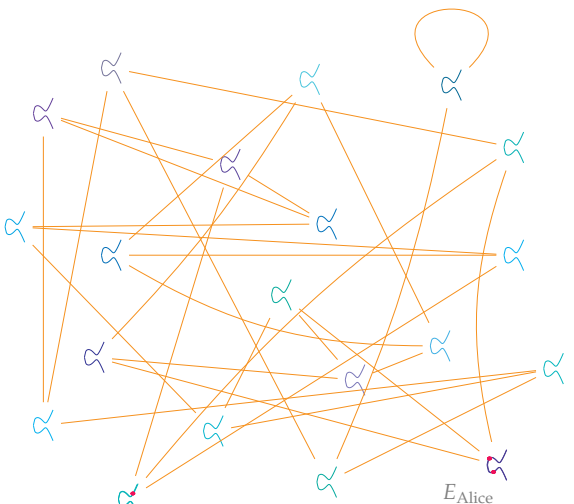
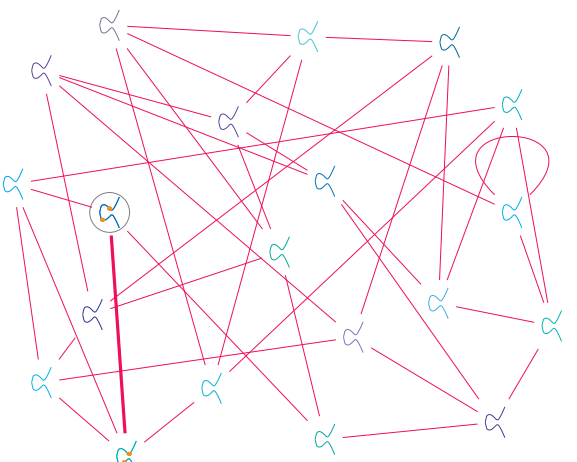


E_{start}

ALICE

2-isogeny

secret path



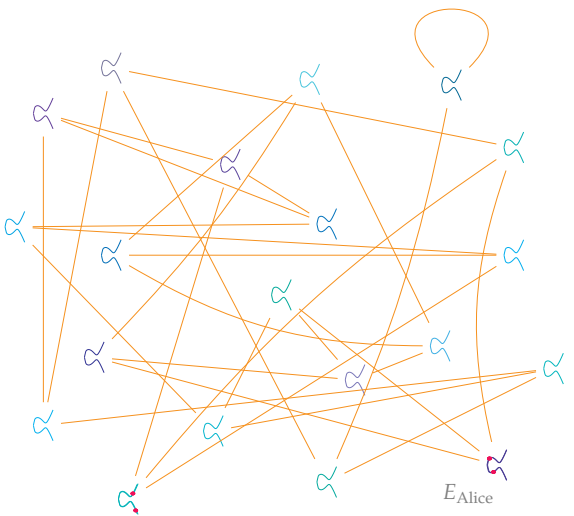
E_{start}

BOB

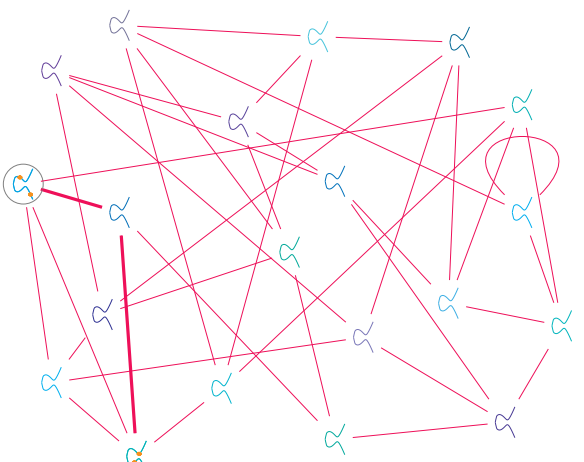
3-isogeny

secret path

SIDH KEY EXCHANGE

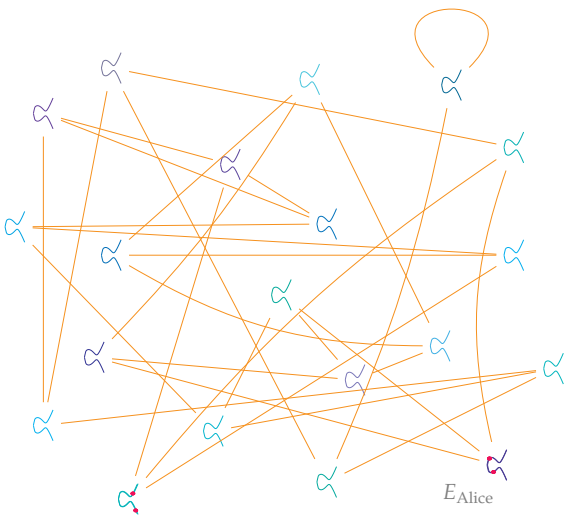


2-isogeny
secret path



3-isogeny
secret path

SIDH KEY EXCHANGE

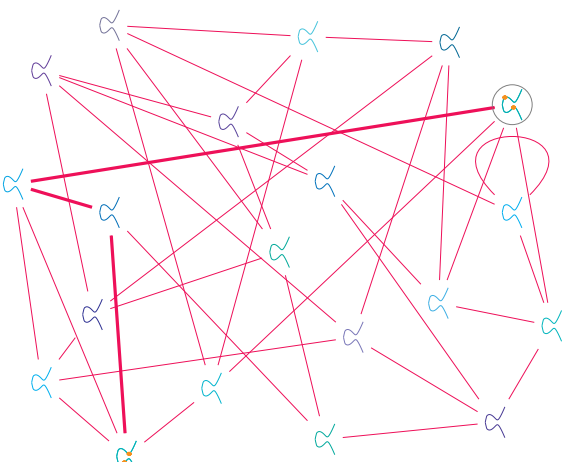


E_{start}

ALICE

2-isogeny

secret path



E_{start}

BOB

3-isogeny

secret path

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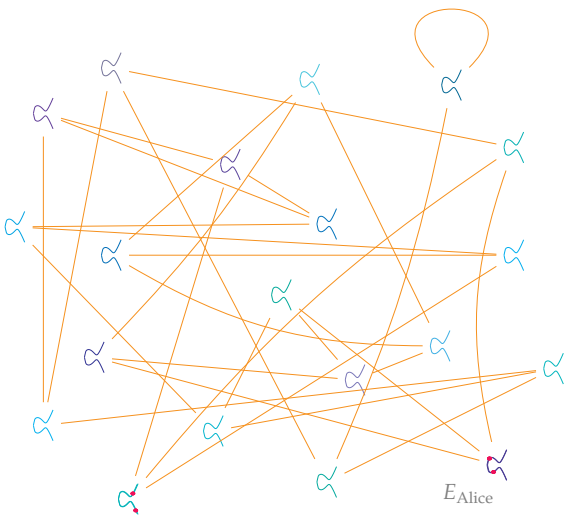
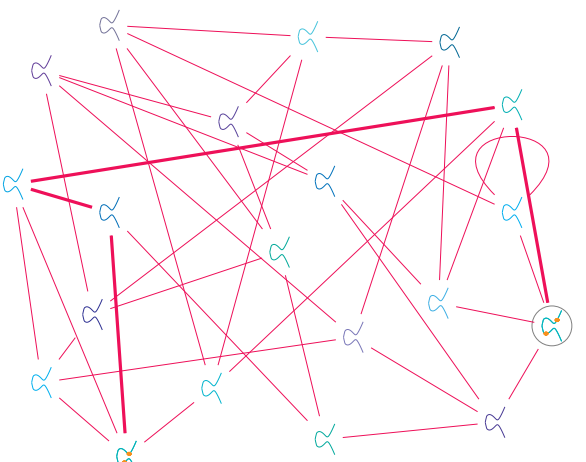


E_{start}

ALICE

2-isogeny

secret path



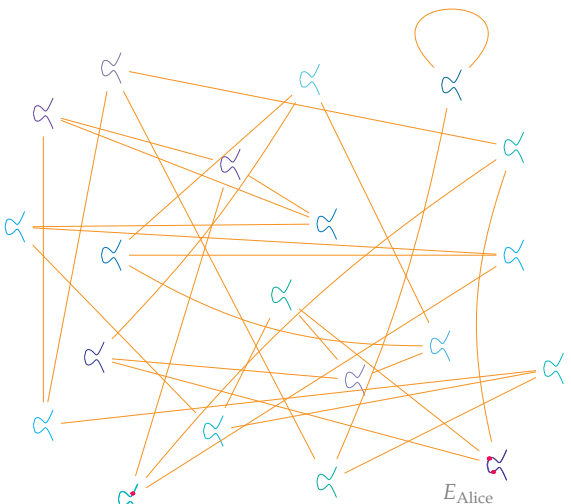
E_{start}

BOB

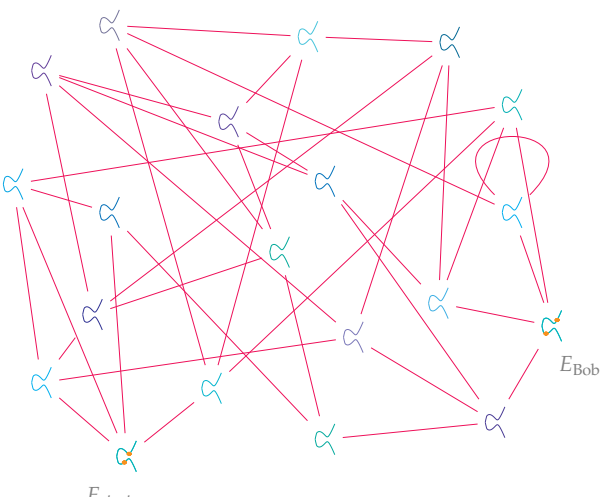
3-isogeny

secret path

SIDH KEY EXCHANGE



2-isogeny
secret path



3-isogeny
secret path

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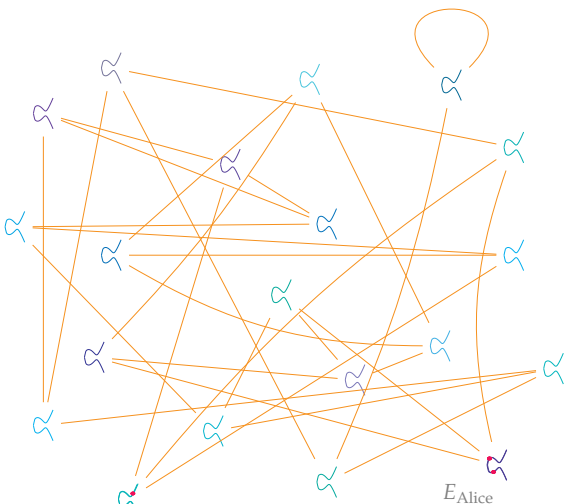
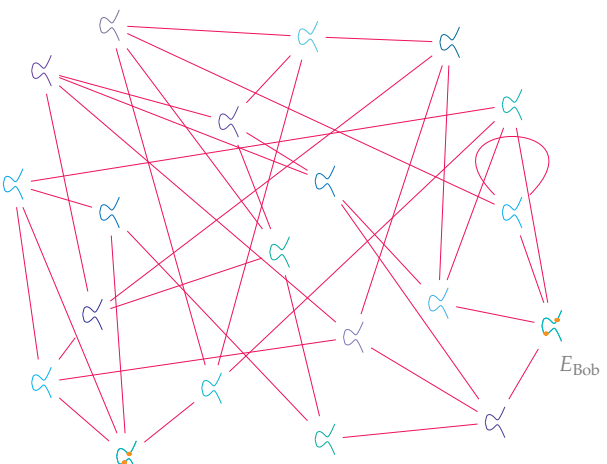




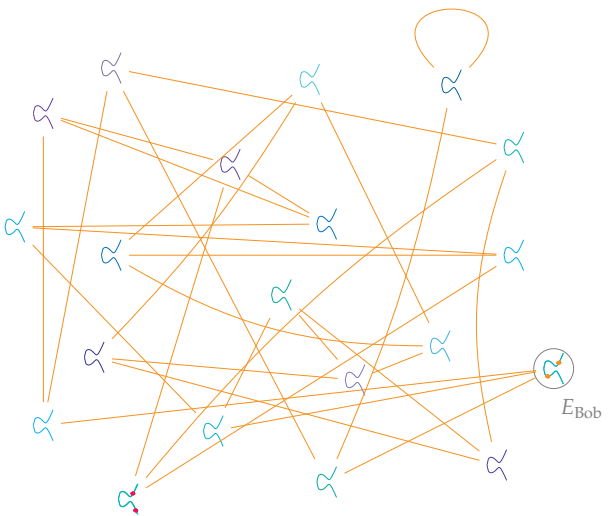
2-isogeny
secret path

E_{Alice}

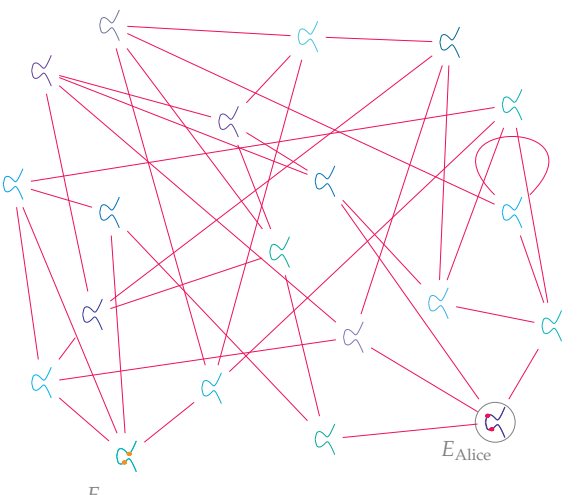
E_{Bob}

3-isogeny
secret path

SIDH KEY EXCHANGE



2-isogeny
secret path



3-isogeny
secret path

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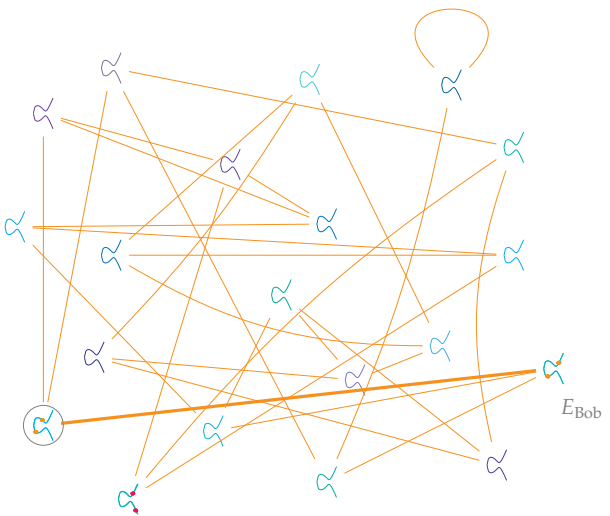
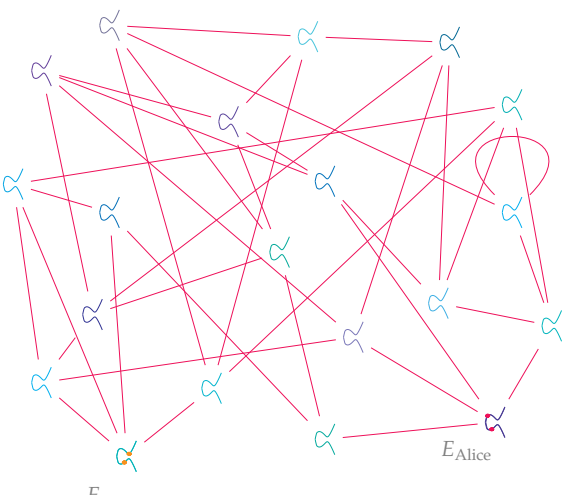


2-isogeny
secret path



3-isogeny
secret path

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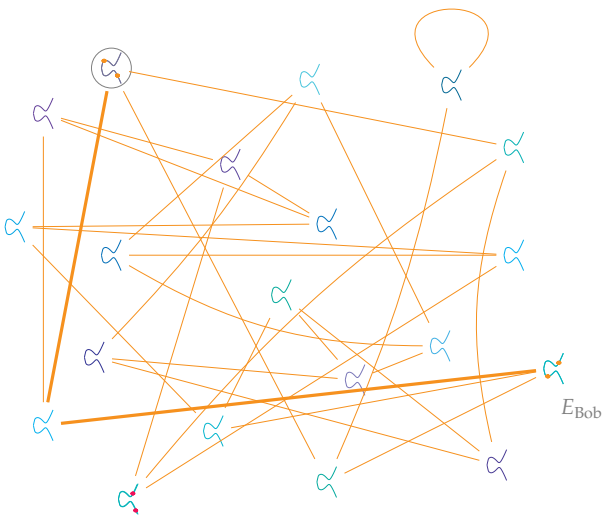
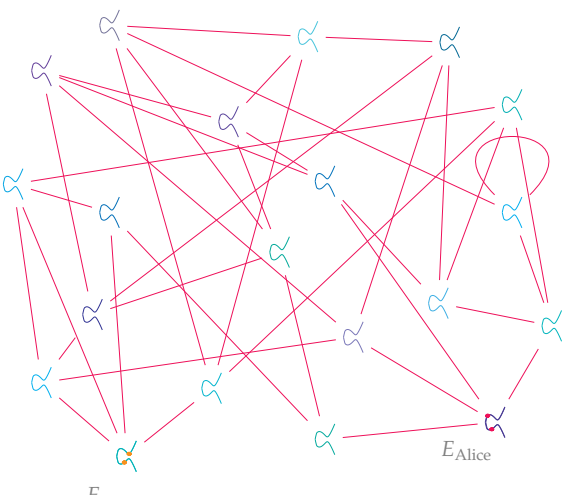


2-isogeny
secret path



3-isogeny
secret path

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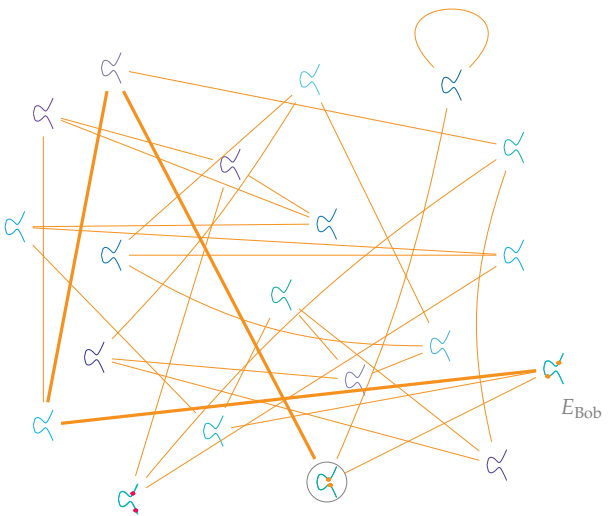
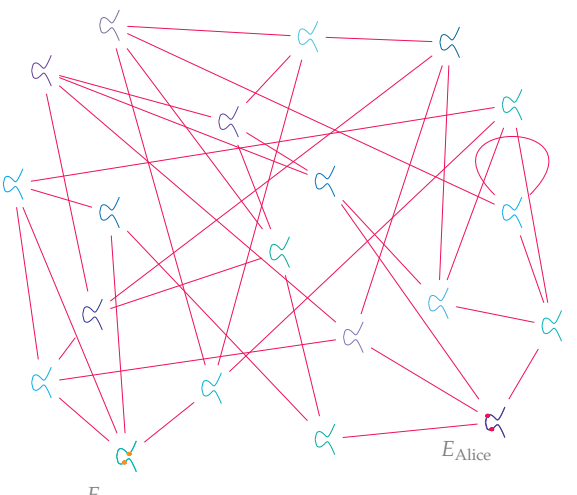


2-isogeny
secret path



3-isogeny
secret path

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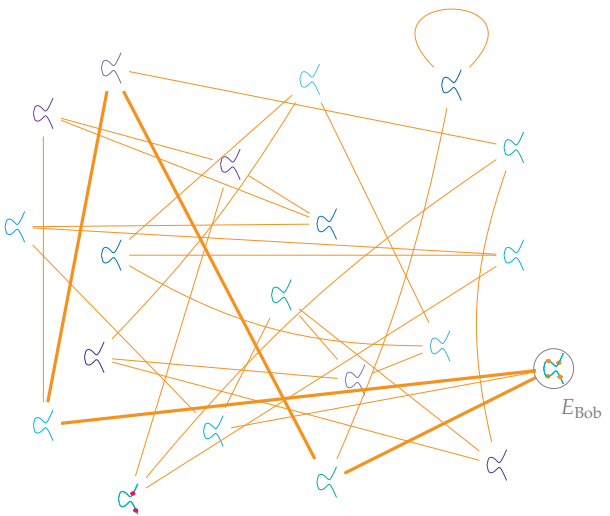

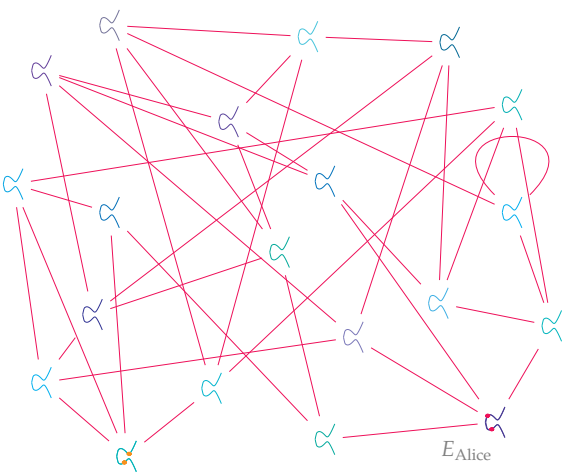

2-isogeny
secret path

3-isogeny
secret path

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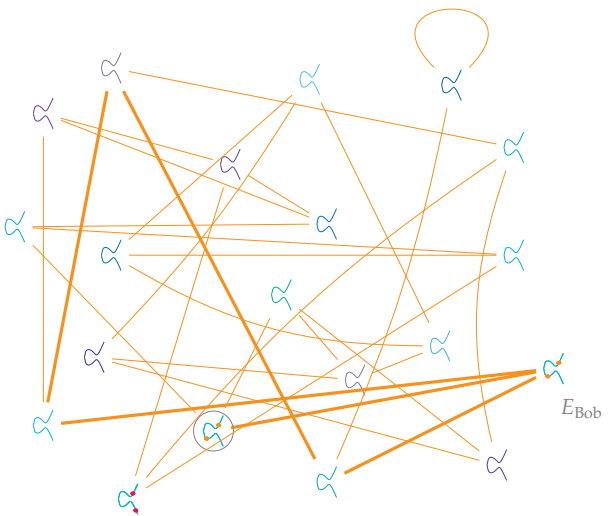

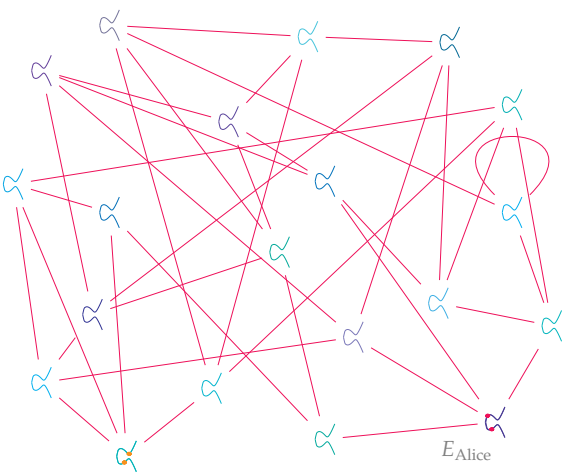

2-isogeny
secret path

3-isogeny
secret path

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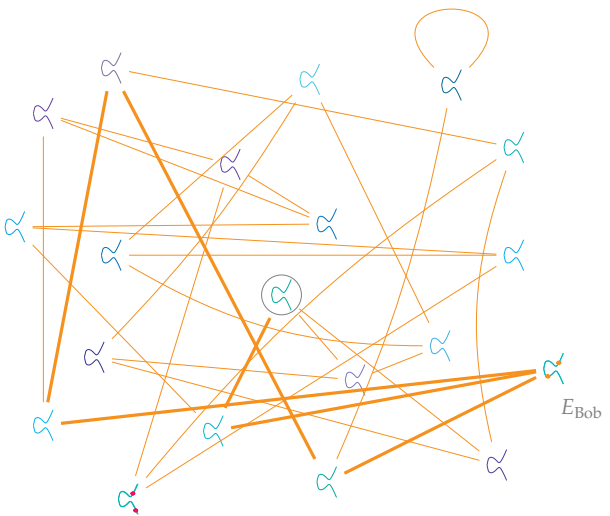

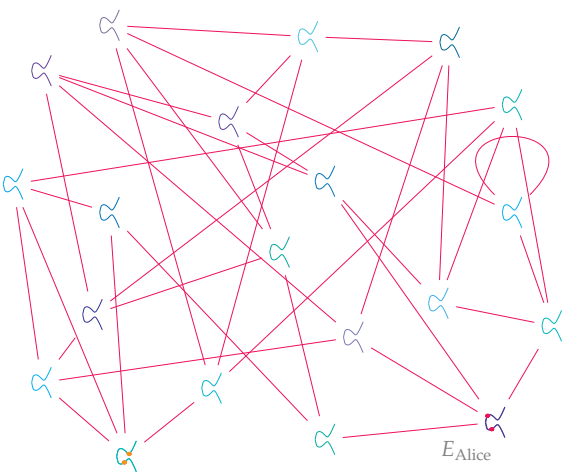

2-isogeny
secret path

3-isogeny
secret path

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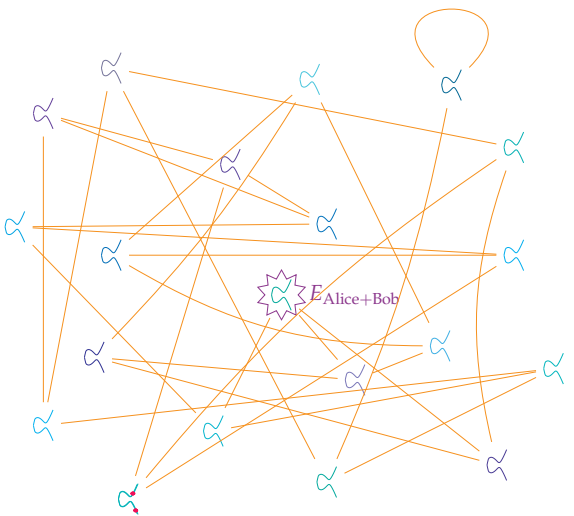
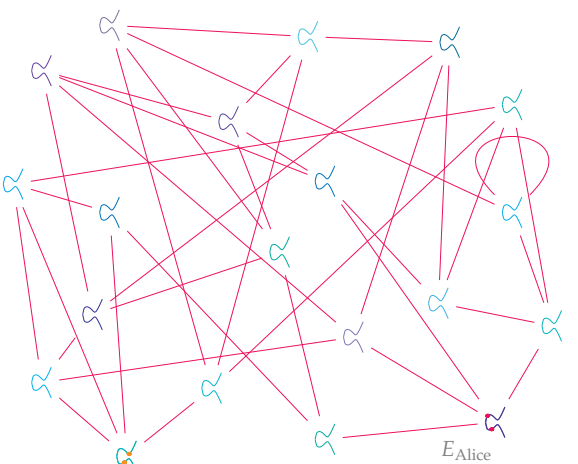


2-isogeny
secret path



3-isogeny
secret path

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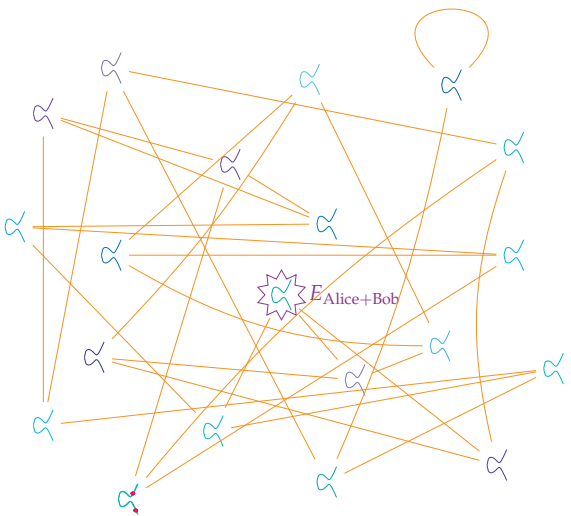
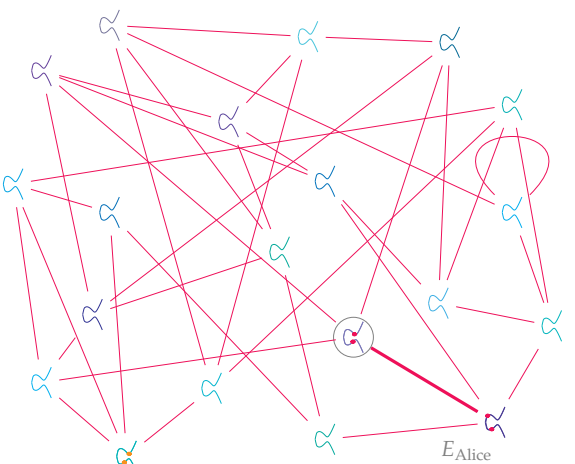


2-isogeny
secret path



3-isogeny
secret path

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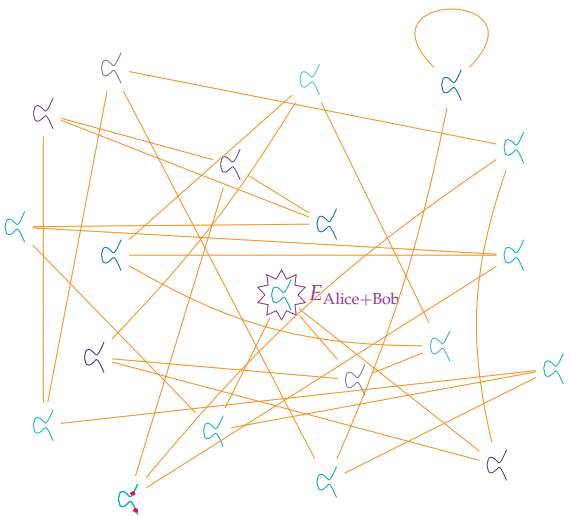

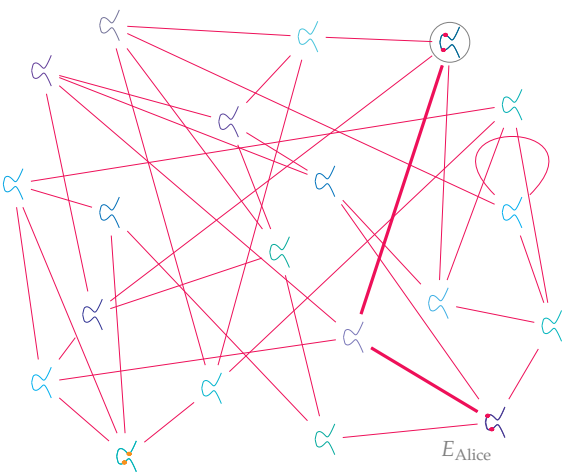

2-isogeny
secret path

3-isogeny
secret path

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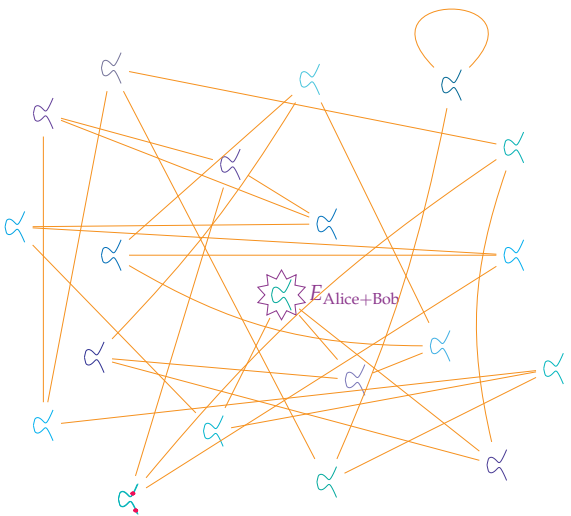
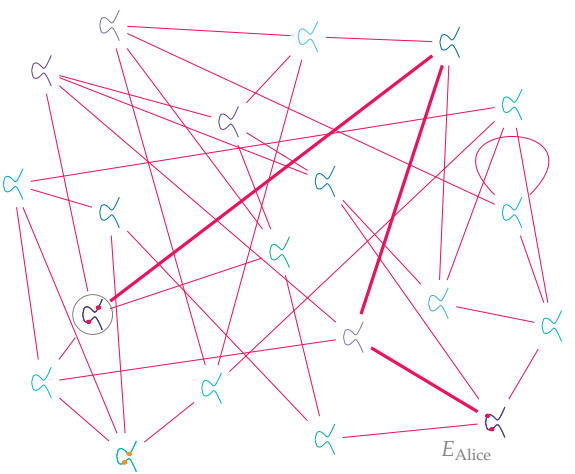


2-isogeny
secret path



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secret path

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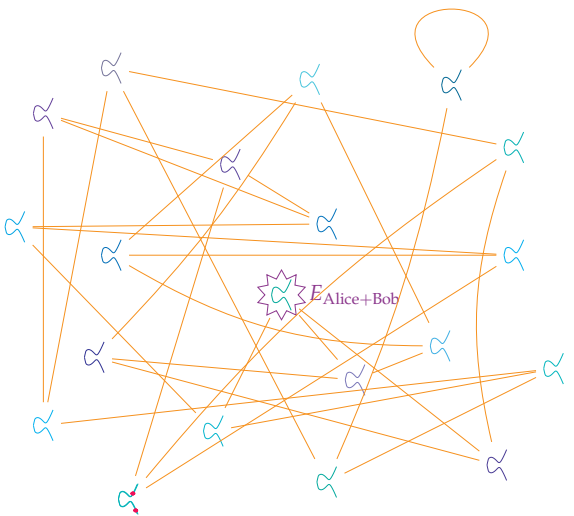
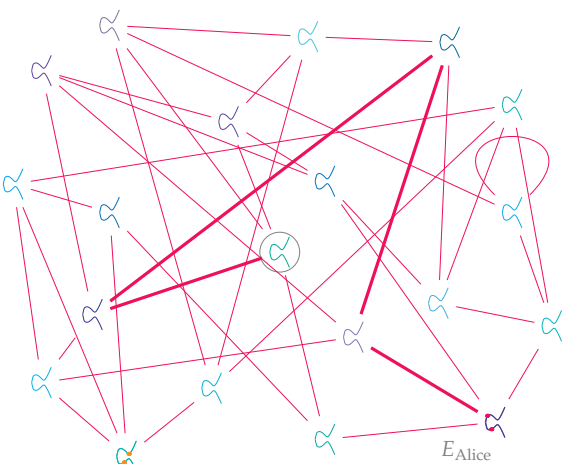


2-isogeny
secret path



3-isogeny
secret path

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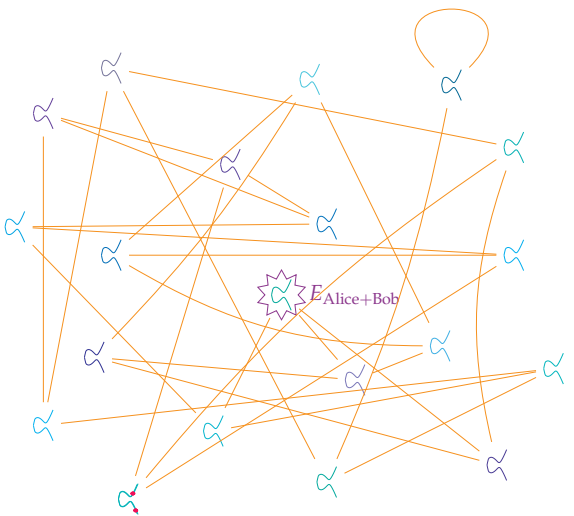
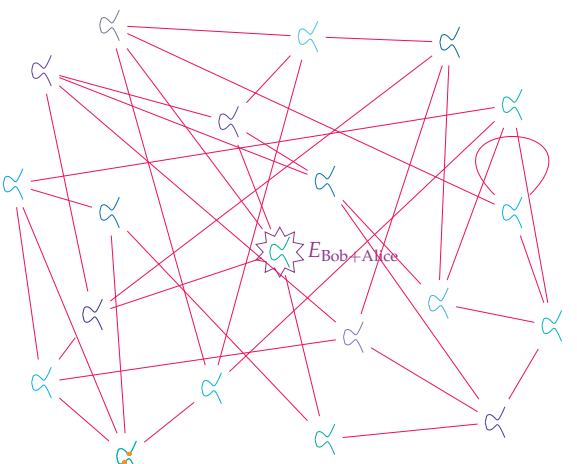
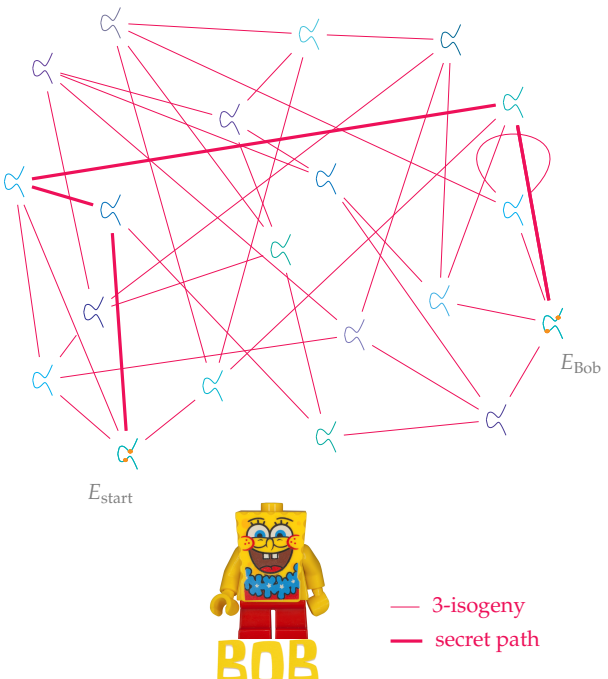


2-isogeny
secret path



3-isogeny
secret path

Retrieving Bob's key



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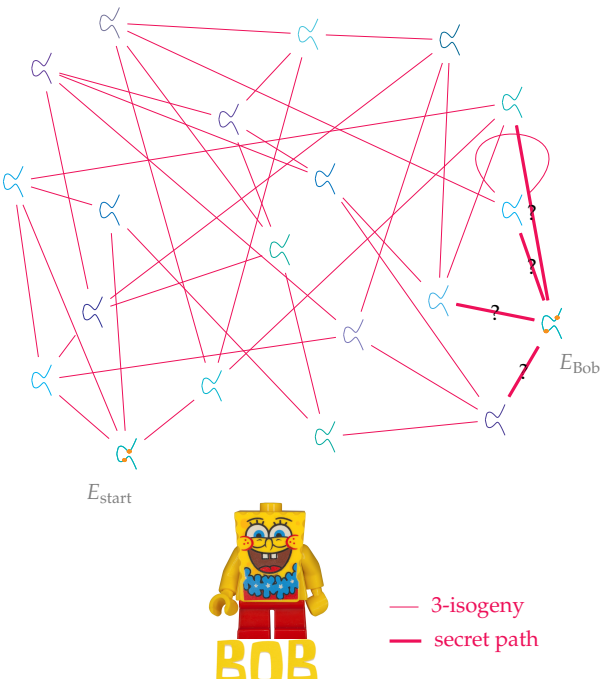
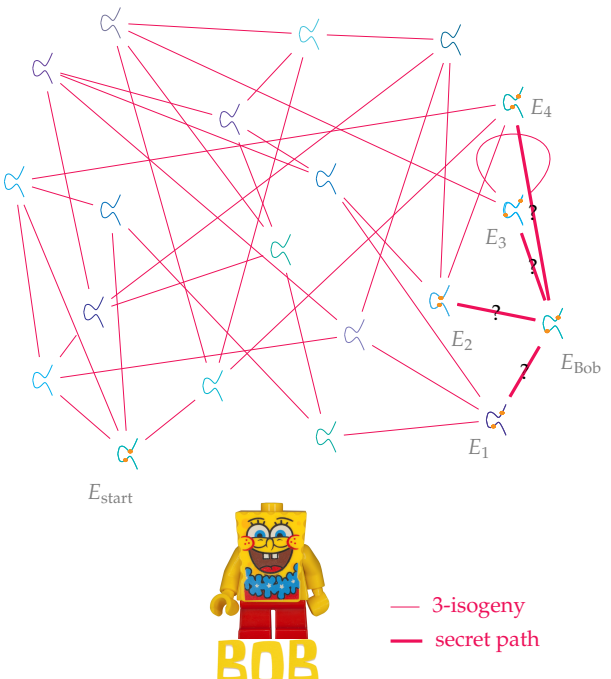
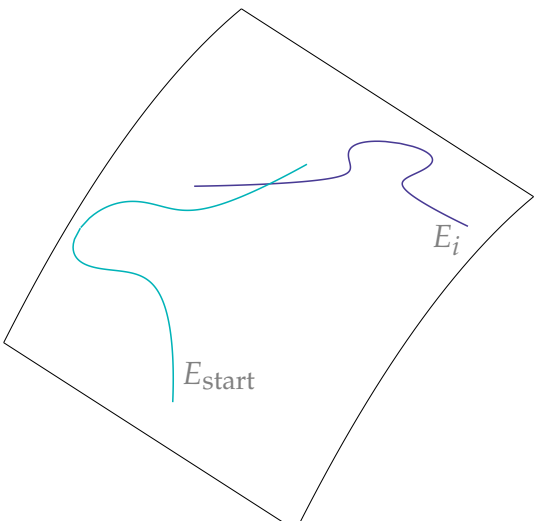


Retrieving Bob's key



Abelian surfaces



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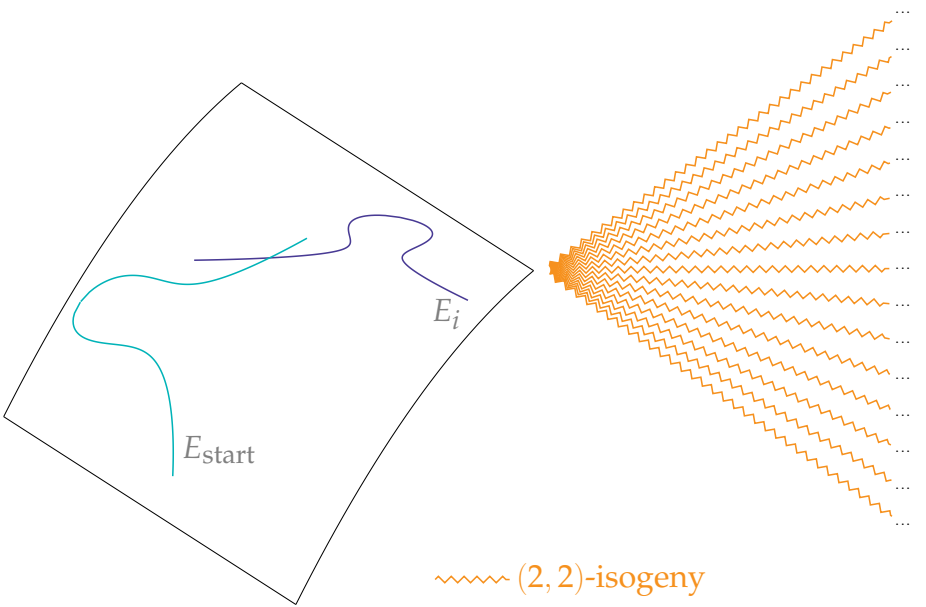

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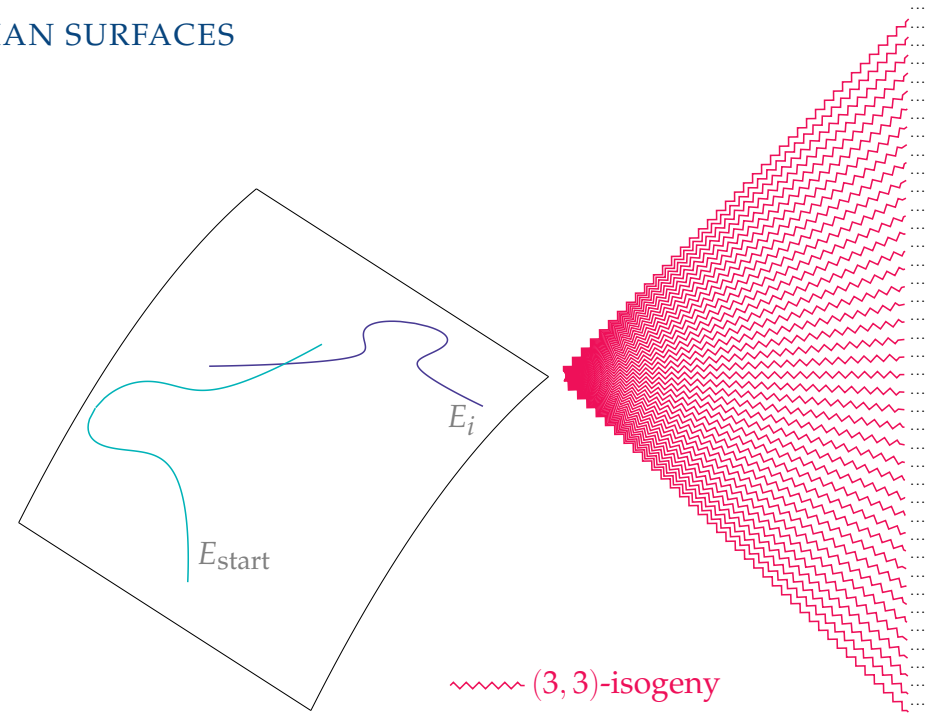

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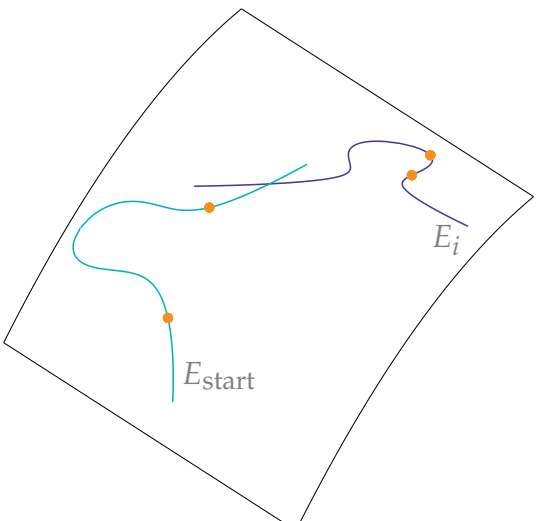
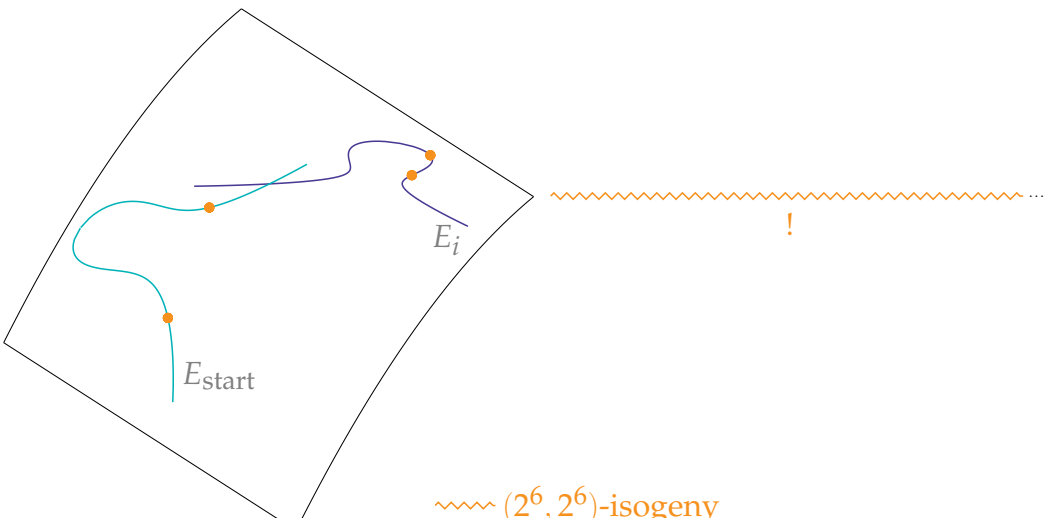
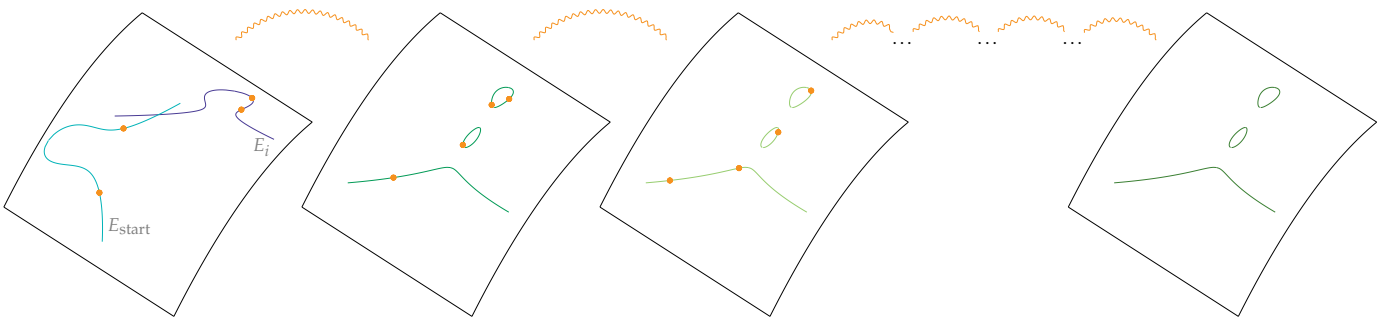


ABELIAN SURFACES



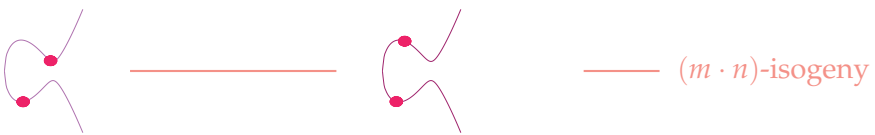
Abelian surfaces from hyperelliptic curves



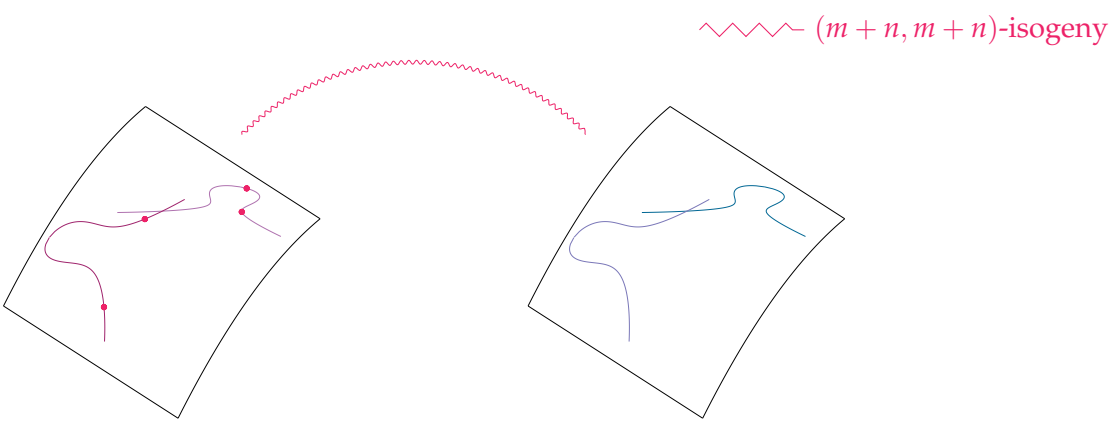
$(2,2)$-isogeny

KANI'S REDUCIBILITY CRITERION (1997)

The one-dimensional isogeny



determines the unique two-dimensional isogeny



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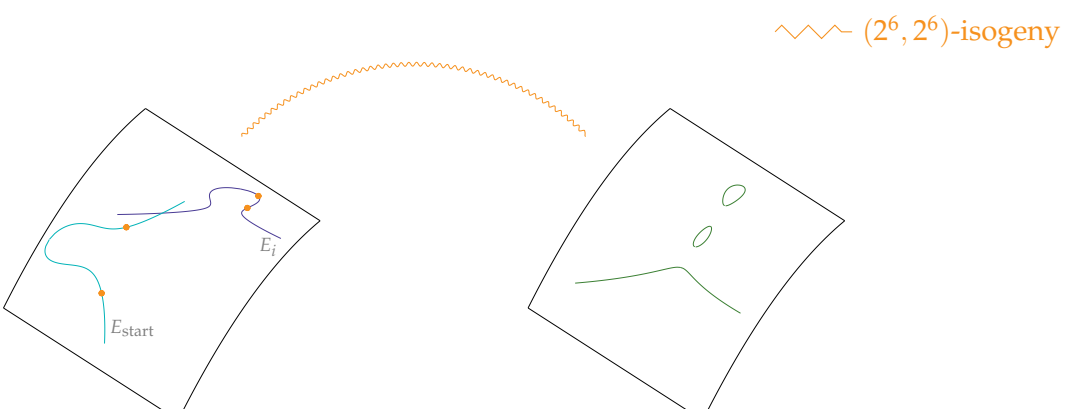

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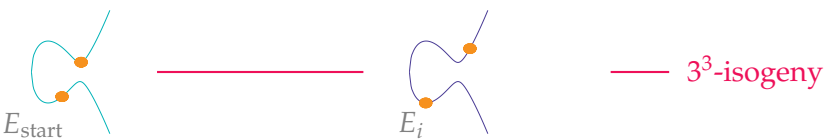

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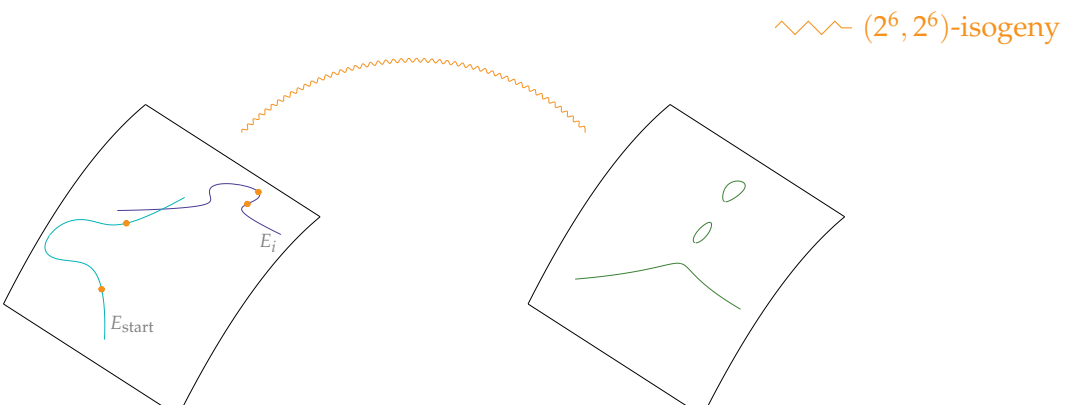

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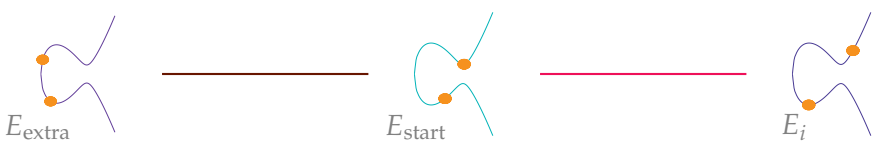
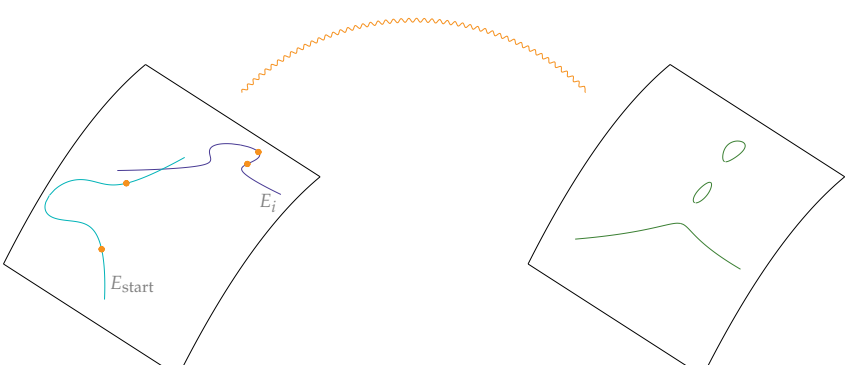


$(2^6 - 3^3)$-isogeny

3^3-isogeny

determines the unique two-dimensional isogeny

$(2^6, 2^6)$-isogeny



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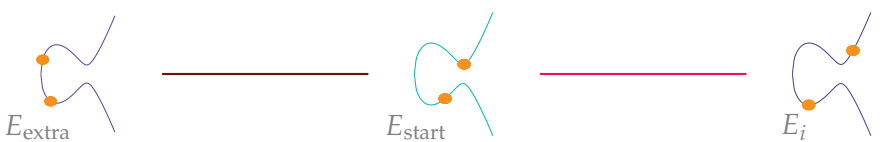

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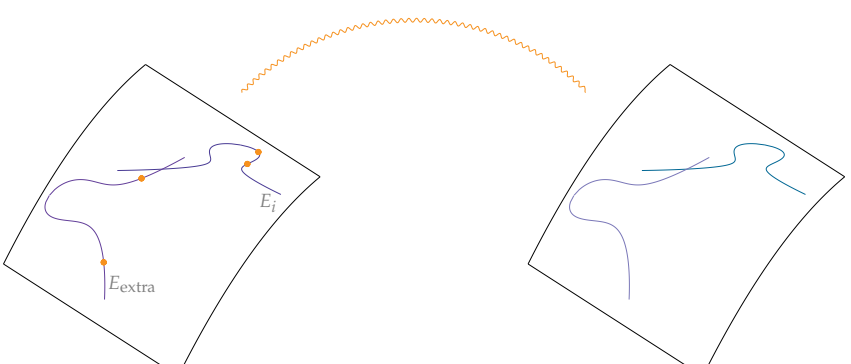
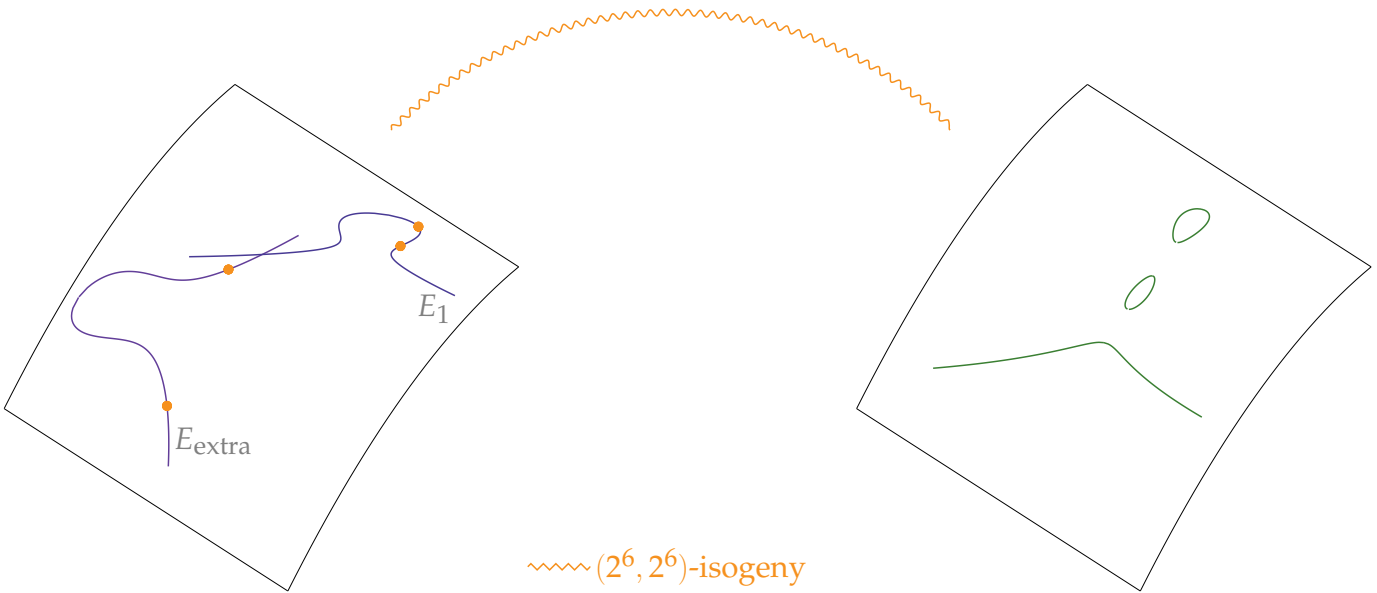


Trial and error



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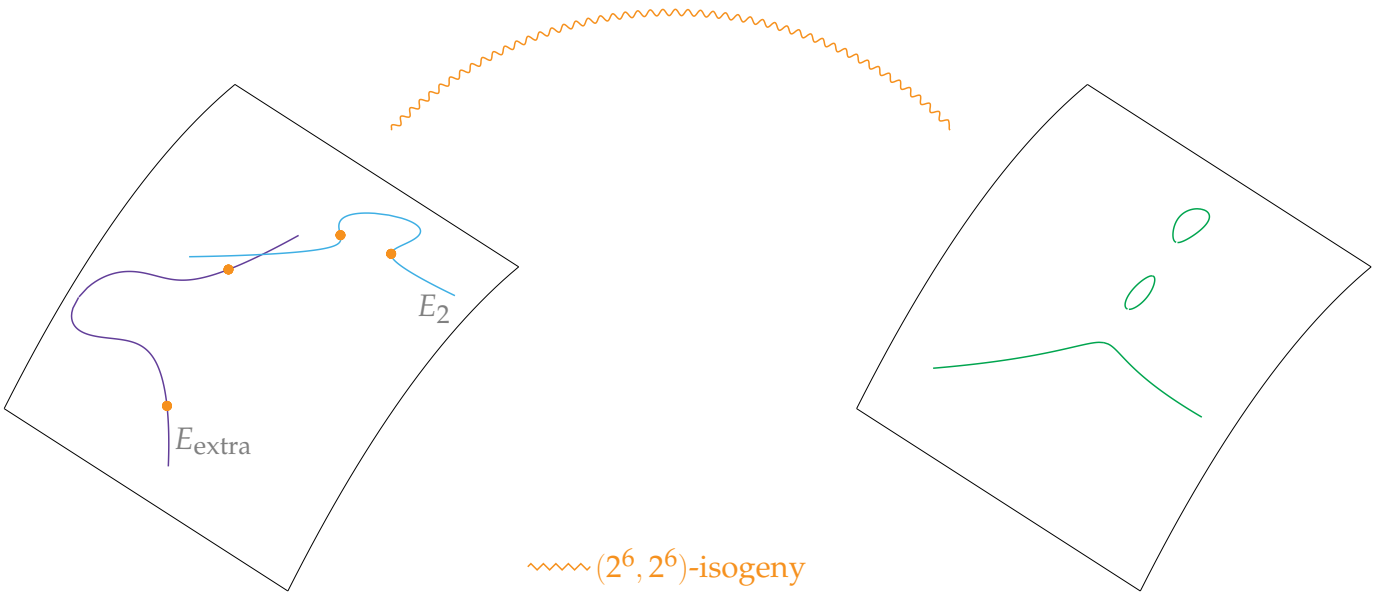

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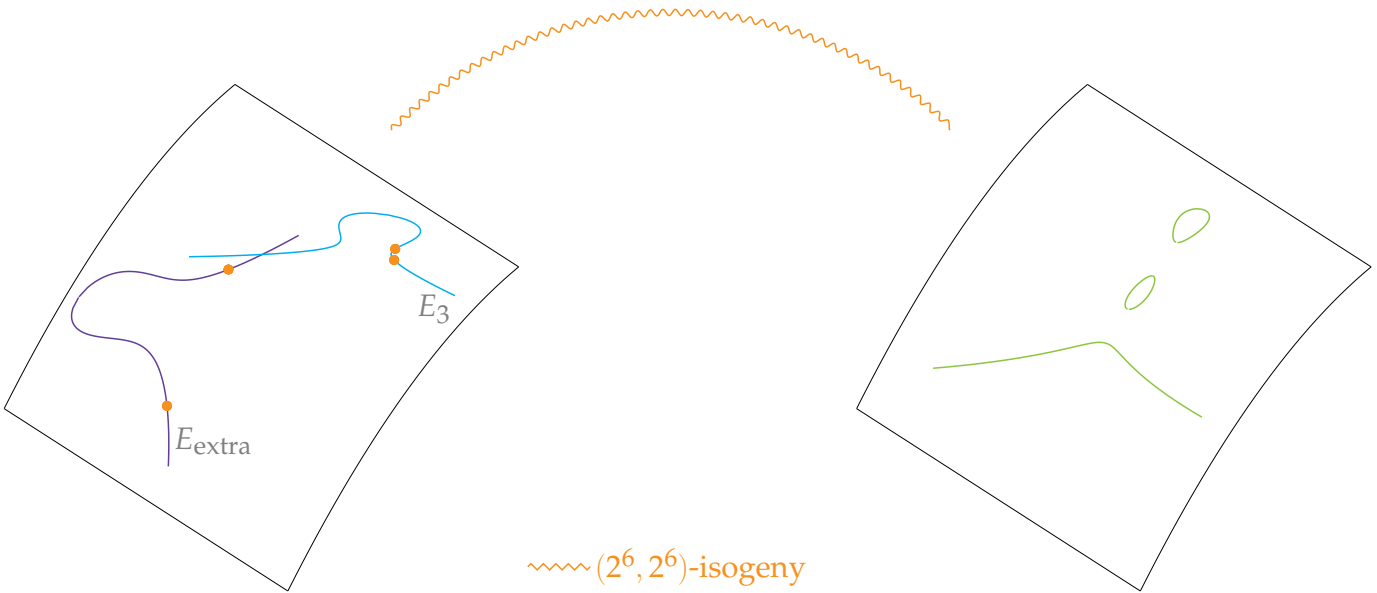

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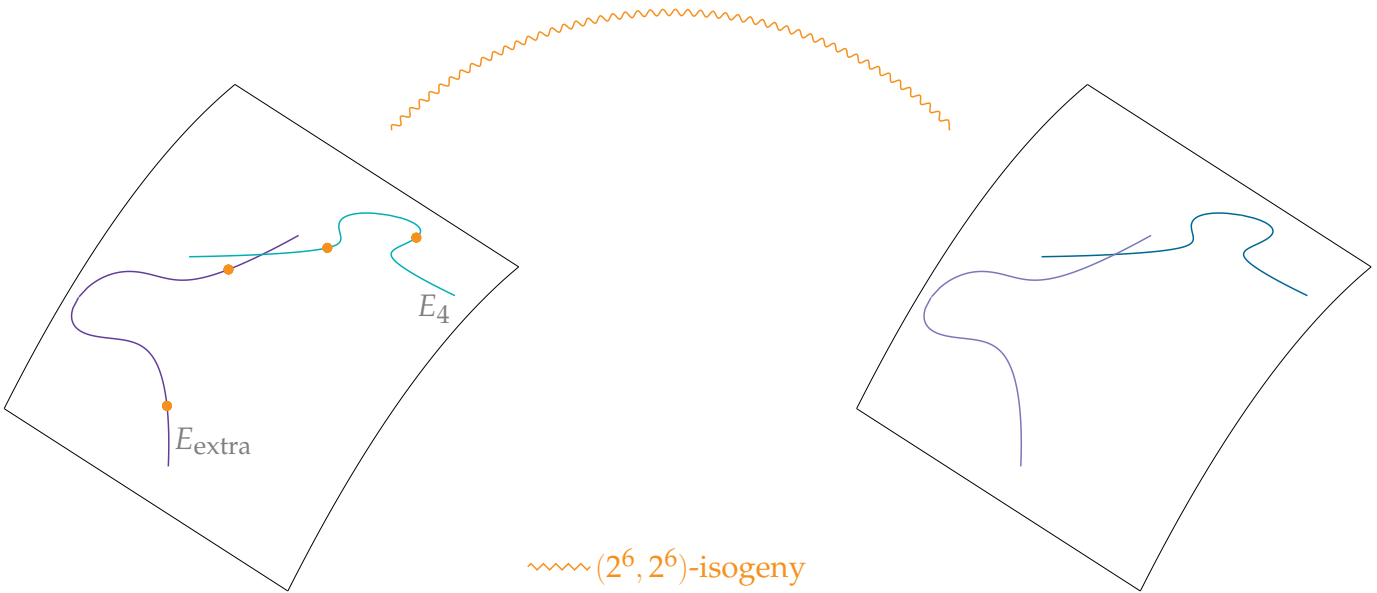

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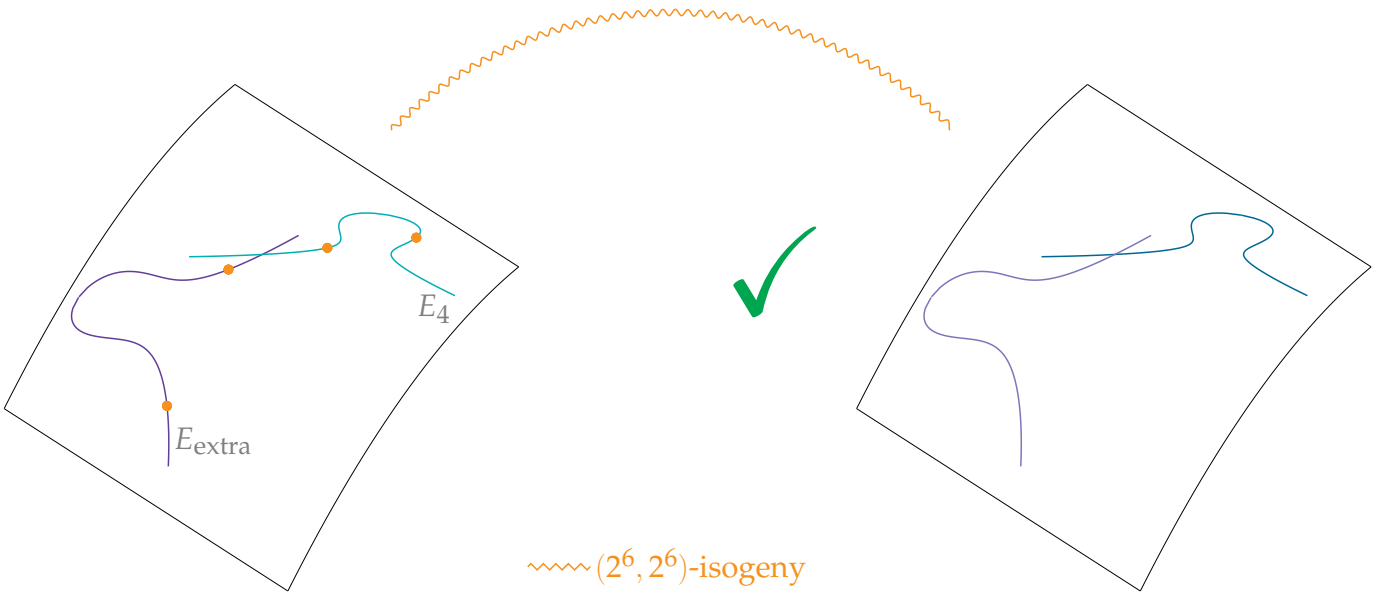
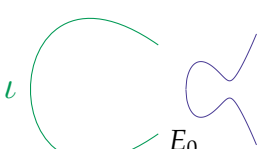


Computing $(2^a - 3^b)$-isogenies in SIKE

- $E_0 : y^2 = x^3 + x$ was used in initial SIKE submission.
  This allows for an easy $(2^a - 3^b)$-isogeny from ι if we write

$$2^a - 3^b = (u + 2v\sqrt{-1}) \cdot (u - 2v\sqrt{-1}).$$



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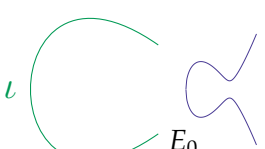

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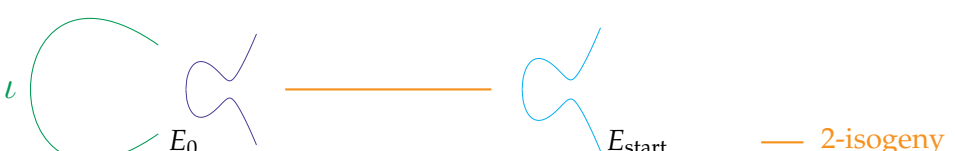

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- If not: hope $2^a - 3^b$ is smooth.
- Leeway allows us to find a smooth (positive) expression of the form

$$d \cdot 2^{a-i} - e \cdot 3^{b-j},$$

which leads to a subexponential attack.

TIMINGS

	Quantum security level	Time
$IKEp182	-	55s
$IKEp217	-	85s
SIKEp434	NIST 1	10m
SIKEp503	NIST 2	20m
SIKEp610	NIST 3	55m
SIKEp751	NIST 5	3h15m

Key ingredients for our attack

Requirements:

- degrees of isogenies are fixed and known (i.e. 2^a-isogeny and 3^b-isogeny);
- Alice and Bob exchange extra point images under their respective isogenies.

This attack does *not* apply to other isogeny-based protocols such as CSIDH, SQISign, M(D)-SIDH, etc.

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Blogpost of this talk: