Hull Attacks on the Lattice Isomorphism Problem

Léo Ducas [1,2] **Shane Gibbons** [1,2]

[1]Cryptology Group, CWI Amsterdam

[2]Mathematical Institute, Leiden University

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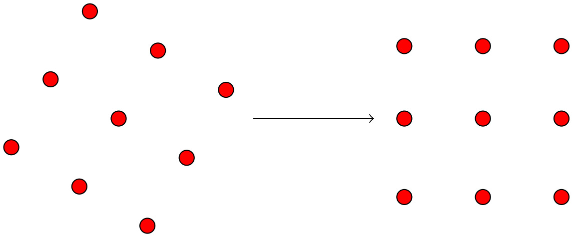
Lattice Isomorphism



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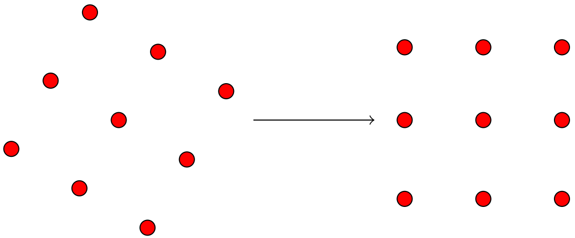

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ΔLIP proposed by [BGPSD23, DvW22] for cryptography, while [DPPW22] propose LIP.

Context and Motivation

Lattice Isomorphism

Let $L, L' \subseteq \mathbb{R}^n$ be lattices. Then L and L' are *isomorphic* if there exists an $O \in \mathcal{O}_n(\mathbb{R})$ such that

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Lattice Isomorphism Problem

Let $m \leq n$. Let $B, B' \in \mathbb{R}^{n \times m}$ be bases of lattices L, L' that are isomorphic. Find an invertible $U \in \mathrm{GL}_m(\mathbb{Z})$ and orthonormal $O \in \mathcal{O}_n(\mathbb{R})$ such that

$$OBU = B'.$$

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BKZ reduction with blocksize β runs in time $2^{0.292\beta + o(\beta)}$ [BDGL16].

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Conjecture [DvW22] (informal)

The best attack against ΔLIP for lattices L, L' requires solving f-approx SVP in both lattices, where

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Our attack: We make the gap larger, by extracting the sublattice $\mathbb{Z}^n$, then solving $\mathbb{Z}$LIP.

Plan of Attack

- Lattice Hulls
- Construction A
- Solving LIP via $\mathbb{Z}$LIP and Code Equivalence

Hull of a Lattice

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Code Hull

Given an $[n, k]_q$ linear code C over $\mathbb{F}_q$, the *hull of* C *is*

$$\mathcal{H} := C \cap C^{\perp},$$

where $C^{\perp} := \left\{ y \in \mathbb{F}_q^n : y \cdot x = 0 \quad \forall x \in C \right\}$.

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Lattice Hull

Let $s \in \mathbb{R}^{\times}$, and let $L \subseteq \mathbb{R}^n$ be a lattice. The s-hull of L is the sublattice

$$H_s(L) = L \cap sL^*,$$

where $L^* := \{x \in \mathrm{span}(L) : \langle x, L \rangle \subseteq \mathbb{Z}\}$.

Construction A Lattices

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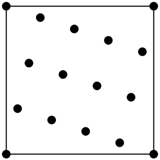

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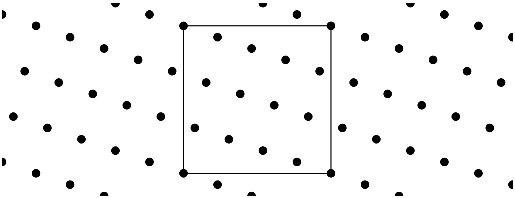

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Solve LIP via the Hull

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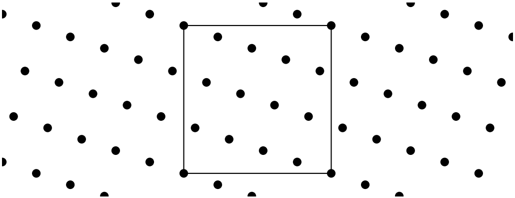

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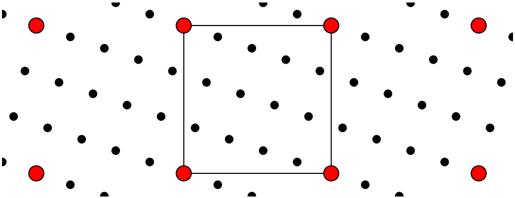

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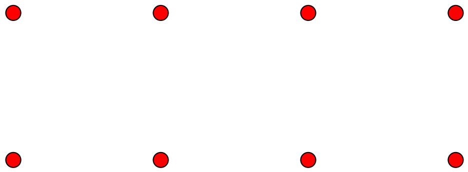

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Isomorphism of the Hull

Fix a rate 1/2 code C with trivial hull $\mathcal{H} = \{0\}$.

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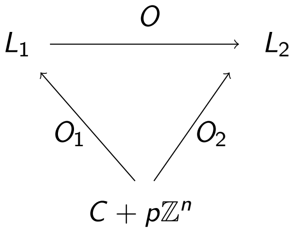

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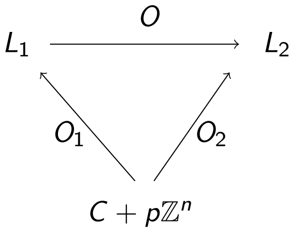

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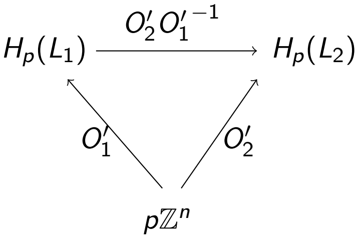

Figure: Isomorphism of Hulls

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We find this automorphism by solving a code equivalence problem between $(O_1')^{-1}L_1 \mod p$ and $(O_2')^{-1}L_2 \mod p$.

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- Signed permutation equivalence (SPEP)
- Permutation equivalence (PEP)

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- Take the p-hull of L_1 and L_2.
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- Solve the easy instance of code equivalence between $(O_1')^{-1}L_1 \mod p$ and $(O_2')^{-1}L_2 \mod p$

Conclusion

We restate the conjecture from [DvW22]

Updated conjecture (Informal)

The best attack against ΔLIP for lattices L, L' requires solving f-approx SVP in both lattices, where

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