Discretization Error Reduction for High Precision Torus Fully Homomorphic Encryption

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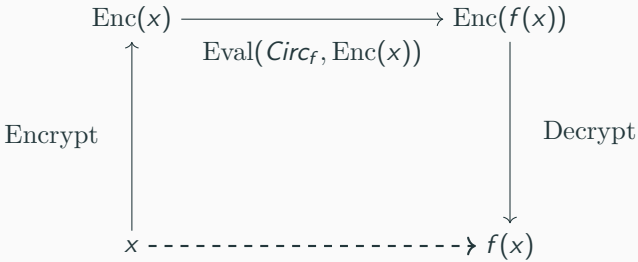
Introduction

Fully Homomorphic Encryption

- Encryption scheme that allows to operate with encrypted data



- Usually comes with a *recryption* procedure, called **bootstrapping**.

Notations

- $n \in \mathbb{Z}$: TLWE dimension
- N : power of 2. (Ring Dimension)
- $\mathbb{T} = \mathbb{R}/\mathbb{Z} = \left[-\frac{1}{2}, \frac{1}{2}\right)$
- $\mathbb{B} = \{0, 1\}$
- $\mathbb{Z}_N[X] = \mathbb{Z}[X]/\langle X^N + 1\rangle$
- $\mathbb{T}_N[X] = \mathbb{T}[X]/\langle X^N + 1\rangle$
- $\mathbf{s} \in \mathbb{B}^n$
- $\mathcal{K} \in \mathbb{B}_N[X]$
- ν : Extension Parameter

TFHE

FHE scheme introduced in [CGGI20][1], based on the LWE (Learning With Errors) problem.

Works with three ciphertext spaces:

- TLWE : $\mathrm{TLWE}_{\mathbf{s}}(m) = (\mathbf{a}, b = \mathbf{a} \cdot \mathbf{s} + m + e) \in \mathbb{T}^{n+1}$
- TRLWE : $\mathrm{TRLWE}_{\mathcal{K}}(m(X)) = (a(X), b(X) = a(X)\mathcal{K}(X) + m(X) + e(X)) \in \mathbb{T}_N[X]^2$
- TRGSW :

$$\mathrm{TRGSW}_{\mathcal{K}}(z(X)) = \left[\begin{array}{c} \mathrm{TRLWE}_{\mathcal{K}}(z(X)/B) \\ \vdots \\ \mathrm{TRLWE}_{\mathcal{K}}(z(X)/B^d) \\ \hline \mathrm{TRLWE}_{\mathcal{K}}(z(X)\cdot(-\mathcal{K})/B) \\ \vdots \\ \mathrm{TRLWE}_{\mathcal{K}}(z(X)\cdot(-\mathcal{K})/B^d) \end{array}\right], \; z(X) \in \mathbb{Z}_N[X].$$

[1] Ilaria Chillotti, Nicolas Gama, Mariya Georgieva, and Malika Izabachène. TFHE: fast fully homomorphic encryption over the torus. In: *Journal of Cryptology* 33.1 (2020), pp. 34–91

TFHE (continued)

TFHE supports *external product* $\boxdot$, a multiplication between TRGSW and TRLWE ciphertext:

$$\mathrm{TRGSW}_{\mathcal{K}}(z(X)) \boxdot \mathrm{TRLWE}_{\mathcal{K}}(m(X)) = \mathrm{TRLWE}_{\mathcal{K}}(z(X)m(X)).$$

Bootstrapping homomorphically decrypts the TLWE ciphertext by calculating **CMux** circuit for $i \in [\![0, n-1]\!]$:

CMux : Homomorphic Selection

$$\mathbf{CMux}\left(\mathrm{BSK}_i = \mathrm{TRGSW}_{\mathcal{K}}(\mathbf{s}_i), X^{\bar{\mathbf{a}}_i}\mathrm{ACC}, \mathrm{ACC}\right) = \mathrm{TRGSW}_{\mathcal{K}}(\mathbf{s}_i) \boxdot ((X^{\bar{\mathbf{a}}_i} - 1) \cdot \mathrm{ACC}) + \mathrm{ACC}$$

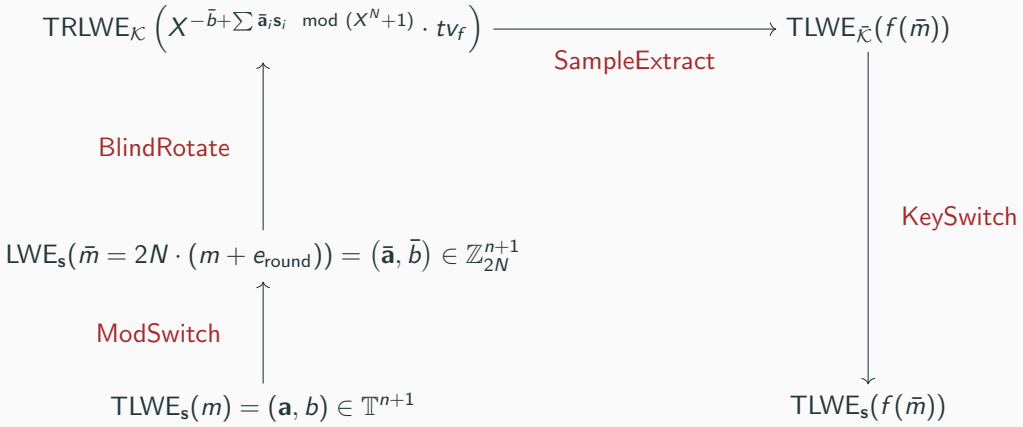
which gives

$$X^{-\bar{b}+\sum \bar{\mathbf{a}}_i\mathbf{s}_i \mod (X^N+1)} \cdot \mathrm{ACC} = \mathrm{TRLWE}_{\mathcal{K}}\left(X^{-\bar{b}+\sum \bar{\mathbf{a}}_i\mathbf{s}_i \mod (X^N+1)} \cdot m(X)\right).$$

Extended BootStrapping for TFHE

TFHE (Functional) Bootstrapping

Given a function $f : \mathbb{T} \to \mathbb{T}$, the (functional) TFHE bootstrapping is a series of algorithms:



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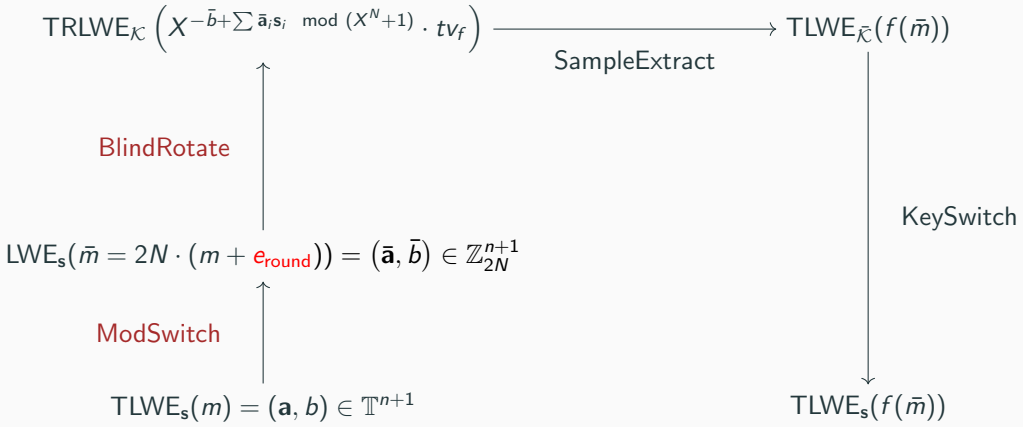
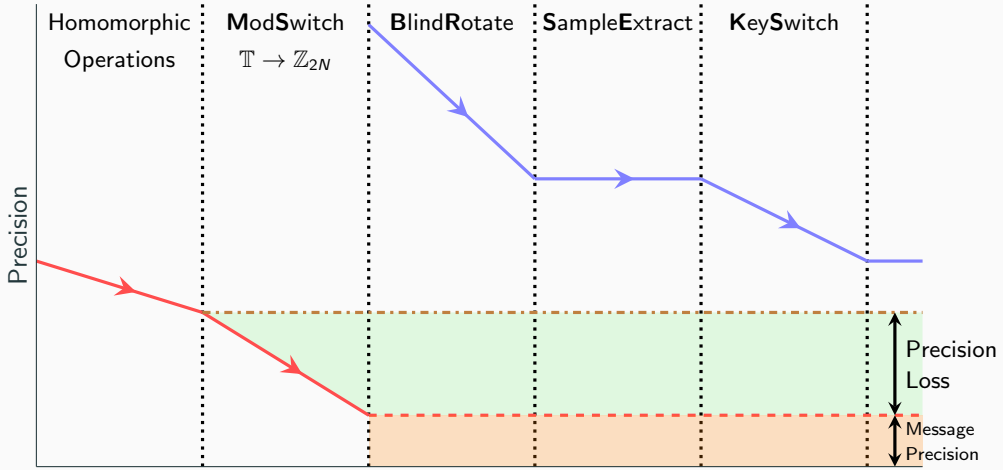
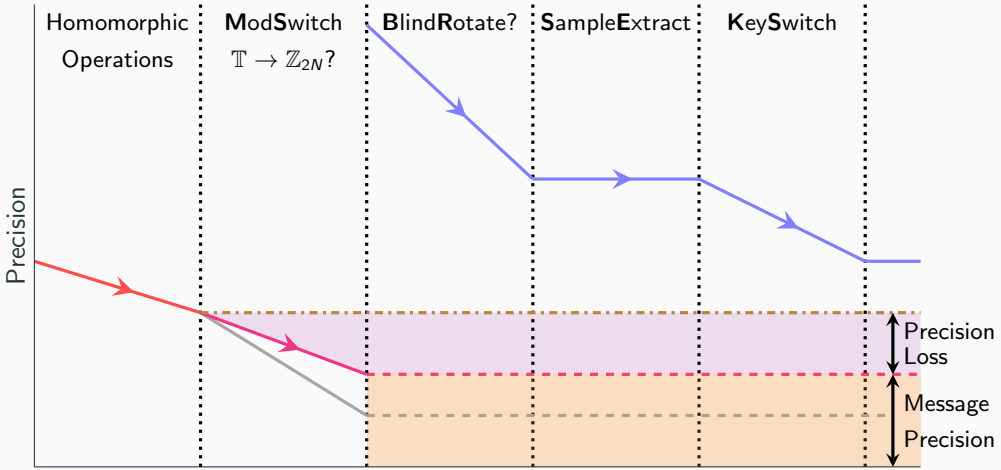


Precision in TFHE Bootstrapping

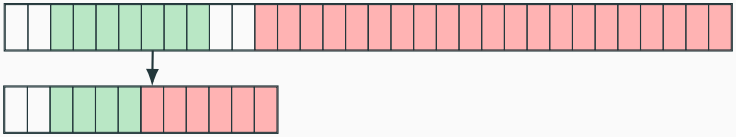


High Precision TFHE : Reduce the ModSwitch (Discretization) Error



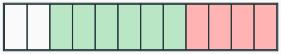
High Precision TFHE : Reduce the ModSwitch (Discretization) Error

ModSwitch to $\mathbb{Z}_{2N}$ adds error:



To reduce this, there are some approaches:

- Reduce Hamming weight of secret key $\mathbf{s}$, Ham($\mathbf{s}$).



- **Increase** $N \to 2^k N$

Enlarging $N \to N' = 2^k N$

Advantages

- Can use small TRLWE/TRGSW error $\Rightarrow$ BlindRotate noise decreases
- Error from ModSwitch decreases $\Rightarrow$ precision increases by k bits

Disadvantages

- Quasilinear growth for polynomial multiplication complexity : $O(N' \log N')$
- (Possibly) Larger keyswitch noise
- (Possibly) Larger public key / ciphertext size

Our Work 1 - Ciphertext Extension

Module homomorphism $\iota : \mathbb{T}_N[X] \to \mathbb{T}_{2^\nu N}[X]$

$$\iota : \mathbb{T}_N[X] \longrightarrow \mathbb{T}_{2^\nu N}[X],$$
$$p(x) = \sum_{i=0}^{N-1} p_i X^i \longmapsto p_{\mathsf{ext}}(X) = \sum_{i=0}^{N-1} p_i X^{2^\nu i}$$

Toy Example: $\mathbb{T}_2[X] \to \mathbb{T}_8[X]$

$$\iota(0.5X + 0.7) = 0.5X^4 + 0.7$$

Ciphertext Extension by Zero Padding

$$\iota\left(\mathsf{TRLWE}^N_{\mathcal{K}}(p(X))\right) = \mathsf{TRLWE}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(p_{\mathsf{ext}}(X)) \text{ for } p(X) \in \mathbb{T}_N[X],$$
$$\iota\left(\mathsf{TRGSW}^N_{\mathcal{K}}(q(X))\right) = \mathsf{TRGSW}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(q_{\mathsf{ext}}(X)) \text{ for } q(X) \in \mathbb{Z}_N[X],$$

Our Work 1 - Extended BlindRotate

The external product $\boxdot$ on ring dimension $2^\nu N$ naturally follows:

$$\mathsf{TRGSW}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(q_{\mathsf{ext}}(X)) \boxdot \mathsf{TRLWE}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(p(X)) = \mathsf{TRLWE}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(p(X) \cdot q_{\mathsf{ext}}(X)).$$

Which enables to evaluate the **CMux** circuit on ring dimension $2^\nu N$ while original ring dimension stays in N:

$$\mathsf{ACC} \leftarrow \mathsf{TRGSW}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(\mathbf{s}_i) \boxdot \left((X^{\bar{\mathbf{a}}_i} - 1) \cdot \mathsf{ACC}\right) + \mathsf{ACC}.$$

Thus, we **ModSwitch** to $\mathbb{Z}_{2^{\nu+1}N}$, gaining additional ν bits of precision.

Our Work 2 - Parallel External Product

Module isomorphism $\tau : \mathbb{T}_{2^\nu N}[X] \to \mathbb{T}_N^{2^\nu}[X]$

$$p(x) = \sum_{i=0}^{2^\nu N-1} p_i X^i \longmapsto \left(p^{(0)}(X), \cdots, p^{(2^\nu-1)}(X)\right)$$
$$= \left(\sum_{i=0}^{N-1} p_{2^\nu i} X^i, \cdots, \sum_{i=0}^{N-1} p_{2^\nu i+2^\nu-1} X^i\right).$$

Toy Example: $\mathbb{T}_8[X] \to \mathbb{T}_2^4[X]$

$$\tau(0.2X^7 + 0.1X^6 + 0.5X^4 + 0.3X + 0.7) = (0.7 + 0.5X, 0.3, 0.1X, 0.2X)$$

Ciphertext Folding

$$\tau\left(\mathsf{TRLWE}_{\mathcal{K}_{\text{ext}}}^{2^\nu N}(m)\right) = \left(\mathsf{TRLWE}_{\mathcal{K}}^{N}(m_0), \cdots, \mathsf{TRLWE}_{\mathcal{K}}^{N}(m_{2^\nu-1})\right) \text{ for } m(X) \in \mathbb{T}_N[X].$$

Our Work 2 - Parallel External Product

For special case, we can parallelize the external product over dimension $2^\nu N$ into 2^ν external products in dimension N:

Special case of External Product $\boxdot$

- **C1.** $\mathcal{K}_{ext}$ should be an extended key of $\mathcal{K}$.
- **C2.** $\mathsf{TRGSW}^{2^\nu N}_{\mathcal{K}_{ext}}(z)$ is an extension of $\mathsf{TRGSW}^{N}_{\mathcal{K}}(z)$, and $z \in \mathbb{Z}$.

$$\mathsf{TRGSW}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(z) \boxdot \mathsf{TRLWE}^{2^\nu N}_{\mathcal{K}_{\mathsf{ext}}}(m) \cong \begin{bmatrix} \mathsf{TRGSW}^{N}_{\mathcal{K}}(z) \boxdot \mathsf{TRLWE}^{N}_{\mathcal{K}}(m_0) \\ \mathsf{TRGSW}^{N}_{\mathcal{K}}(z) \boxdot \mathsf{TRLWE}^{N}_{\mathcal{K}}(m_1) \\ \vdots \\ \mathsf{TRGSW}^{N}_{\mathcal{K}}(z) \boxdot \mathsf{TRLWE}^{N}_{\mathcal{K}}(m_{2^\nu-1}) \end{bmatrix}$$

$$\cong \left(\mathsf{TRLWE}^{N}_{\mathcal{K}}(z \cdot m_0), \cdots, \mathsf{TRLWE}^{N}_{\mathcal{K}}(z \cdot m_{2^\nu-1})\right)$$

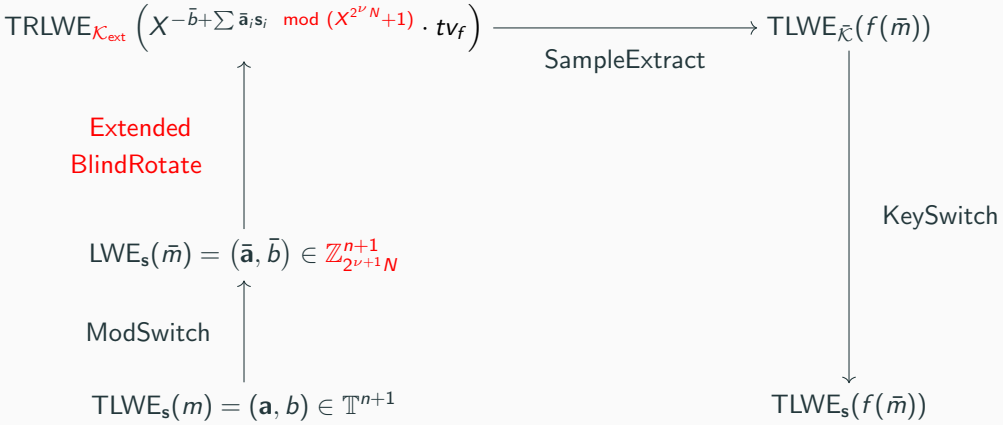
Our Work 2 - Parallel CMux / BlindRotate

With the parallel external product, we can build parallel CMux circuit:

$$\overrightarrow{\mathsf{ACC}} \leftarrow \begin{bmatrix} \mathsf{BSK}_i \boxdot \left(\tau\left(X^{\bar{a}_i} \cdot \tau^{-1}\left(\overrightarrow{\mathsf{ACC}}\right)\right)_0 - \overrightarrow{\mathsf{ACC}}_0\right) + \overrightarrow{\mathsf{ACC}}_0 \\ \mathsf{BSK}_i \boxdot \left(\tau\left(X^{\bar{a}_i} \cdot \tau^{-1}\left(\overrightarrow{\mathsf{ACC}}\right)\right)_1 - \overrightarrow{\mathsf{ACC}}_1\right) + \overrightarrow{\mathsf{ACC}}_1 \\ \vdots \\ \mathsf{BSK}_i \boxdot \left(\tau\left(X^{\bar{a}_i} \cdot \tau^{-1}\left(\overrightarrow{\mathsf{ACC}}\right)\right)_{2^\nu-1} - \overrightarrow{\mathsf{ACC}}_{2^\nu-1}\right) + \overrightarrow{\mathsf{ACC}}_{2^\nu-1} \end{bmatrix}$$

Equivalent to running 2^ν CMux circuits!

Summary : Extended BootStrapping



- Gains ν bit of precision
- Able to keep small ring dimension N
- Parallelized bootstrapping

Experimental Results

Implementation

We implemented our **E**xtended **B**oot**S**trapping on top of the TFHE library[2]

We also implemented three state-of-the-art full-domain bootstrapping algorithms ([KS21], [YXSCZ21], [CZBSG22]), which all uses TFHE bootstrapping as sub-algorithm. The codes are all available at our github page[3].

Experimental Environment

- CPU : Intel i9-13900K 5.8GHz 24 core, 32 threads
- RAM : DDR5 128GB
- Ubuntu 22.04

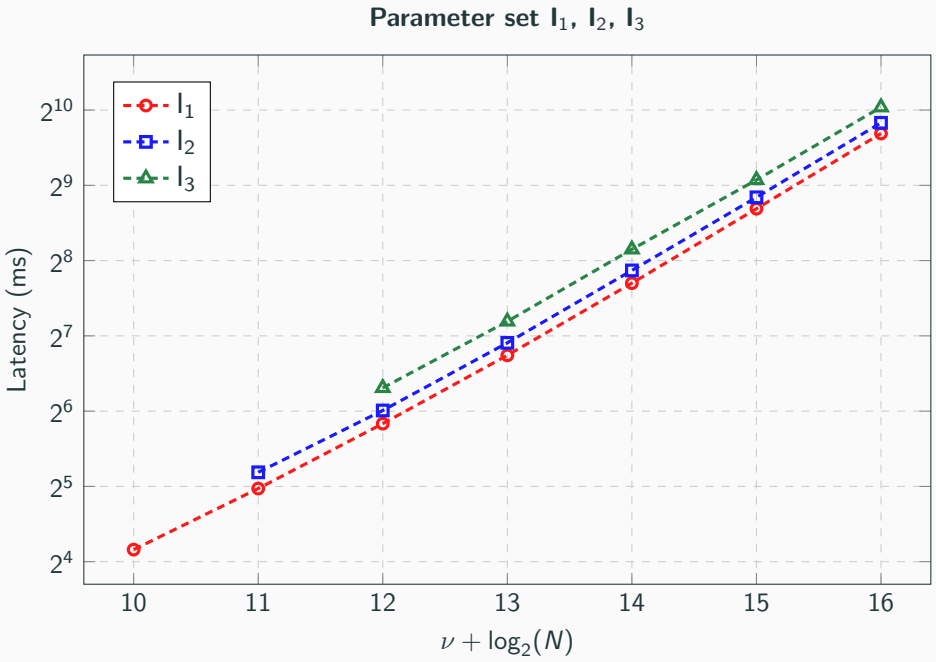
[2] https://tfhe.github.io/tfhe/
[3] https://github.com/Stirling75/Extended-BootStrapping

Parameters

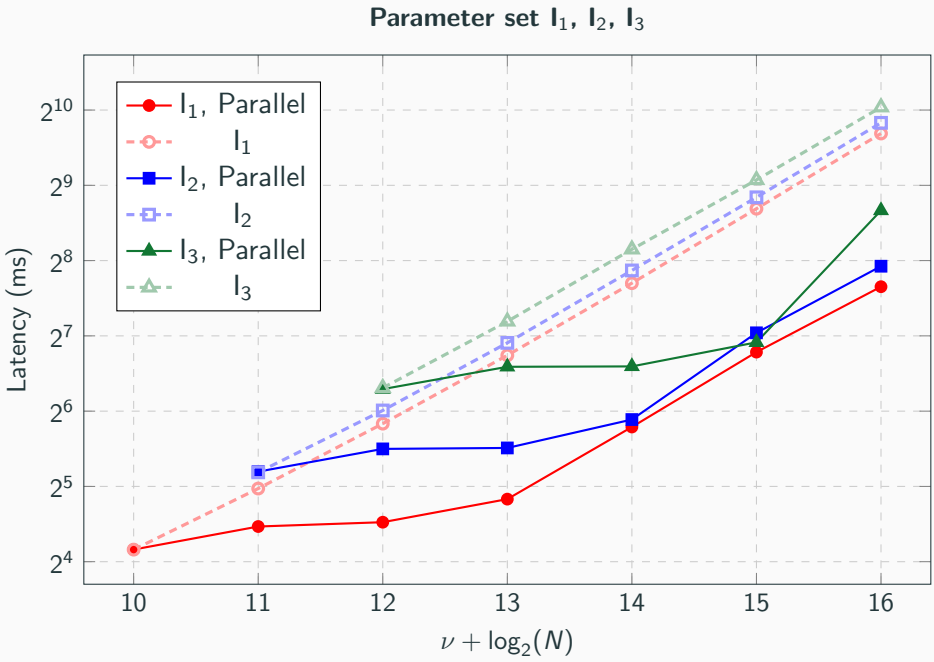
Table 1: TFHE parameter sets. λ indicates the security level of given parameter set.

Param Set	λ	TLWE		TRLWE			KSK		BSK	
		n	σ_{TLWE} $(\log_2)$	N	k	σ_{TRLWE} $(\log_2)$	ℓ_{KS}	B_{KS}	ℓ_{BS}	B_{BS}
I_1	80	750	-21.2	1024	1	-29.3	3	2^8	7	2^4
I_2	80	750	-21.2	2048	1	-32	3	2^8	7	2^4
I_3	80	750	-21.2	4096	1	-32	3	2^8	7	2^4
II	80	900	-25.7	2048	1	-32	5	2^6	7	2^4
III_1	128	670	-12.4	1024	1	-20.1	3	2^5	8	2^3
III_2	128	670	-12.4	2048	1	-32	3	2^5	8	2^3
III_3	128	670	-12.4	4096	1	-32	3	2^5	8	2^3
IV	128	1300	-26.1	2048	1	-32	5	2^6	7	2^4

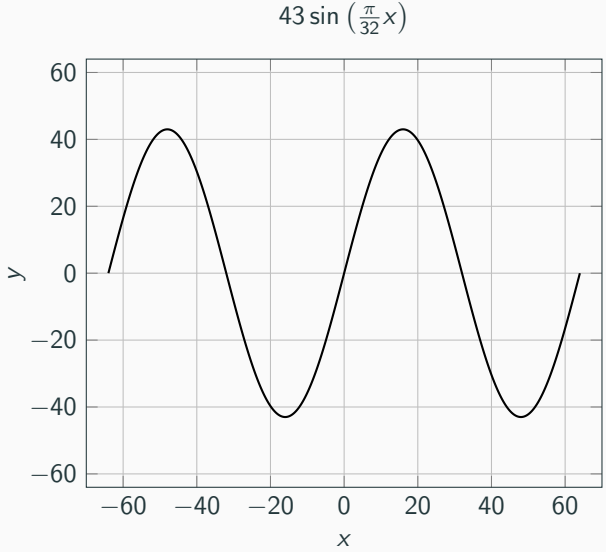
EBS : Performance (Non-Parallelized)



EBS : Performance (Parallelized)

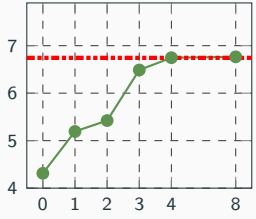
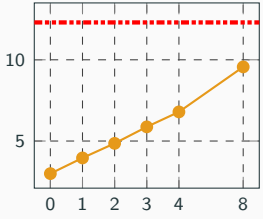
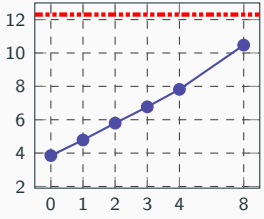
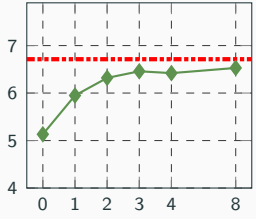
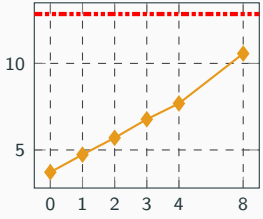
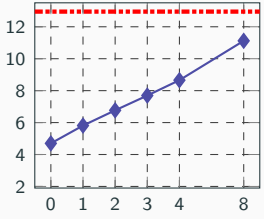


EBS : Function Evaluation



Homomorphic evaluation of function $f(x) = 43 \sin\left(\frac{\pi}{32}x\right)$ on domain $[-64, 64)$.

EBS : Function Evaluation

Parameter	FDFB-EBS [KS21]	TOTA-EBS [YXSCZ21]	Comp-EBS [CZBSG22]
I_1			
II			

Thank You

References i

[CGGI20] Ilaria Chillotti, Nicolas Gama, Mariya Georgieva, and Malika Izabachène. TFHE: fast fully homomorphic encryption over the torus. In: *Journal of Cryptology* 33.1 (2020), pp. 34–91 (cit. on p. 6).

[CZBSG22] Pierre-Emmanuel Clet, Martin Zuber, Aymen Boudguiga, Renaud Sirdey, and Cédric Gouy-Pailler. Putting up the swiss army knife of homomorphic calculations by means of TFHE functional bootstrapping. In: *Cryptology ePrint Archive* (2022) (cit. on pp. 22, 27).

[KS21] Kamil Kluczniak and Leonard Schild. FDFB: Full domain functional bootstrapping towards practical fully homomorphic encryption. In: *arXiv preprint arXiv:2109.02731* (2021) (cit. on pp. 22, 27).

[YXSCZ21] Zhaomin Yang, Xiang Xie, Huajie Shen, Shiying Chen, and Jun Zhou. TOTA: Fully Homomorphic Encryption with Smaller Parameters and Stronger Security. In: *Cryptology ePrint Archive* (2021) (cit. on pp. 22, 27).