Generic Models for Group Actions

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Post-Quantum Cryptography

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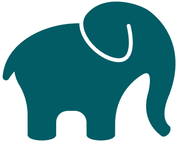
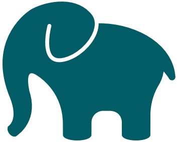


Alice



Bob

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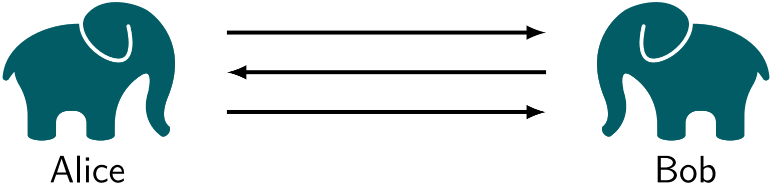

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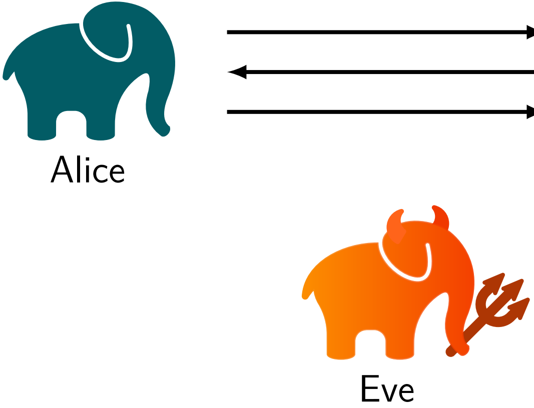

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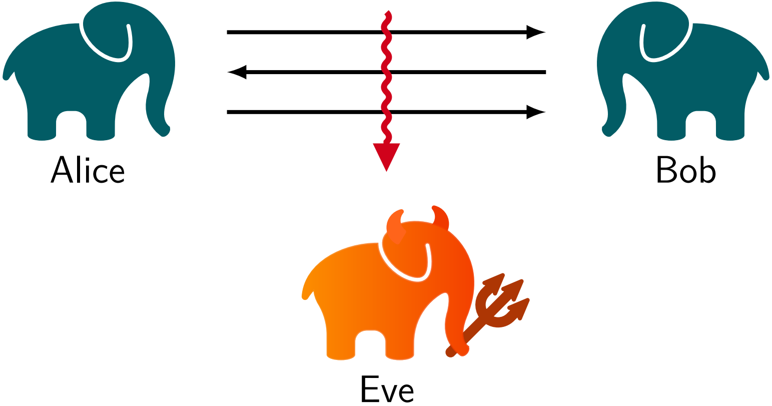

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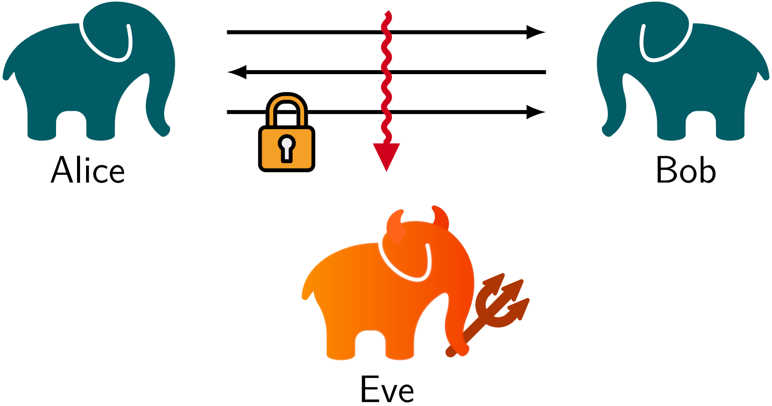

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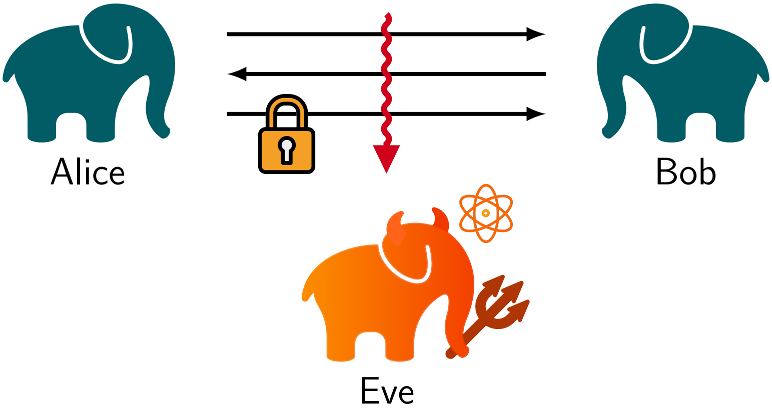

Post-Quantum Cryptography

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- Popular alternative: Cryptographic Group Actions
  - Based on **isogenies**

Group Actions

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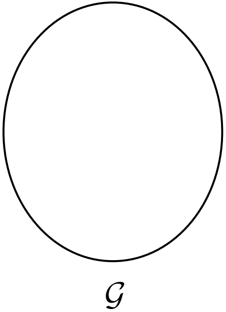

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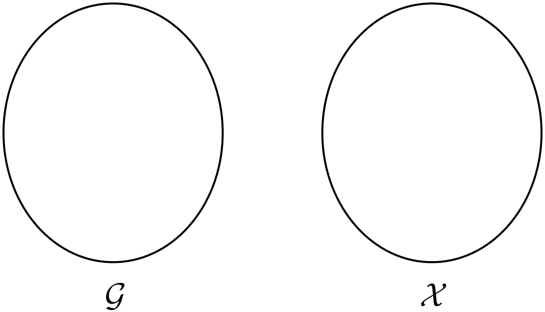

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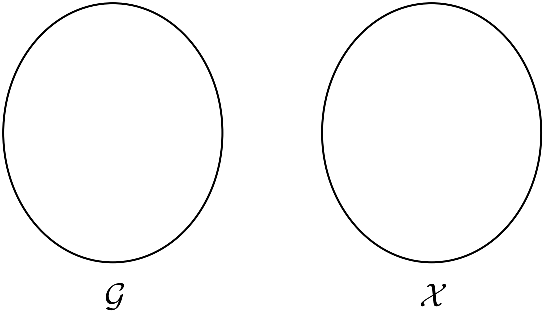

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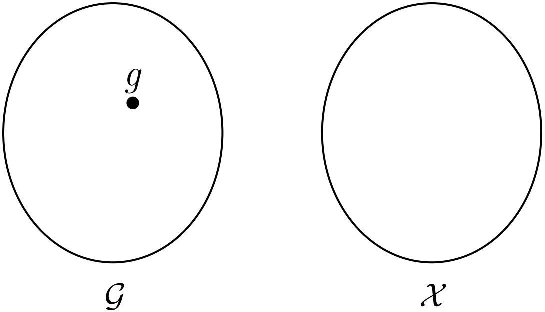

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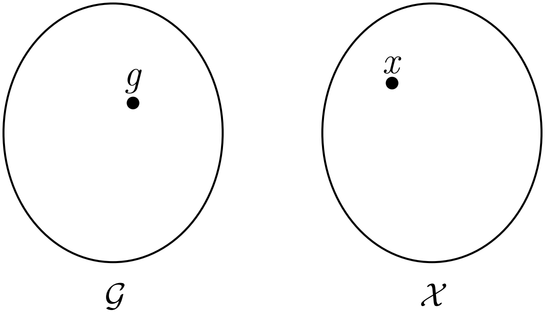

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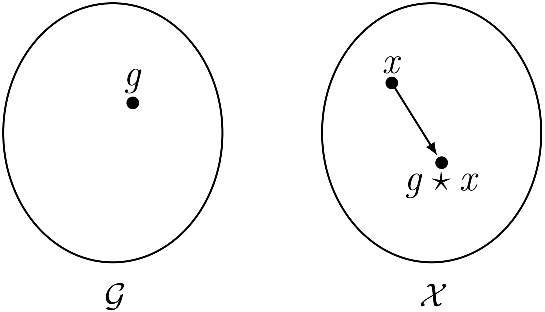

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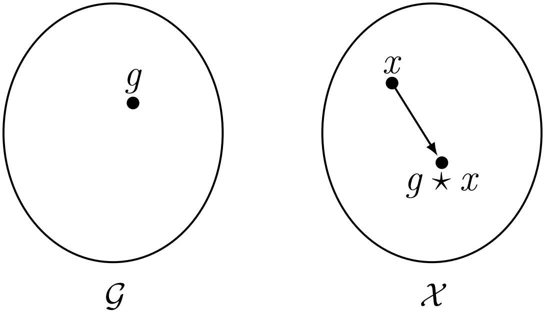

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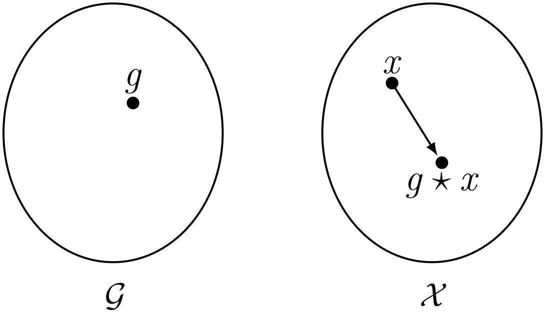

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- Identity:
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- Compatibility:
  $g \star (h \star x) = (g + h) \star x$

Standard Group Action Assumptions

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Quantum Hardness

Kuperberg (subexponential)

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$$\text{DDH}(g \star x, \cdot, \cdot) \qquad \text{and} \qquad \text{DDH}(h \star x, \cdot, \cdot).$$

- Underlies the security of the CSIDH key exchange [DHK+22]

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  - Impossibility of *quantum* lower bounds
- Define the **algebraic** group action model
  - *Classical* and *quantum* reductions between (non-standard) assumptions and DLOG

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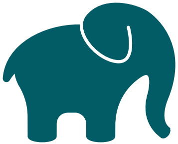

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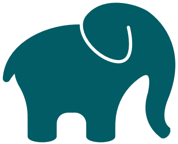

$\mathcal{O}^{\mathsf{exp}}$

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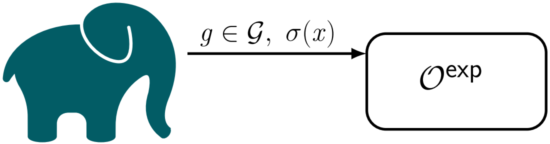

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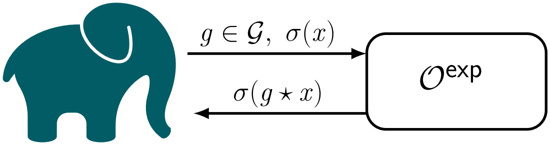

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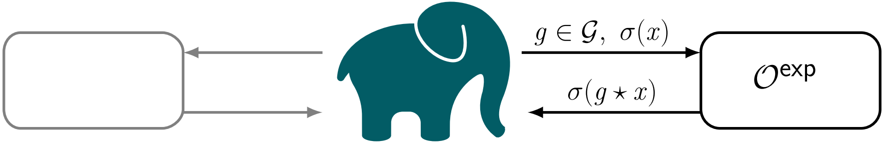

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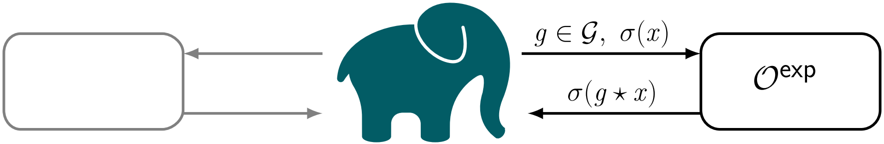

Runtime measured in # oracle queries

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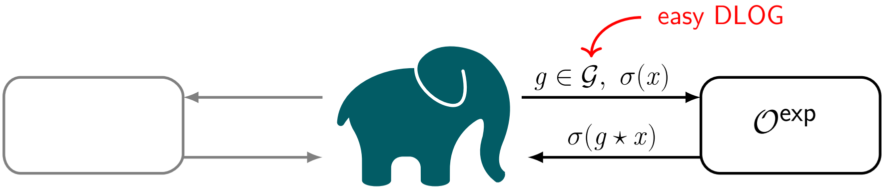

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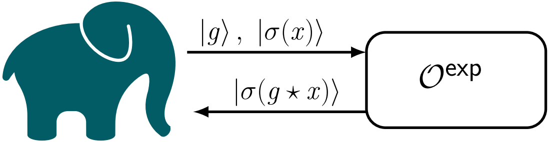

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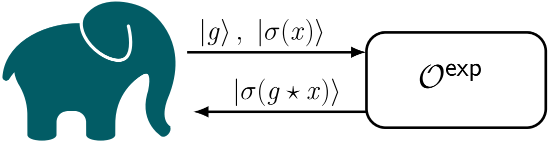
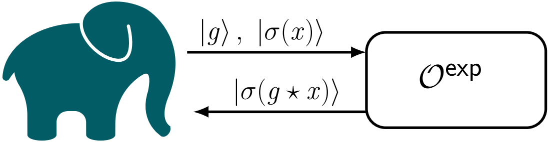


Ettinger-Høyer:



Ettinger-Høyer:

- **Generic** quantum algorithm solving DLOG

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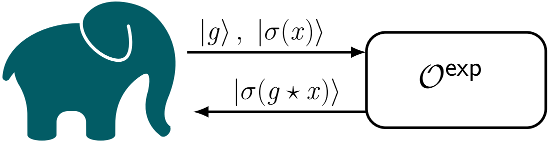

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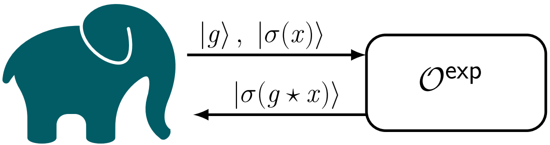

Ettinger-Høyer:

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$\Rightarrow$ Not even DLOG is hard

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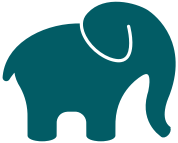

GAME

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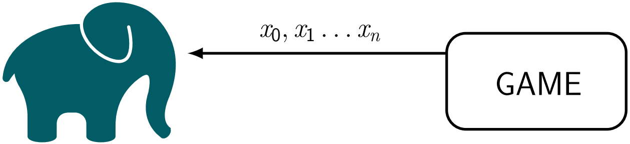

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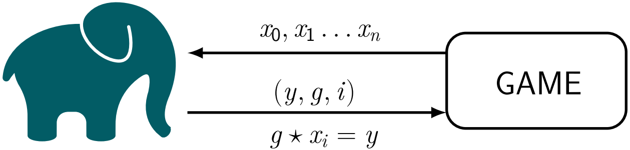

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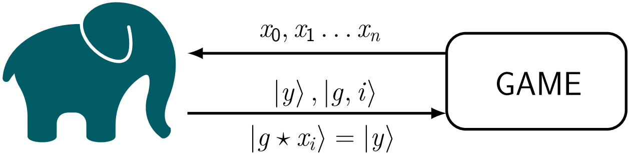

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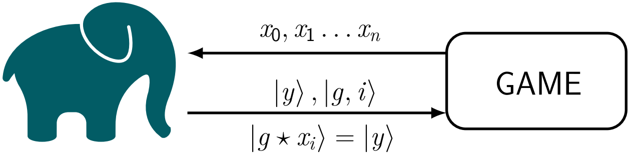

Runtime measured in # unit operations

Results in the QAGAM

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Strong CDH (St-CDH)

Given x, $g \star x$, $h \star x$ compute $(g + h) \star x$ while having access to oracles

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Theorem (DLOG $\Rightarrow$ St-CDH)

For every ***quantum*** *adversary $\mathcal{A}$ in the* ***quantum algebraic*** *group action model against* St-CDH *there exists $\mathcal{B}, \mathcal{C}$ against* DLOG *with*

$$\epsilon_{\mathcal{A}} \leq \sqrt{(q+1) \cdot \epsilon_{\mathcal{B}}} + \epsilon_{\mathcal{C}}.$$

Summary

- Adapted the GGM and AGM to the group action setting.
  - Include further algebraic properties of isogenies like **twists**.
- Proved information-theoretic lower bounds in the **generic** group action model.
- Gave algebraic reductions between non-standard assumptions and DLOG in the **algebraic** group action model.

https://ia.cr/2023/186

References I

[AEK+22] Michel Abdalla, Thorsten Eisenhofer, Eike Kiltz, Sabrina Kunzweiler, and Doreen Riepel. Password-authenticated key exchange from group actions. In Yevgeniy Dodis and Thomas Shrimpton, editors, *CRYPTO 2022, Part II*, volume 13508 of *LNCS*, pages 699–728. Springer, Heidelberg, August 2022.

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