

# A Thorough Treatment of Highly-Efficient NTRU Instantiations

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8. May 2023

# NTRU

first practical lattice-based encryption scheme [HPS98]

$\mathcal{R} = \mathbb{Z}_q[X]/(f(X))$ ,  $pk = \mathbf{g}/\mathbf{f}$ ,  $sk = \mathbf{f}$ ,  $c = \mathbf{r} \cdot pk + \vec{m}$ , narrow distribution  $\eta$

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NTRU $_{\eta}$  assumption:  $pk \approx_c \mathbf{h} \leftarrow_{\$} \mathcal{R}$

$\mathcal{R}$ -LWE $_{\eta}$  assumption: difficult to compute  $\vec{m}$  given  $c = \mathbf{r} \cdot \mathbf{h} + \vec{m}$ , where  $\mathbf{h} \leftarrow_{\$} \mathcal{R}$  and  $\vec{m}, \mathbf{r} \leftarrow_{\$} \eta$ .

3 of 17 first round Lattice NIST PQC candidates used NTRU Variant

## Decryption Errors and Compactness

$$\begin{aligned}\text{Dec}(\mathbf{f}, \text{Enc}(\mathbf{h}, \vec{m})) &= (\mathbf{f}\mathbf{c} \bmod^{\pm} q) \bmod^{\pm} 3 \\ &= \underbrace{(3(\mathbf{g}\mathbf{r} + \mathbf{f}'\vec{m}))}_{\text{correctness term}} + \vec{m} \bmod^{\pm} q \bmod^{\pm} 3\end{aligned}$$

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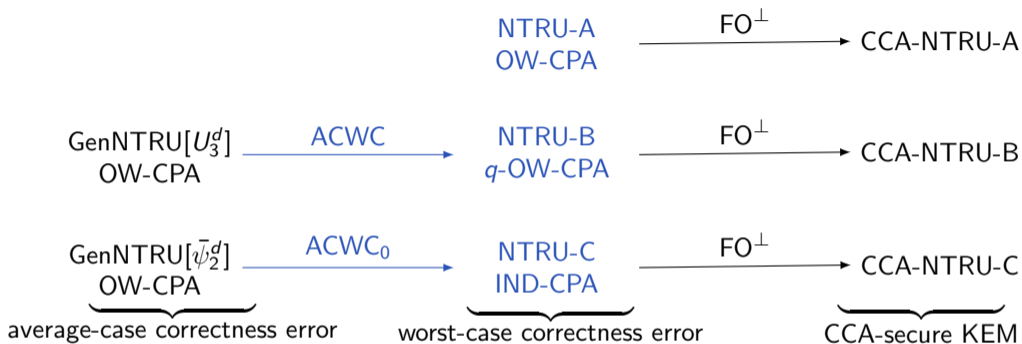
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our solution: apply error-reducing transform  $\xrightarrow{\text{FO}}$  CCA-KEM

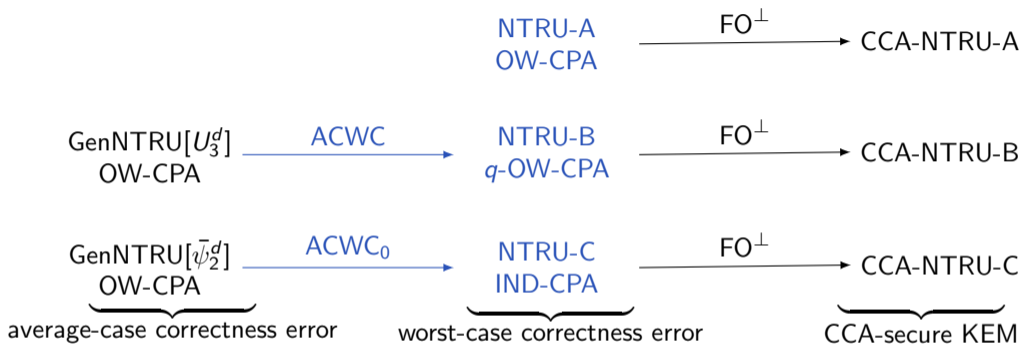
Advantage: smaller  $q \implies$  better security for NTRU assumption + more compact

# This work



3 NTRU variants

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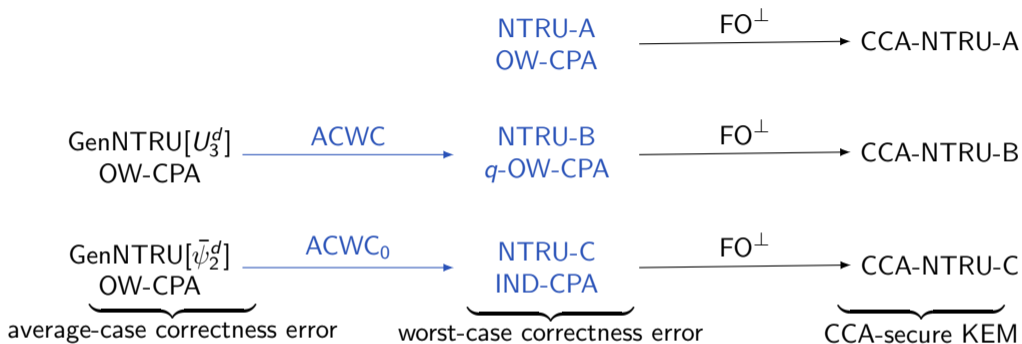


3 NTRU variants

NTRU-A with worst-case correctness

2 error-reducing transformations, analyzed in (Q)ROM

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3 NTRU variants

NTRU-A with worst-case correctness

2 error-reducing transformations, analyzed in (Q)ROM

analysis of the worst-case correctness

obtain CCA-KEMs through FO

## Practical application

Instantiated with NTT-friendly Rings  $\mathbb{Z}_q[X]/(X^d - X^{d/2} + 1)$  [LS19], our scheme is

- 15% more compact

- 15x improvement in ephemeral round-trip time

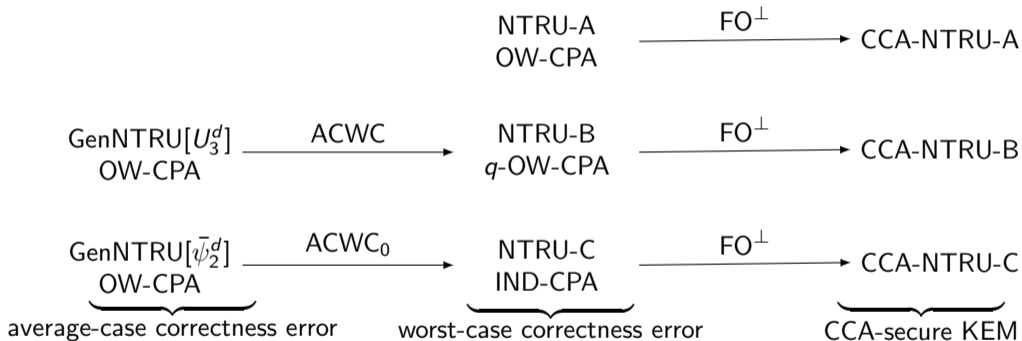
- 35x faster key-generation

- 6x faster key-encapsulation

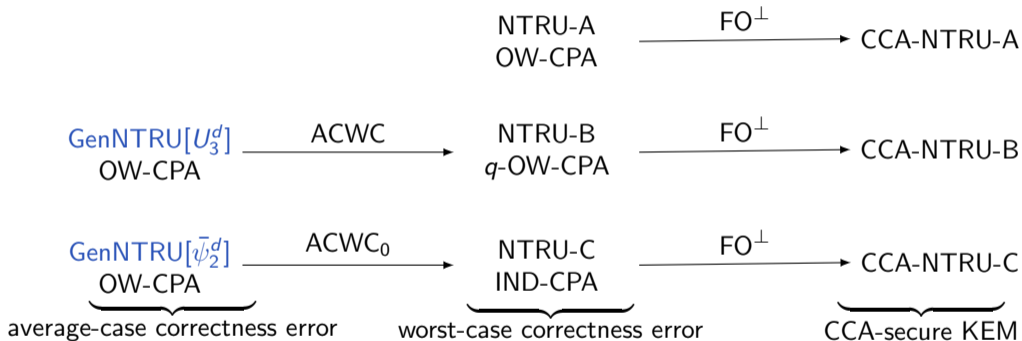
- 9x faster key-decapsulation

than NIST-Finalist NTRU-HRSS-701 [HRSS17]

# Overview



# Overview



# GenNTRU $[\eta]$

KeyGen()

01  $\mathbf{f}', \mathbf{g} \leftarrow \eta$

02  $\mathbf{f} := 3\mathbf{f}' + 1$

03 **if**  $\mathbf{f}$  or  $\mathbf{g} \notin \mathcal{R}^\times$ , restart

04 **return**  $(pk, sk) = (3\mathbf{g}\mathbf{f}^{-1}, \mathbf{f})$

Enc( $\mathbf{h} \in \mathcal{R}, \vec{m} \in \{-1, 0, 1\}^d$ )

05  $\mathbf{r} \leftarrow \eta$

06 **return**  $\mathbf{c} := \mathbf{h}\mathbf{r} + \vec{m}$

Dec( $\mathbf{f} \in \mathcal{R}, \mathbf{c} \in \mathcal{R}$ )

07 **return**  $\vec{m} := (\mathbf{c}\mathbf{f} \bmod \pm q) \bmod \pm 3$

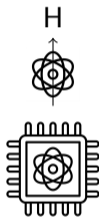
$h \bmod \pm q$  to mean the integer from the set  $\left\{-\frac{q-1}{2}, \dots, \frac{q-1}{2}\right\}$  which is congruent to  $h$  modulo  $q$

Randomness Recoverable:  $\mathbf{r} = \mathbf{h}^{-1}(\mathbf{c} - \vec{m})$



# Quantum Random Oracle Model [BDF<sup>+</sup>11]

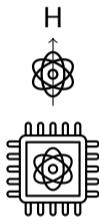
$$H\left(\frac{1}{\sqrt{2}}|0^n\rangle + \frac{1}{\sqrt{2}}|1^n\rangle\right)$$



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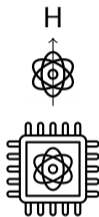
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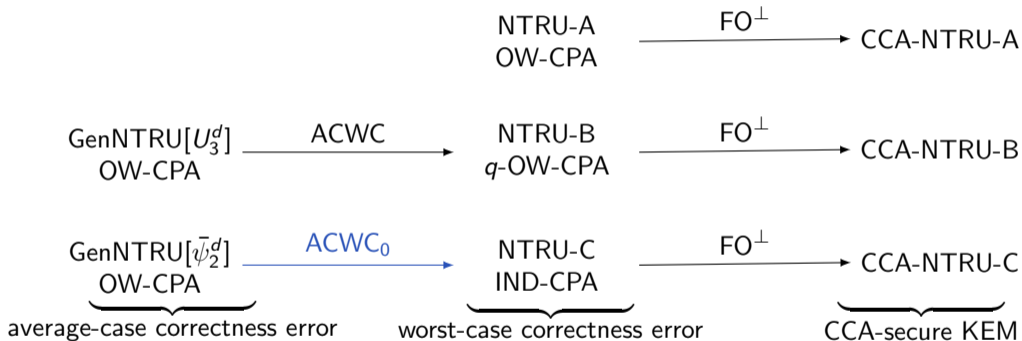
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common lemma: Oneway-to-hiding [AHU19]

# Overview



# ACWC<sub>0</sub>[PKE]

Enc'(pk, m ∈ {0, 1}<sup>λ</sup>)

01 pick random  $r$

02 **return** (Enc(pk, r), F(r) ⊕ m)

Dec'(sk, (c, u))

03  $r := \text{Dec}(sk, c)$

04 **return** F(r) ⊕ u

Intuition: Correctness independent of  $m \implies$  worst-case correct

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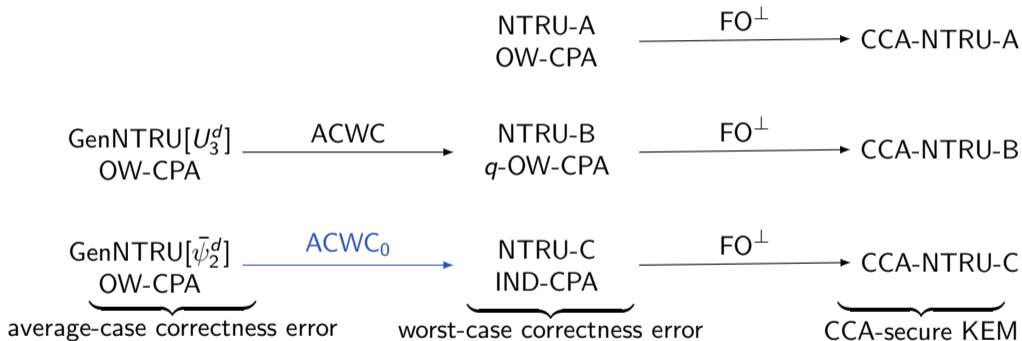
$\text{Enc}'(pk, m \in \{0, 1\}^\lambda)$	$\text{Dec}'(sk, (c, u))$
01 pick random $r$	03 $r := \text{Dec}(sk, c)$
02 <b>return</b> $(\text{Enc}(pk, r), F(r) \oplus m)$	04 <b>return</b> $F(r) \oplus u$

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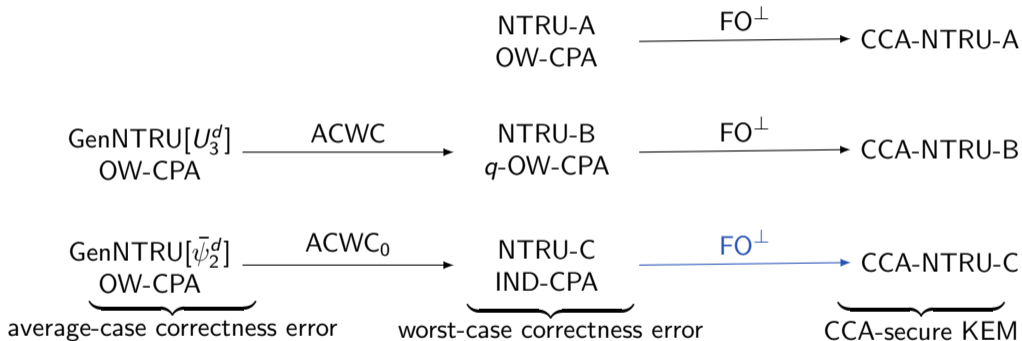
Thm: If PKE is oneway, then ACWC<sub>0</sub>[PKE] is IND-CPA secure in the (Q)ROM.

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# Fujisaki-Okamoto Transform with Explicit Rejection

$$\text{Encaps}_{pk}(\cdot; r) = (\underbrace{\text{Enc}_{pk}(r; G(r))}_{\text{ciphertext}}, \underbrace{H(r)}_{\text{key}})$$

$$\text{Decaps}_{sk}(c) = \begin{cases} H(r') & \text{if } \text{Enc}_{pk}(r'; G(r')) \stackrel{?}{=} c \\ & \text{where } r' := \text{Dec}_{sk}(c) \\ \perp & \text{else} \end{cases}$$

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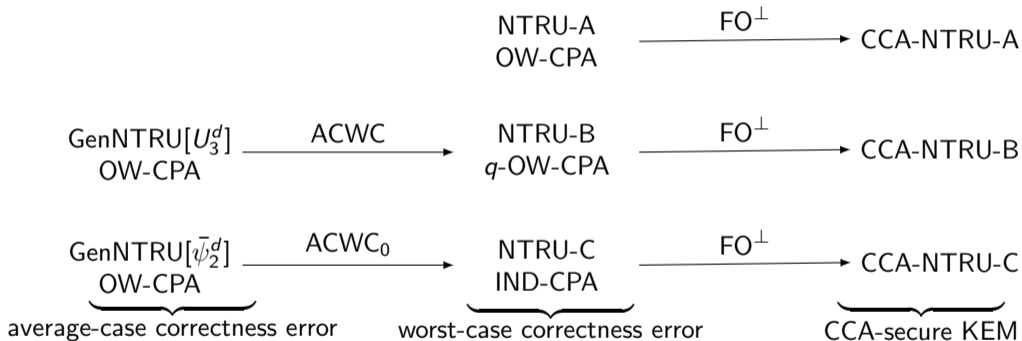
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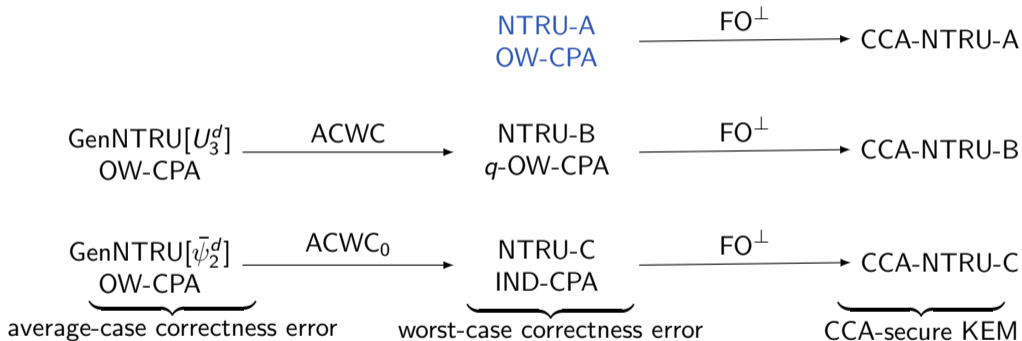
Explicit Rejection has been proven secure in the QROM [DFMS21]

more efficient decapsulation than implicit rejection

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## Distribution used in NTRU-A

define distribution  $\psi_2^d$  over  $\mathbb{Z}^d$  as  $\vec{b}_1 + \vec{b}_2 - \vec{b}_3 - \vec{b}_4$ , for  $\vec{b}_i \leftarrow_{\$} \{0, 1\}^d$

alternative generation of  $\psi_2^d$

$$\vec{b} = \left( \vec{b}_1 \underbrace{-2\vec{b}_2 \odot \vec{b}_3}_{0 \pmod{2}} \right) \odot \left( 1 \underbrace{-2\vec{b}_4}_{0 \pmod{2}} \right),$$



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Idea of NTRU-A: use message  $m$  as  $\vec{b}_1 = \vec{b} \pmod{2}$ , and sample  $\vec{b}_2, \vec{b}_3, \vec{b}_4$  random

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$\implies$  adversary only controls  $\mathbf{e} \pmod{2}$  in  $\mathbf{c} = \mathbf{hr} + \mathbf{e}$

$\implies$  worst-case decryption errors  $\approx$  average-case errors

$\implies$  no additional error-reducing transform necessary

# NTRU-A

Enc( $\mathbf{h} \in \mathcal{R}, \vec{m} \in \{0, 1\}^d$ )

01  $\mathbf{r} := \text{Gen1}()$

02  $\vec{b}_2, \vec{b}_3, \vec{b}_4 \leftarrow \{0, 1\}^d$

03  $\mathbf{e} := (\vec{m} - 2\vec{b}_2 \odot \vec{b}_3) \odot (1 - 2\vec{b}_4)$

04 **return**  $\mathbf{hr} + \mathbf{e}$

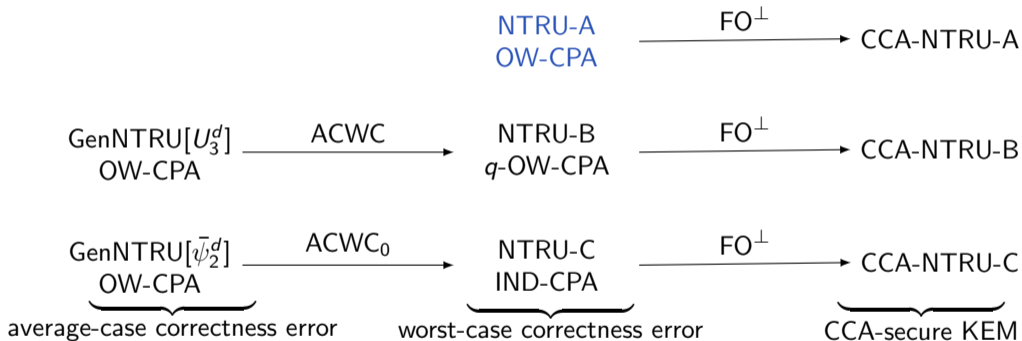
Dec( $\mathbf{f} \in \mathcal{R}, \mathbf{c} \in \mathcal{R}$ )

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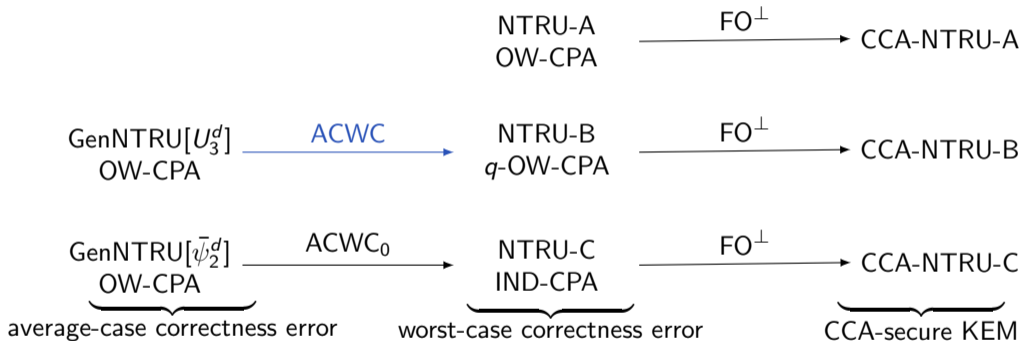
OW-CPA secure based on  $\text{NTRU}_{\psi_2^d}$  and  $\mathcal{R}\text{-LWE2}_{\psi_2^d}$  assumption

$\mathcal{R}\text{-LWE2}_{\psi_2^d}$  assumption: given  $(\mathbf{h}, \mathbf{hr} + \mathbf{e})$  for  $\mathbf{h} \leftarrow \mathcal{R}$  and  $\mathbf{r}, \mathbf{e} \leftarrow_{\$} \psi_2^d$  difficult to compute  $\mathbf{e} \bmod 2$

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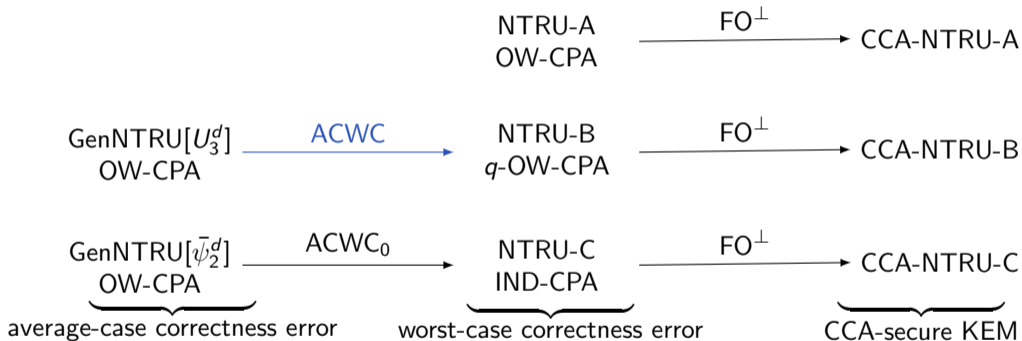
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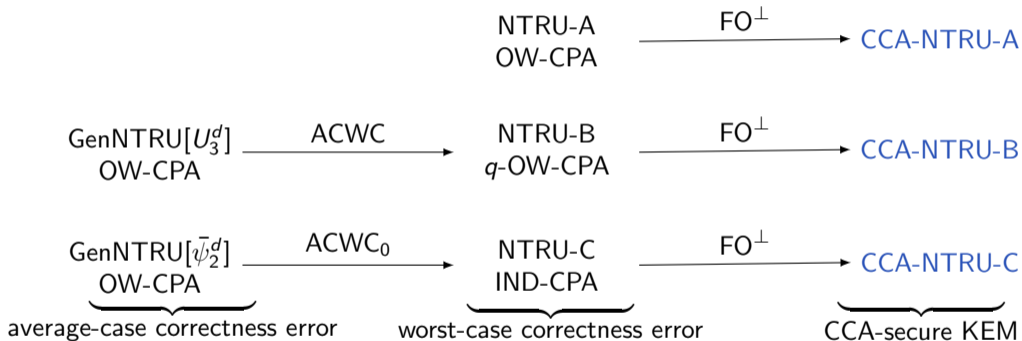
worst-case error bound uses Hoeffding Bound

see Paper for details

# Overview



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## Results

Scheme	KeyGen	Encaps	Decaps	pk (B)	c (B)	security
CCA-NTRU-A <sub>2917</sub> <sup>648</sup>	6.2K	5.6K	7.3K	972	972	180
NTRU-HRSS-701	220.3K	34.6K	65K	1138	1138	166
NTTRU	6.4K	6.1K	7.9K	1248	1248	183
Kyber-512 (90's)	6.2K	7.9K	9.2K	800	768	148
Kyber-768 (90's)	11K	13.1K	14.8K	1184	1088	212

Table: Number of cycles (on a Skylake machine) for various operations of a CCA-secure KEM.

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Thank you!





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