Ideal-SVP is hard for small-norm uniform prime ideals

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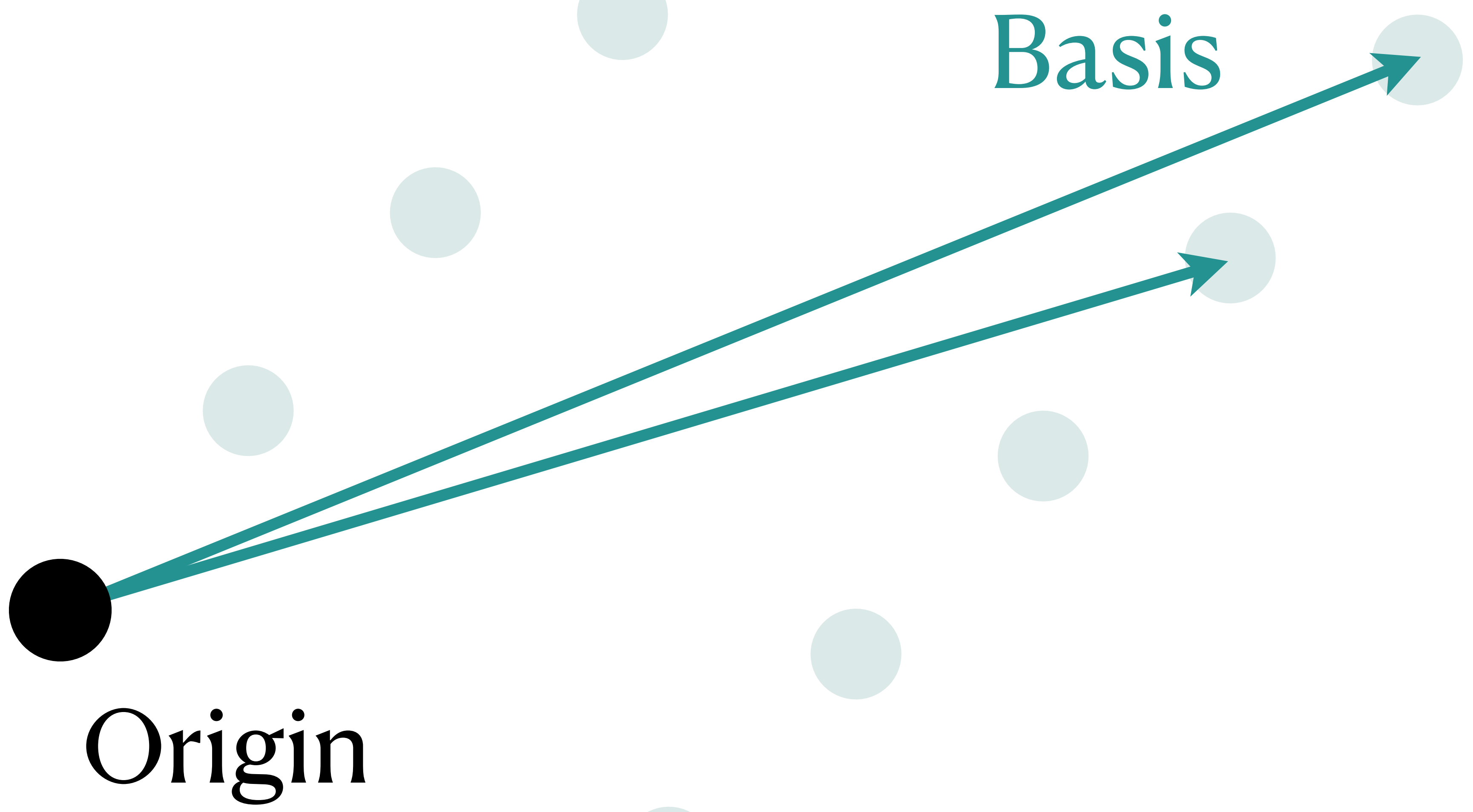
Ideal-SVP

- Any basis $(b_1, \ldots, b_n)$ of a vector space describes a lattice $L = \mathbb{Z}b_1 + \ldots + \mathbb{Z}b_n$



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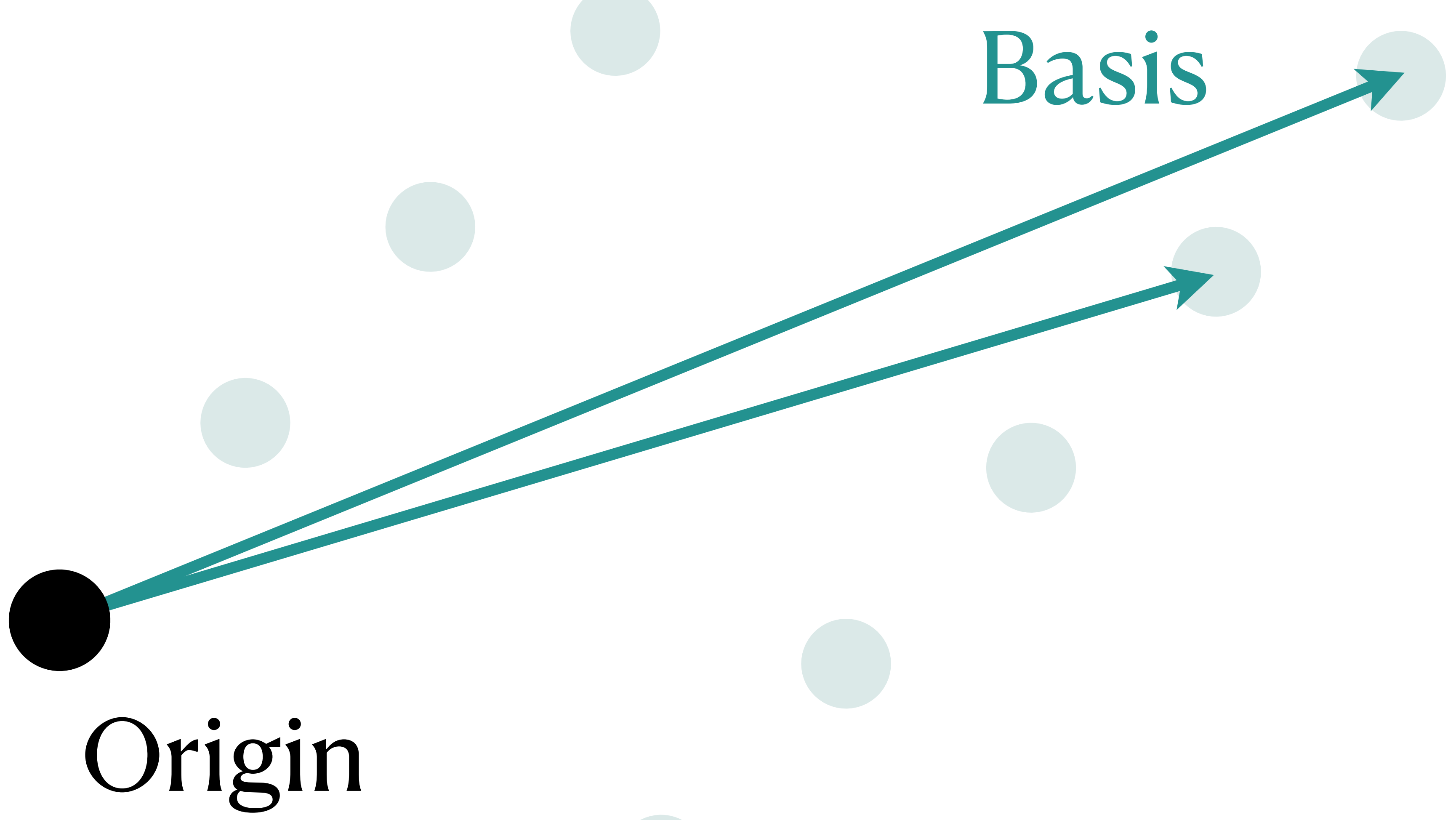

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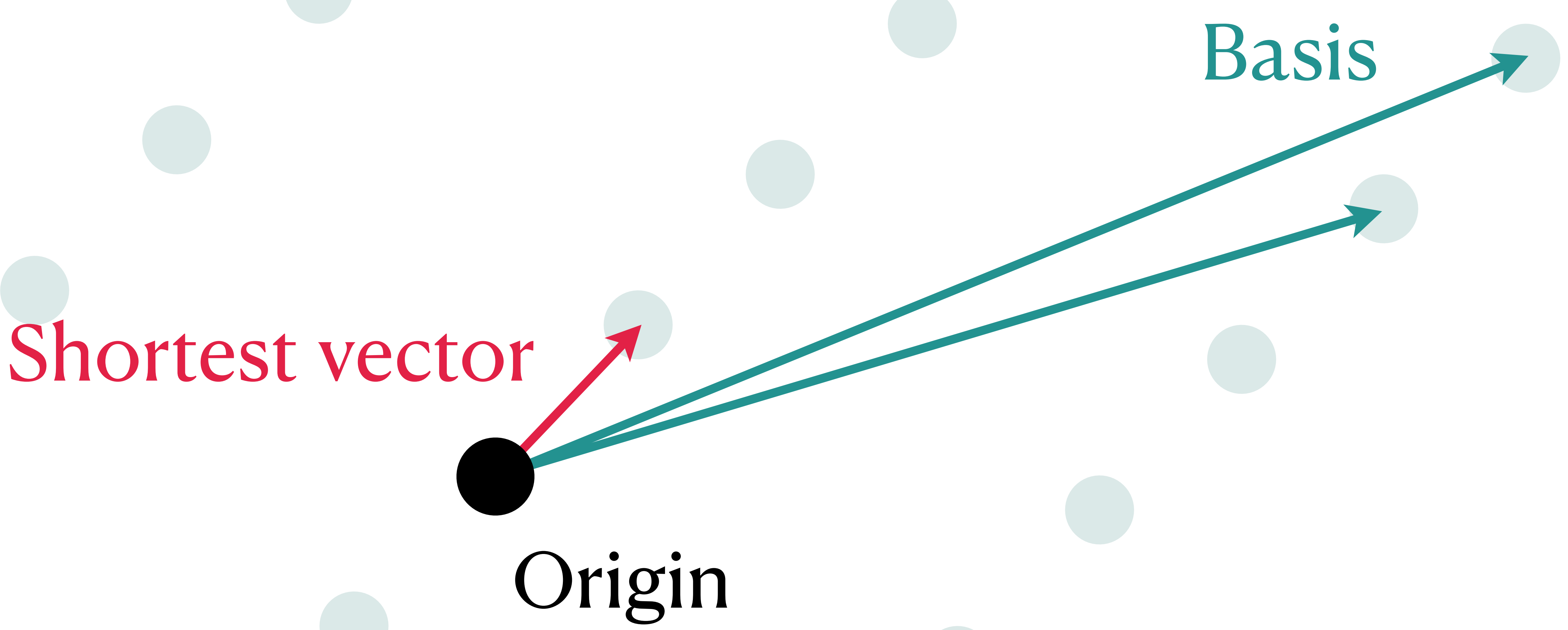
- **SVP**: find the non-zero point in L closest to the origin (up to approx. factor…)



Ideal-SVP

- Some lattices have "something extra"...
- $K = \mathbb{Q}[X] / (X^n + 1)$ is a **field**, and a $\mathbb{Q}$**-vector space** of dimension $n = 2^r$
- $\mathcal{O} = \mathbb{Z}[X] / (X^n + 1)$ is a lattice in K... and a **subring**!
- An **ideal lattice** is a lattice $\mathfrak{a} \subset K$ such that $\mathcal{O}\mathfrak{a} \subset \mathfrak{a}$

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- An integral ideal is **prime** if it is not a product of other integral ideals
- Any ideal lattice is of the form $\mathfrak{a} = \prod \mathfrak{p}_i^{e_i}$ where $\mathfrak{p}_i \subset \mathcal{O}$ is prime and $e_i \in \mathbb{Z}$

Ideal-SVP

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- Ideal lattices are **special**: is SVP as hard?
    - There are specific algorithms for SVP in ideal lattices
    - Ideal-SVP still considered hard, but one can reach better approximation factors than SVP in generic lattices

Average-case Ideal-SVP

Is Ideal-SVP hard on average?

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Haar-Ideal-SVP

- This work: **uniformly random prime ideal** of small norm

P-Ideal-SVP

  - *Integral*, unlike [Gentry09], and *small*, unlike [BDPW20]
  - Composes with NTRU reductions! Links NTRU to worst-case Ideal-SVP

Method

Worst-case Ideal-SVP $\overset{\text{[Gentry09]}}{\leq}$ P^{-1}-Ideal-SVP $\overset{\text{New}}{\leq}$ P-Ideal-SVP

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poly-time given a factoring oracle

P^{-1}-Ideal-SVP reduces to P-Ideal-SVP

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1. Let $s_{\mathfrak{p}} \in \mathfrak{p}$ small (solve P-Ideal-SVP for $\mathfrak{p}$, *uniform prime of bounded norm*);
2. Let $(\mathfrak{b}, y) \leftarrow \texttt{SampleIdeal}(\mathfrak{p}, s_{\mathfrak{p}})$

   ➡ $\mathfrak{b}$ is a uniform integral ideal of bounded norm, and $y \in (\mathfrak{b}\mathfrak{p})^{-1}$ is small
3. If $\mathfrak{b}$ is not prime, **abort**;
4. Let $s_{\mathfrak{b}} \in \mathfrak{b}$ small (solve P-Ideal-SVP for $\mathfrak{b}$, *uniform prime of bounded norm*);
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$$\text{NTRU for } D^{\text{NTRU}} \overset{\text{[PS21]}}{\geq} \mathfrak{p}\text{-Ideal-SVP} \overset{\text{New}}{\geq} \text{Worst-case Ideal-SVP}$$

- First distribution over NTRU instances with a polynomial modulus whose hardness is supported by a worst-case lattice problem
- Caveat: sampling $D^{\mathbf{NTRU}}$ needs factoring oracle