

Generalized Special-Sound Interactive Proofs and their Knowledge Soundness

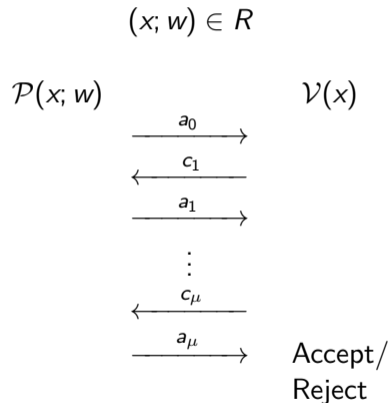
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Preliminaries - Interactive Proofs (IPs)

Goal of an Interactive Proof (of Knowledge):

- Prove that a statement x admits a witness, or
- Prove knowledge of a witness w for a public statement x .



Desirable Security Properties:

- Completeness: *Honest provers always succeed in convincing a verifier.*
- **(Knowledge) Soundness: *Dishonest provers (almost) never succeed.***
- Zero-Knowledge: *No information about the witness is revealed.*

Knowledge soundness \iff existence of a *knowledge extractor*.

Knowledge extractor

- Input: Statement x and oracle access to a prover \mathcal{P}^* attacking the protocol.
- Goal: Compute a witness w for statement x .

Another Notion for IPs - Special-Soundness

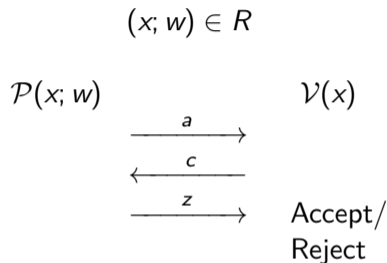
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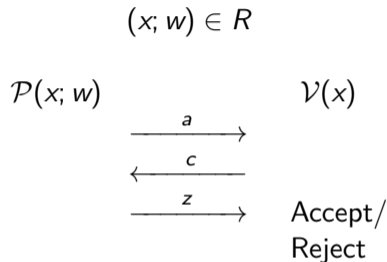
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2-out-of- N special-soundness implies knowledge soundness with knowledge error $1/N$.

- $1/N$ matches the trivial cheating probability.



Natural Generalizations of Special-Soundness

- ① k -out-of- N special-soundness \implies knowledge error $(k - 1)/N$.
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 - Requires k accepting transcripts with common first message a .
- ② (k_1, \dots, k_μ) -out-of- (N_1, \dots, N_μ) special-sound multi-round interactive proofs:
 - Require a tree of transcripts to extract.

Non-Special-Sound Interactive Proofs - Amortization (1/2)

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Proving Knowledge of n Pre-Images \mathbb{Z}_q -Module Homomorphism Ψ

$$\Psi(x_1) = P_1, \dots, \Psi(x_n) = P_n$$

$$\mathcal{P}(x_1, P_1, \dots, x_n, P_n)$$

$$\mathcal{V}(P_1, \dots, P_n)$$

$$\xleftarrow{c_1, \dots, c_n}$$

$$c_1, \dots, c_n \leftarrow_R \mathbb{Z}_q$$

$$z = \sum_i c_i x_i$$

$$\xrightarrow{z}$$

$$\Psi(z) \stackrel{?}{=} \sum_i c_i P_i$$

Non-Special-Sound Interactive Proofs - Amortization (2/2)

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 - Transcripts $(c_1, z_1), \dots, (c_n, z_n)$ s.t. c_1, \dots, c_n is a basis of \mathbb{Z}_q^n .

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- This IP is $(q^{n-1} + 1)$ -special-sound;
 - q is typically exponentially large \implies generic extractor is inefficient.

Proving Knowledge of Opening x_1, \dots, x_n of Merkle Tree Commitment P

$\mathcal{P}(x_1, \dots, x_n, P)$

$\mathcal{V}(P)$

$\xleftarrow{i_1, \dots, i_k}$

$i_1, \dots, i_k \leftarrow_R \{1, \dots, n\}$

$\xrightarrow{x_{i_1}, \dots, x_{i_k}}$
+Validation Paths

Check local openings.

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- This IP is $((n-1)^k + 1)$ -special-sound;
 - \implies generic knowledge extractor is inefficient.

- A more general notion of special-soundness,
 - capturing the above examples;
- A novel knowledge extractor;
- A generalization to multi-round interactive proofs;
- Parallel repetition theorem;
- An application to the FRI-protocol (IOP).

A Generalized Special-Soundness Notion

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- $A \subseteq B \subseteq \mathcal{C}$ and $A \in \Gamma$ implies $B \in \Gamma$.

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A 3-round interactive proof with challenge set \mathcal{C} is Γ -**out-of- \mathcal{C} special-sound**, if there exists an efficient algorithm to extract a witness from accepting transcripts $(a, c_1, z_1), \dots, (a, c_k, z_k)$ with $\{c_1, \dots, c_k\} \in \Gamma$.

Examples:

- k -special-sound IPs:
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- Merkle tree IP:
 - Challenge set $\{A \subseteq \{1, \dots, n\} : |A| \leq k\}$;
 - $\Gamma = \{S \subseteq \mathcal{C} : \cup_{A \in S} A = \{1, \dots, n\}\}$.

Extractor for Γ -Special-Sound IPs (1/2)

Key Observation:

- At any stage the extractor can partition \mathcal{C} into a set of “useful” and “useless” challenges.

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- Merkle tree IP:
 - $\mathcal{C} = \{S \subseteq \{1, \dots, n\} : |S| \leq k\}$;
 - $U_\Gamma(A) = \{B \in \mathcal{C} : B \not\subseteq \cup_{S \in A} S\}$.

We have to be careful when formally defining the useful challenge function U_Γ .

Formal Definition

$$U_\Gamma: 2^{\mathcal{C}} \rightarrow 2^{\mathcal{C}}, \quad S \mapsto \{c \in \mathcal{C} \setminus S : \exists A \in \Gamma \text{ s.t. } S \subset A \wedge A \setminus \{c\} \notin \Gamma\}$$

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Main idea.

- To find the ℓ -th transcript:
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More precisely, we adapt the extractor for k -special-sound IPs.

- Fails for the extractor from [ACK21];
- Works for the extractor introduced to handle parallel repetition [AF22].

Crucial parameter in the analysis:

$$t_{\Gamma} := \max \left\{ k \in \mathbb{N}_0 : \begin{array}{l} \exists c_1, \dots, c_k \in \mathcal{C} \text{ s.t.} \\ c_i \in U_{\Gamma}(\{c_1, \dots, c_{i-1}\}) \quad \forall i \end{array} \right\}$$

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Expected running time grows linearly in t_{Γ} :

- \implies knowledge soundness if t_{Γ} is polynomial.

Examples

k_Γ : Threshold special-soundness parameters.

t_Γ : Refined special-soundness parameters.

	Challenge Set	k_Γ	t_Γ
k -special-sound Π	\mathcal{C}	k	k
Amortization	\mathbb{Z}_q^n	$q^{n-1} + 1$	n
Merkle Tree Opening	$\{1, \dots, n\}^k$	$(n-1)^k + 1$	$n - k + 1$
t -fold Parallel of Π^1	\mathcal{C}^t	$(k-1)^t + 1$	$t \cdot (k-1) + 1$

Another example: Local Special-Soundness

¹Correction of the paper. Parallel repetition *is* appropriately captured by this generalization.

This presentation:

- Introduced a more general notion of special-soundness,
 - together with the tools to analyze this property;
- Claimed that it implies knowledge soundness.

In the paper:

- The knowledge extractor;
- A generalization to multi-round interactive proofs;
- Parallel repetition theorem;
- An application to the FRI-protocol (IOP).
 - With a discussion on the limitations of this approach.

Next:

- Analyzing the Fiat-Shamir transformation of $(\Gamma_1, \dots, \Gamma_\mu)$ -special-sound IPs.
 - Follow-up work to appear soon.
- Improve our FRI-extractor.
 - Open question.

Thanks!

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