An Algorithmic Approach to $(2,2)$-isogenies in the Theta Model and Applications to Isogeny-based Cryptography

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Motivation

- SIDH attacks relied on the computation of chains of 2-isogenies between elliptic products.
  - $\hookrightarrow$ In dimension two: Richelot formulae and specific algorithms for gluing and splitting.
- SIDH attacks have introduced a new representation for isogenies between elliptic curves.
  - $\hookrightarrow$ KEMs: FESTA and QFESTA.
  - $\hookrightarrow$ SQIsign variants.

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- SIDH attacks have introduced a new representation for isogenies between elliptic curves.
  - ↪ KEMs: FESTA and QFESTA.
  - ↪ SQIsign variants.

Problem

We needed a faster way to compute $(2,2)$-isogenies between elliptic products.

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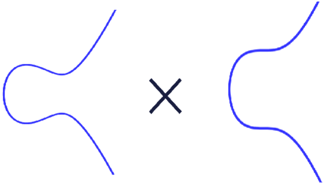

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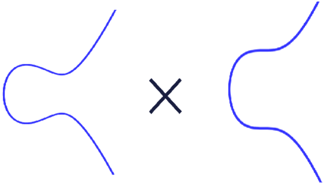

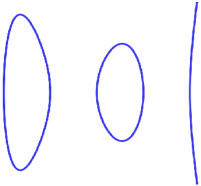

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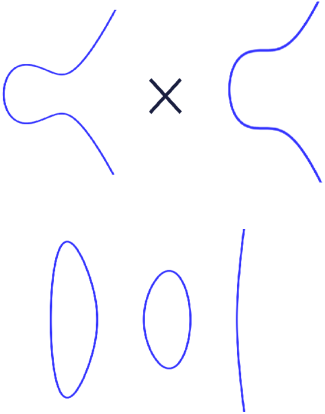
  - Products of elliptic curves,
  - Jacobians of genus-2 curves.
- Isogenies between PPASes have kernels of rank two.
- An (N, N)-isogeny is an isogeny between PPASes whose kernel $\simeq \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.



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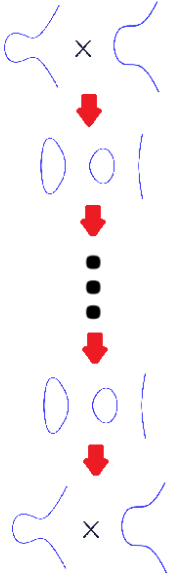
$$\Phi = \Phi_n \circ \ldots \circ \Phi_1$$



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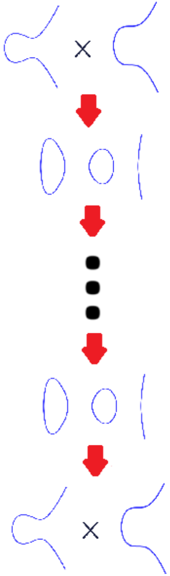
- Gluing isogeny $\Phi_1 : E_1 \times E_2 \to \mathsf{Jac}(\mathcal{C})$ (Howe, Leprévost, and Poonen, 2000).
- Splitting Isogeny $\Phi_n : \mathsf{Jac}(\mathcal{C}) \to E_1' \times E_2'$ (Smith, 2005).
- Richelot Isogenies $\Phi_i : \mathsf{Jac}(\mathcal{C}_i) \to \mathsf{Jac}(\mathcal{C}_{i+1})$, for $i = 2, \ldots, n-1$ (Smith, 2005).



Efficient formulae for $(2, 2)$-isogenies

- Represent PPASes via the *theta model*.
- Very efficient formulae to perform arithmetic.
- We adapt these formulae to our use case.
- Compared to the state of the art:
  - Codomain computation is **ten** times faster.
  - Isogeny evaluation is **twenty** times faster.
- We can now compute "cryptographic-size" isogenies in matter of ms.

Theta structures

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- $e(S_1', T_1') = e(S_2', T_2') = \mu$,
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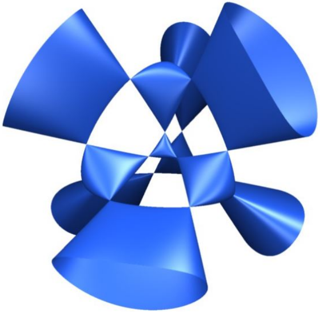
$\langle S_1', S_2' \rangle \oplus \langle T_1', T_2' \rangle \rightsquigarrow \theta_{00}, \theta_{10}, \theta_{01}, \theta_{11}$

$$P \in \mathcal{A} \to (\theta_{00}(P) : \theta_{10}(P) : \theta_{01}(P) : \theta_{11}(P)) \in \mathbb{P}^3$$



Taken from nLab.

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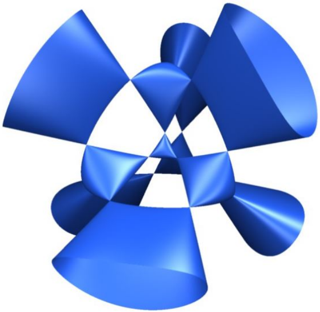
The projective point $(\theta_{00}(0) : \theta_{10}(0) : \theta_{01}(0) : \theta_{11}(0))$ is enough to describe $\mathcal{A}$.



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Some operators

The Hadamard transform

$$\mathcal{H}(x, y, z, w) := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

We define $(\tilde{\theta}_{00}(P) : \tilde{\theta}_{10}(P) : \tilde{\theta}_{01}(P) : \tilde{\theta}_{11}(P)) = \mathcal{H}\left(\theta_{00}(P), \theta_{10}(P), \theta_{01}(P), \theta_{11}(P)\right)$ to be the *dual coordinates* of P.
Also $\mathcal{H} \circ \mathcal{H}(x, y, z, w) = (x, y, z, w)$.

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The $\star$ operator:

$$(x,y,z,w) \star (x', y', z', w') = (xx', yy', zz', ww').$$

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Let $\Phi : \mathcal{A} \to \mathcal{B}$, $\ker \Phi = \langle T_1, T_2 \rangle$, where $T_i = [2]T_i'$.
For all $P, Q \in \mathcal{A}$

$$\left(\theta_i^{\mathcal{A}}(P+Q)\right)_i \star \left(\theta_i^{\mathcal{A}}(P-Q)\right)_i = \mathcal{H}\left(\left(\tilde{\theta}_i^{\mathcal{B}}(\Phi(P))\right)_i \star \left(\tilde{\theta}_i^{\mathcal{B}}(\Phi(Q))\right)_i\right).$$

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We can obtain addition formulae:

- Differential addition: $8\mathbf{S} + 17\mathbf{M}$,
- Doubling: $8\mathbf{S} + 6\mathbf{M}$.

The same formulae as in (Gaudry, 2005).

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One can prove:

$$\mathcal{H} \circ \mathcal{S}(\theta_{00}^{\mathcal{A}}(T_1''), \theta_{10}^{\mathcal{A}}(T_1''), \theta_{01}^{\mathcal{A}}(T_1''), \theta_{11}^{\mathcal{A}}(T_1'')) = (x\alpha, x\beta, y\gamma, y\delta),$$
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for some unknown x, y, z, w.

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for some unknown x, y, z, w.
Hence, we can recover the dual theta-null point $(\alpha : \beta : \gamma : \delta)$ for $\mathcal{B}$, and in turn the theta-null point $\mathcal{H}(\alpha : \beta : \gamma : \delta)$ on $\mathcal{B}$.

Operation counting

Isogeny Type	Doubling	Codomain		Evaluation
		Precomputations	Codomain	
Normalised	$8\mathbf{S} + 6\mathbf{M}$	$4\mathbf{S} + 24\mathbf{M} + 1\mathbf{I}$	$8\mathbf{S} + 10\mathbf{M} + 1\mathbf{I}$	$4\mathbf{S} + 3\mathbf{M}$
Projective	$8\mathbf{S} + 8\mathbf{M}$	$5\mathbf{S} + 14\mathbf{M}$	$8\mathbf{S} + 7\mathbf{M}$	$4\mathbf{S} + 4\mathbf{M}$
Gluing	$12\mathbf{S} + 12\mathbf{M}$	—	$8\mathbf{S} + 13\mathbf{M} + 1\mathbf{I}$	$8\mathbf{S} + 10\mathbf{M} + 1\mathbf{I}$

Details I skated over

- The formulae I showed assume we have T_1'' and T_2'' such that $\ker(\Phi) = [4]\langle T_1'', T_2'' \rangle$.
- The correction formula requires $100\mathbf{M} + 8\mathbf{S} + 4\mathbf{I}$
- At the end of the chain, we are left with an elliptic product in theta coordinates.
- Switching to the Montgomery model for the two curves is not expensive.

Performance

Table 1: Running times of computing the codomain and evaluating a $(2^n, 2^n)$-isogeny between elliptic products over the base field $\mathbb{F}_{p^2}$. Times were recorded on a Intel Core i7-9750H CPU with a clock-speed of 2.6 GHz with turbo-boost disabled.

		Codomain			Evaluation		
$\log p$	n	Theta Rust	Theta SageMath	Richelot SageMath	Theta Rust	Theta SageMath	Richelot SageMath
254	126	**2.13 ms**	108 ms	1028 ms	**161 μs**	5.43 ms	114 ms
381	208	**9.05 ms**	201 ms	1998 ms	**411 μs**	8.68 ms	208 ms
1293	632	**463 ms**	1225 ms	12840 ms	**17.8 ms**	40.8 ms	1203 ms

Conclusion

- We have shown formulae to compute $(2^n, 2^n)$-isogenies between elliptic products.
- Significant improvements in isogeny-based cryptography.
- Generalisation to four-dimensional elliptic products (Dartois, 2024).

Thanks for your attention!

Questions?

An example – Elliptic products

Elliptic curves

In the case of an elliptic curve E:

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Product theta structure on $E_1 \times E_2$

$$(P_1, P_2) \in E_1 \times E_2 \mapsto (\theta_0^{E_1}(P_1)\theta_0^{E_2}(P_2) : \theta_1^{E_1}(P_1)\theta_0^{E_2}(P_2) : \theta_0^{E_1}(P_1)\theta_1^{E_2}(P_2) : \theta_1^{E_1}(P_1)\theta_1^{E_2}(P_2))$$

The isogeny formula – Evaluation

We can also evaluate the isogeny Φ at any point P:

$$(\tilde{\theta}^{\mathcal{B}}_{00}(\Phi(P)), \tilde{\theta}^{\mathcal{B}}_{10}(\Phi(P)), \tilde{\theta}^{\mathcal{B}}_{01}(\Phi(P)), \tilde{\theta}^{\mathcal{B}}_{11}(\Phi(P))) = (\alpha^{-1}, \beta^{-1}, \gamma^{-1}, \delta^{-1}) \star \mathcal{H} \circ \mathcal{S}\left((\theta^{\mathcal{A}}_i(P))_i\right),$$

from which we can compute

$$(\theta^{\mathcal{B}}_{00}(\Phi(P)), \theta^{\mathcal{B}}_{10}(\Phi(P)), \theta^{\mathcal{B}}_{01}(\Phi(P)), \theta^{\mathcal{B}}_{11}(\Phi(P))) = \mathcal{H}(\tilde{\theta}^{\mathcal{B}}_{00}(\Phi(P)), \tilde{\theta}^{\mathcal{B}}_{10}(\Phi(P)), \tilde{\theta}^{\mathcal{B}}_{01}(\Phi(P)), \tilde{\theta}^{\mathcal{B}}_{11}(\Phi(P))).$$