On the Semidirect Discrete Logarithm Problem in Finite Groups

Analysis of a candidate problem in post-quantum cryptography

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Outline

Introduction to SDLP

Definitions

Semidirect Product

Let G be a finite group and $Aut(G)$ its group of automorphisms. We define $G \rtimes Aut(G)$ to be the group of pairs in $G \times Aut(G)$ equipped with the following multiplication:

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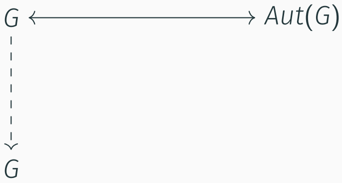
Notice



$$\begin{aligned}
(g, \phi)^2 &= (g\phi(g), \phi^2) \\
(g, \phi)^3 &= (g, \phi)(g\phi(g), \phi^2) \\
&= (g\phi(g)\phi^2(g), \phi^3) \\
(g, \phi)^4 &= (g, \phi)(g\phi(g)\phi^2(g), \phi^3) \\
&= (g\phi(g)\phi^2(g)\phi^3(g), \phi^4)
\end{aligned}$$

Definitions

$\rho_{g,\phi}$

Fix $(g, \phi) \in G \rtimes Aut(G)$. Define $\rho_{g,\phi} : G \to G$ by

$$\rho_{g,\phi}(h) = g\phi(h)$$

We have seen that

$$\rho^{x}_{g,\phi}(1_G) = g\phi(g)...\phi^{x-1}(g)$$

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SDLP

Fix $G \rtimes Aut(G)$ and a pair (g, ϕ). Suppose we are given $\rho^{x}_{g,\phi}(1_G)$ for some $x \in \mathbb{Z}$. The **S**emidirect **D**iscrete **L**ogarithm **P**roblem is to recover x.

Background

- Natural; outside the mainstream; feasibly post-quantum
- Turns out semidirect product cryptography can be described via commutative group actions*
- Commutative group actions give us Diffie-Hellman-style key exchanges (NIKEs)†, and digital signatures‡
- Recent fast algorithms for SDLP in certain classes of group§

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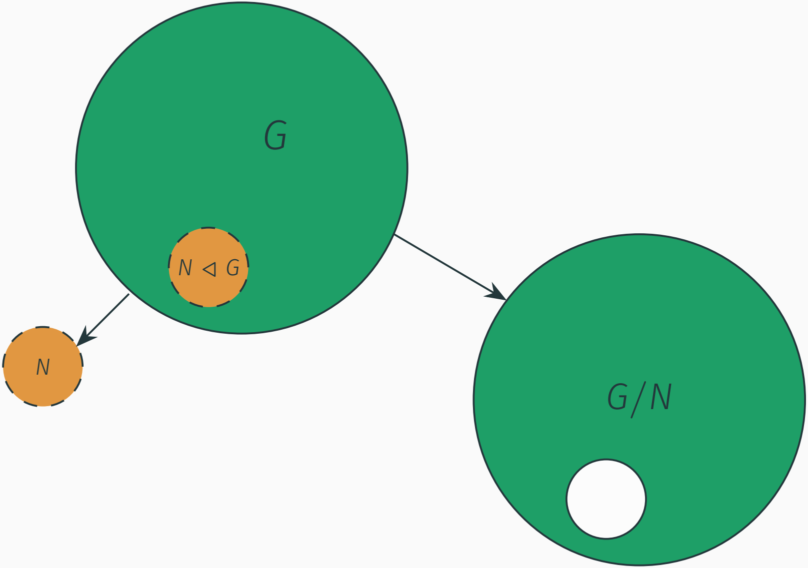
*B. et al. 2023a.
†Habeeb et al. 2013.
‡B. et al. 2023b.
§Mendelsohn et al. 2023; Imran and Ivanyos 2024.

Reduction to Simple Groups

Intuition



The Decomposition Tool

Imran and Ivanyos 2024, Theorem 3

Consider SDLP with respect to a pair $(g, \phi) \in G \rtimes Aut(G)$. Given a ϕ-invariant normal subgroup N of G, the solutions of SDLP are a linear combination of solutions of an instance of SDLP in G/N and an instance of SDLP in N.

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- Our contribution: reduce an arbitrary instance of SDLP in a finite group to instances of SDLP in simple groups, then solve those with the Classification. Requires a couple of (justified) computational group theory oracles.
- To complete the reduction need to compute invariant subgroups and check the recursion terminates.

Computing the Invariant Subgroup

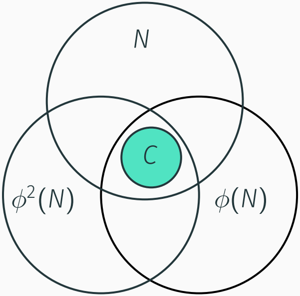

Figure 1: The ϕ-invariant subgroup cannot be smaller than the characteristic subgroup.

- Suppose we can compute a maximal normal subgroup of G, say N.
- Imran and Ivanyos 2024 show that the intersection

$$N \cap \phi(N) \cap \ldots \cap \phi^i(N) \cap \ldots$$

  stabilises with a ϕ-invariant subgroup[a]
- The algorithm doesn't terminate in the trivial subgroup if N contains a characteristic subgroup C (see left)
- We show there is a characteristic subgroup if and only if *every* maximal normal subgroup contains a characteristic subgroup.

[a] No proof that this is not the trivial subgroup.

Characteristically Simple Groups

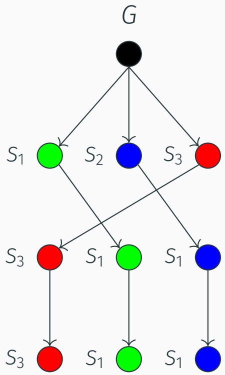

Figure 2: An automorphism of S^3.

- Well-known that groups with no characteristic subgroups (characteristically simple groups) are exactly of the form S^k for some simple group S.
- We show the algorithm for computing ϕ-invariant normal subgroups terminates in the identity exactly when $G = S^k$ and ϕ acts transitively on these components.
- In turn this gives us k^2 SDLP instances in S to solve.

Recursion to (Characteristically) Simple Groups

At each step of the recursion if the ϕ-invariant subgroup algorithm outputs trivial subgroup, call a simple/characteristically simple SDLP solver on that group.

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Correspondence theorem: the subgroups of G/N are of the form N'/N where $N \subset N' \triangleleft G$; and $(G/N)/(N'/N) \cong G/N'$

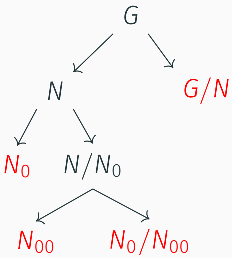

Figure 3: A recursion tree whose nodes are simple or characteristically simple.

Simple Groups Analysis

Simple Groups

Theorem (Classification of Finite Simple Groups)

Every finite simple group is isomorphic to a member of one of four infinite classes:

1. the **cyclic groups** of prime order,
2. the **alternating groups** of degree at least 5,
3. the **classical groups** of Lie type,
4. the **exceptional groups** of Lie type

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Corollary

The Semidirect Discrete Logarithm Problem (SDLP) in any finite group is **not a secure assumption** for quantum resistant primitives.

Let G be a cyclic group of prime order, then for any $g \in G$ and $\phi \in \mathsf{Aut}(G)$ we have $\phi(g) = g^a$ for some $a \in \mathbb{N}$, so:

$$s_{g,\phi(x)} = g\phi(g)\cdots\phi^x(g) = g \cdot g^a \cdots g^{a^x} = g^{\sum_{i=0}^{x} a^i} .$$

Cyclic Groups

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With a Quantum Computer we can recover

$$\sum_{i=0}^{x} a^i = \frac{a^{x+1}-1}{a-1}$$

then use again it again to solve SDLP.

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Linear Groups Analysis

Matrix Power Problem

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Matrix Power Problem

Given vectors $\mathbf{a}, \mathbf{b} \in V$ and a matrix $\mathbf{T} \in \mathrm{GL}(V)$ find $x \in \mathbb{N}$ such that:

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Result: We can do the same for projective linear groups $G \leq \mathbb{P}\mathrm{GL}$.

Reduction to Inner Automorphisms

Theorem (Kohl 2003)

If G is a non-abelian finite simple group, then for all $\phi \in \mathrm{Aut}(G)$ there exists an integer $x \leq \log_2 |G|$ such that $\phi^x \in \mathrm{Inn}(G)$.

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Memo: by Imran and Ivanyos 2024, we can solve SDLP(G, ϕ) by solving most y instances of SDLP(G, ϕ^y).

Consequence

We can limit ourselves to solve SDLP for inner authormorphism, i.e. conjugations.

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2. the **alternating groups** of degree at least 5, <- Linear
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Like for DLOG with division over $\mathbb{Z}/p\mathbb{Z}$, this do not directly implies that SDLP is broken.

Constructive Recognition Problem

Black-Box Groups

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Since Lie groups and alternating groups are defined as (projective) linear groups the SDLP reduces to the following:

Constructive Recognition Problem, Babai and Beals 1999

Given a simple black-box group G, the problem require to find a computationally efficient isomorphism between G and an explicitly defined simple group.

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     3.2 solved for any BBG, up to DLOG in Borovik and Yalçınkaya 2020
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$$G_2(q), q \geqslant 3; F_4(q); E_6(q); {}^2E_6(q); {}^3D_4(q); E_7(q); E_8(q)$$

$${}^2B_2\left(2^{2n+1}\right), n \geqslant 1; {}^2G_2\left(3^{2n+1}\right), n \geqslant 1; {}^2F_4\left(2^{2n+1}\right), n \geqslant 1$$

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In Kantor and Magaard 2013 and 2015 reduce the problem to $\mathbb{P}\mathrm{SL}(2, q)$, using *number theory oracles*.

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*solved if q is odd

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3. the other are referred as the six *pariahs*, and have cardinality $\leq 2^{67}$

Breaking Sporadic Groups

1. **Result:** Baby-Step Giant-Step algorithm can be adapted to SDLP, cutting the bit security of $\mathbb{M}$ to 89.6;

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5. For G one of the six pariahs $b(G) = 67^2 \approx 2^{13}$;

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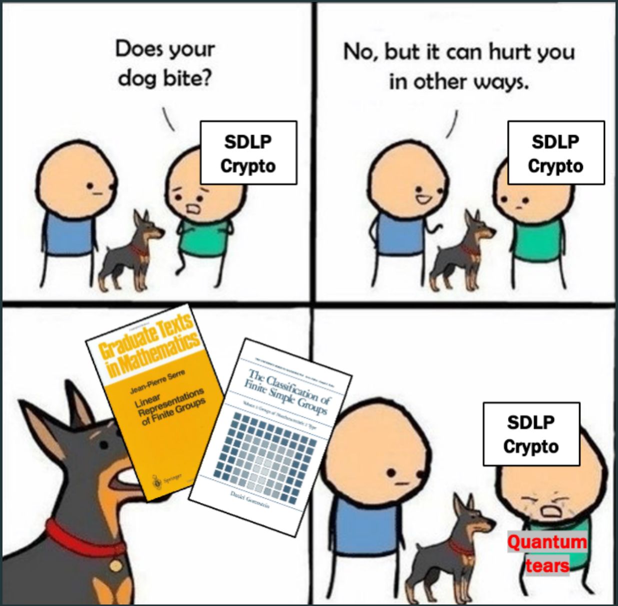
Meme


Does your dog bite?
SDLP Crypto
No, but it can hurt you in other ways.
SDLP Crypto
Graduate Texts in Mathematics
Jean-Pierre Serre
Linear Representations of Finite Groups
The Classification of Finite Simple Groups
SDLP Crypto
Quantum tears

Thank you for your attention!

`eprint.iacr.org/2024/905`

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Additional Material

Finding Maximal Normal Subgroups

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We can solve it via computing a composition series:

$$\{e\} = G_n \leq G_{n-1} \leq ... \leq G_1 \leq G \ ,$$

a well know problem in computational group theory

Finding Maximal Normal Subgroups

The task of finding a maximal normal subgroup is more subtle, since it depends on the particular implementation of the black-box group G.

We can solve it via computing a composition series:

$$\{e\} = G_n \leq G_{n-1} \leq ... \leq G_1 \leq G \ ,$$

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However, this branch of literature typically wishes to achieve much stronger results and are thwarted by DLOG computation - we do not impose this limitation since we assume QCs.

Finding Maximal Normal Subgroups : Possible Solutions:

1. if we know the particular structure of the group G, we can use it to construct for any subgroup S the smallest normal subgroup containing $\langle S^G \rangle$ in linear time, as explained in Babai et al. 1991.

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2. Use Imran and Ivanyos 2024, but requires that every non-Abelian composition factor of G possesses a faithful small permutation representation;
3. Otherwise, with Babai and Beals 1999 and a QC we can find $G_1 \leq G$, with G/G_1:

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   3.1 simple and nonabelian, so G_1 is Maximal Normal Subgroup;
   3.2 or abelian, so we can use point 1 to get the maximal normal subgroup $A_1 \lhd G/G_1$[¶] and A_1G_1 will be a maximal normal in G by the correspondence theorem.

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SDLP on Matrix Groups (Imran and Ivanyos 2024)

Consider $G \leq \mathrm{GL}_n(\mathbb{F})$ and $\phi \in \mathrm{Inn}(G)$ such that $\phi(\mathsf{G}) = \mathsf{S}\mathsf{G}\mathsf{S}^{-1}$, then:

$$s_{\mathsf{G},\phi}(x) = \mathsf{G} \cdot \mathsf{S}\mathsf{G}\mathsf{S}^{-1} \cdot \mathsf{S}^2\mathsf{G}\mathsf{S}^{-2} \cdots \mathsf{S}^{x-1}\mathsf{G}\mathsf{S}^{-x+1} \cdot \mathsf{S}^x\mathsf{G}\mathsf{S}^{-x} =$$

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So if we vectorize the matrices we get:

$$\mathrm{vec}(s_{\mathsf{G},\phi}(x)) = \mathrm{vec}\left((\mathsf{GS})^x \cdot \mathsf{G} \cdot \mathsf{S}^{-x}\right)$$

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