Direct FSS Constructions For Branching Programs and More from PRGs with Encoded-Output Homomorphism

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December 11, 2024

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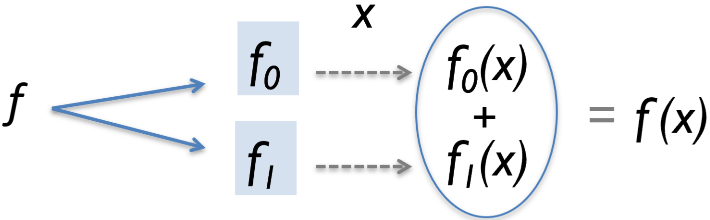

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Definition (FSS)

Given $F := \{f\colon D \to R\}$, FSS consists of a pair of PPT algorithms $(\mathsf{Gen}, \mathsf{Eval})$:

- $(k_0, k_1) \leftarrow \mathsf{Gen}(f, 1^\lambda)$.
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Hiding: k_b alone hides f.

Correctness: $y_0 + y_1 = f(x)$ for all $x \in D$.

Compactness: $|k_b|$ scales with the description size of f

In this work, we focus on the two-party case.

Applications of FSS (Incomplete List)

- Mozillia+Prio: Private data collection [CB17] moz://a
- Writing PIR: Riposte [CBM15]
- SQL query: Splinter [WYG+17]
- ORAM: Floram [Ds17]
- Mixed-mode secure computation [BGI19, BCG+21]
- Private heavy hitters [BBC+21]
- Private set intersection (PSI) [GRS22, GRS23]

Previous Constructions of (Two-Party) FSS

- FSS from one-way functions [GI14, BGI15, BGI16b, BCG+21]
  - for point functions, interval functions, decision trees
- FSS from learning parity with noise [BCG+19, CM21, DIJL23]
  - for low-degree polynomials
- FSS from DDH [BGI16a, BCG+17, BGI+18], DCR [FGJS17, OSY21, RS21], LWE [BKS19, ACK23], class groups [ADOS22]
  - for branching programs
  - via homomorphic secret sharing and universal branching programs
- FSS from multi-key FHE [DHRW16]
  - for all function classes

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Our Results

1. PRG with Encoded-Output Homomorphism (EOH-PRG)
   - New abstraction of (relaxed) homomorphic PRG
   - Can be instantiated from LWE or DCR

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1. PRG with Encoded-Output Homomorphism (EOH-PRG)
   - New abstraction of (relaxed) homomorphic PRG
   - Can be instantiated from LWE or DCR
2. Direct FSS constructions from EOH-PRG
   - for bit-fixing predicates
   - for branching programs (without universal circuits)
   - for deterministic finite automata (DFA)

Our Results - Efficiency

		Assumption	Key Size	Run time (No. of Mul./ Exp.)
LWE	HSS [BKS19]	Ring-LWE	$4(n \cdot w) \cdot w\ell \log q$	$8w^2 \cdot \ell n \log n$
	EOH-PRG(Ours)	Ring-LWE	$2(n + w) \cdot w\ell \log p$	$2(1 + \lceil \frac{w}{n} \rceil) \cdot \ell n \log n$
DCR	HSS [OSY21]	DCR	$7w \cdot w\ell \log N^2$	$14w^2 \cdot \ell$
	EOH-PRG(Ours)	DCR	$2(w + 1) \cdot w\ell \log N^2$	$(3w + 2) \cdot \ell$

- (ℓ, w) length and width of the branching program
- For LWE, n secret dimension, p plaintext modulus and q ciphertext modulus
- For DCR, N RSA modulus

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Recall FSS for Point Functions [GI14, BGI15, BGI16b]

For $\alpha \in \{0,1\}^n, \beta \in R$, $f_{\alpha,\beta}\colon \{0,1\}^n \to R$ is defined as

$$f_{\alpha,\beta}(x) = \begin{cases} \beta, & \text{if } x = \alpha. \\ 0, & \text{otherwise.} \end{cases}$$

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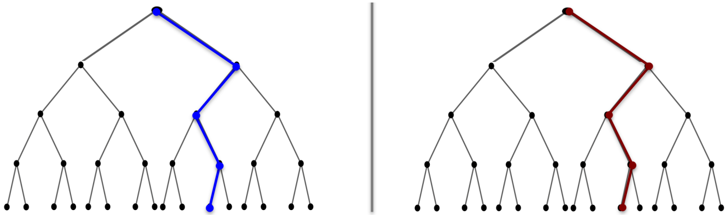

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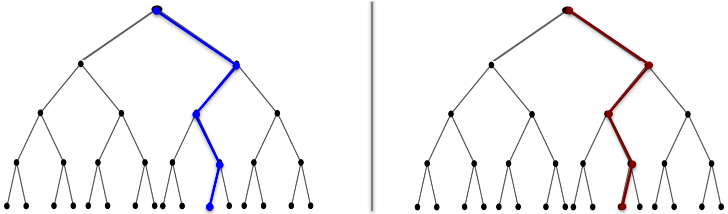

- The shares define two correlated GGM-like trees.
- $CW_i \leftarrow G(s_i) + \mathcal{E}(s_{i+1})$ w.r.t. α with $G(\cdot)$ PRG.
- For each node v, the two parties obtain shares of $(\mathbf{s}_i, 1) \in \{0,1\}^{\lambda+1}$ if v is specified by α and $(\mathbf{0}, 0)$ otherwise.

Towards FSS for Bit-Fixing Predicates

For $\alpha \in \{0,1,*\}^n, \beta \in \{0,1\}$, $f_{\alpha,\beta}\colon \{0,1\}^n \to \{0,1\}$ is defined as

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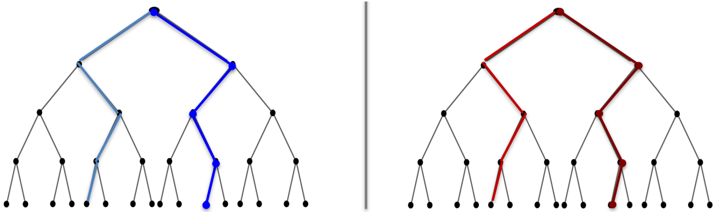

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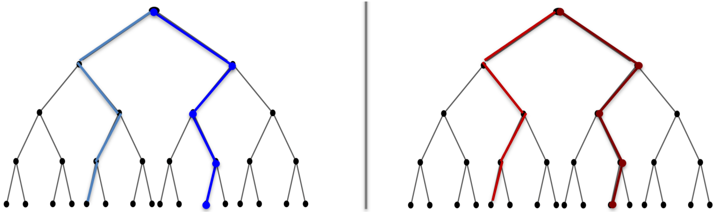
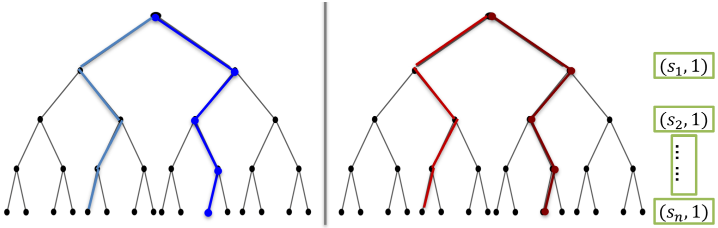


- $\lambda^{\omega(1)}$ matched evaluation paths.
- Key size scales with the number of matched paths.

How to compress the key size?



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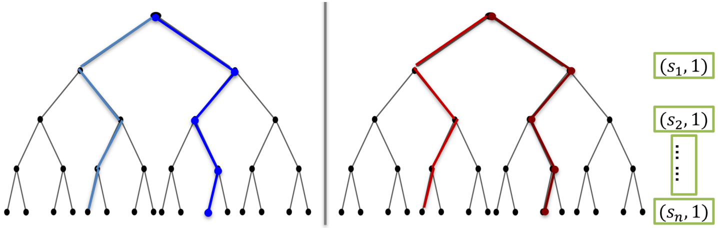

1. Sample a random value for each level
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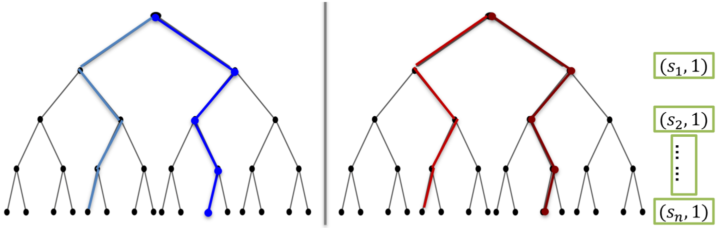

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How to move from current level to next level while maintaining the invariant for each matched node?

Homomorphic PRG (A Little Technical)

Suppose we have a homomorphic PRG satisfying $G(s+t) = G(s) + G(t)$.

1. Assume $CW_i \leftarrow G(s_i) \oplus \mathcal{E}(s_{i+1})$.
2. Assume P_b has $s_{i,b}$ such that $s_{i,0} \oplus s_{i,1} = s_i$.
3. Then P_b obtains $\sigma_b \leftarrow b \cdot CW_i \oplus G(s_{i,b})$.
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Problem

Homomorphic PRG does not exist!

PRG with Encoded-Output Homomorphism(EOH-PRG)

Definition (EOH-PRG)

Given additive secret shares (s_0, s_1) of a seed s, and additive secret shares (y_0, y_1) of a blinded encoding $G(s) + \mathcal{E}(m)$, it holds that

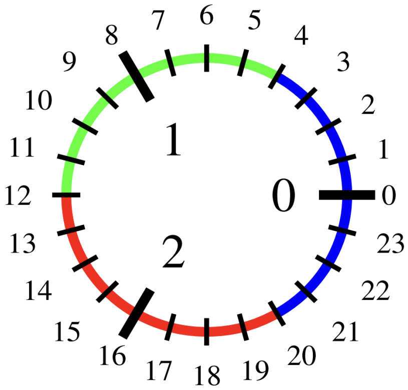
$$\mathsf{Conv}(y_0 - G(s_0)) - \mathsf{Conv}(y_1 - G(s_1)) = m$$

except with negligible probability over the random choice of the secret shares.

Then $(G, \mathsf{Conv}, \mathcal{E})$ is a EOH-PRG.

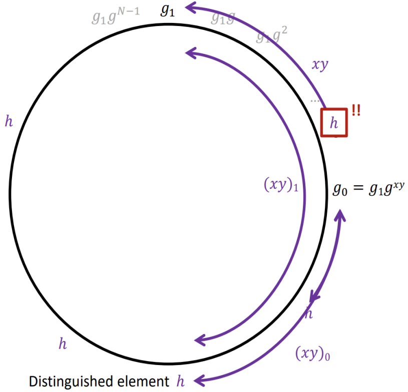
EOH-PRG Instantiatings from LWE or DCR

LWE
Distributed rounding [BKS19]



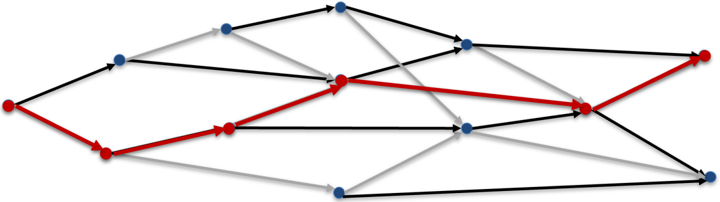
Lifting

DCR
DDLog [OSY21, RS21]



Lifting

FSS for Branching Programs



- Reduce $\lambda^{\omega(1)}$ evaluation paths to $O(w)$ nodes of each level of BP.

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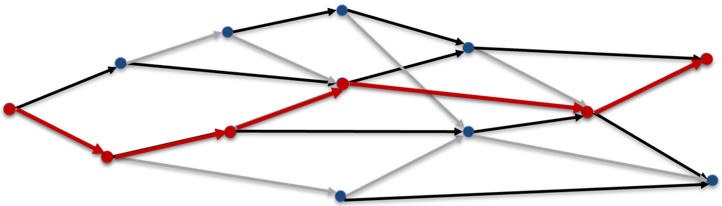

- Reduce $\lambda^{\omega(1)}$ evaluation paths to $O(w)$ nodes of each level of BP.
- Viewing a DFA as a BP jumping to the same level together with KDM security, we obtain FSS for DFAs.

Summary

- New EOH-PRG abstraction.
- FSS for bit-fixing predicates from EOH-PRG.
- FSS for branching programs from EOH-PRG without universal transformations.
- FSS for DFAs from KDM secure EOH-PRG.
    - It is not clear how to achieve it from HSS for BPs.

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  - It is not clear how to achieve it from HSS for BPs.

The Key Takeaway

An ideal cryptographic primitive, even if purely hypothetical, such as a homomorphic PRG, has the potential to significantly simplify the overall construction.


Thank you

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