CDS Composition of Multi-Round Protocols

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Sigma protocol [Cramer'96]

P		V
$(x, w) \in R$		x
$a \leftarrow \mathsf{A}(x; r)$	$\xrightarrow{\quad a \quad}$	
	$\xleftarrow{\quad e \quad}$	$e \leftarrow \$$
$z \leftarrow \mathsf{Z}(x, r, e, w)$	$\xrightarrow{\quad z \quad}$	$1/0 \leftarrow \mathsf{Verif}(x, a, e, z)$

Sigma protocol [Cramer'96]

- Completeness:
  - $R(x, w) = 1 \rightarrow$ Verifier accepts
- 2-Special Soundness:
  - $\mathrm{Ext}(a, e, z, e', z') \rightarrow w$
- (Special) Honest Verifier Zero Knowledge:
  - $\mathrm{Sim}(x, e) \rightarrow (a, z)$
- Generalization:
  - Multi-round, $(k_1, \dots, k_\mu)$-special sound PCIP

The CDS OR-composition [CDS94]

Relation: $(x_1 \vee x_2; w_1)$
Ingredients: Σ-protocols + secret sharing

$x_1; w_1$	x_2	
$a_1 \leftarrow A(x_1; r_1)$	$e_2 \leftarrow \$$ $(a_2, z_2) \leftarrow \mathrm{Sim}(x_2, e_2)$	$\{a_1, a_2\}$ →

$e_1 = e - e_2$ ← $e \leftarrow \$$

$z_1 \leftarrow Z(x_1, r_1, e_1, w_1)$ → $\{e_1, e_2, z_1, z_2\}$

The CDS OR-composition [CDS94]

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Ingredients: Σ-protocols + secret sharing

Linear proof size

$x_1; w_1$	x_2	
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Linear proof size

Security preserving

$x_1; w_1$	x_2
$a_1 \leftarrow A(x_1; r_1)$	$e_2 \leftarrow \$$ $(a_2, z_2) \leftarrow \mathrm{Sim}(x_2, e_2)$

$\{a_1, a_2\}$ →

$e \leftarrow \$$ ←

$$e_1 = e - e_2$$

$$z_1 \leftarrow Z(x_1, r_1, e_1, w_1)$$

$\{e_1, e_2, z_1, z_2\}$ →

The CDS OR-composition [CDS94]

Relation: $(x_1 \vee x_2; w_1)$

Ingredients: Σ-protocols + secret sharing

Linear proof size

Security preserving

Precomputable (if x_2 known)

$x_1; w_1$	x_2	
$a_1 \leftarrow A(x_1; r_1)$	$e_2 \leftarrow \$$ $(a_2, z_2) \leftarrow \mathrm{Sim}(x_2, e_2)$	$\{a_1, a_2\}$ →
$e_1 = e - e_2$		← $e \leftarrow \$$
$z_1 \leftarrow Z(x_1, r_1, e_1, w_1)$		$\{e_1, e_2, z_1, z_2\}$ →

The CDS OR-composition [CDS94]

Relation: $\{x_{i\in[n]}; w_{j\in A\in\Gamma}\}$, $\Gamma \in$ MSP (Monotone Span Program)

Ingredients: n Σ-protocol Π + SS

$i \in \mathrm{A}$	$j \notin \mathrm{A}$	
$a_i \leftarrow A(x_i; r_i)$	$e_j \leftarrow \$$ $(a_j, z_j) \leftarrow \mathrm{Sim}(x_j, e_j)$	$\{a_{i\in[n]}\}$ →
$\{e_{i\in[n]}\} \leftarrow \mathrm{SS.Complete}_{\Gamma^*}(e, \{e_j\})$		← $e \leftarrow \$$
$z_i \leftarrow Z(x_i, r_i, e_i, w_i)$		$\{e_{i\in[n]}, z_{i\in[n]}\}$ →

Generalizations of Sigma-protocols

- Generalizations of Sigma-protocols:
  - 2-special sound $\Longrightarrow$ k-special sound
  - 3-round $\Longrightarrow$ $2\mu + 1$-round

- Why generalizations?
  - Better efficiency
  - SNARKs
  - Compressed Sigma protocol [AC20, ACK21]
  - …

Generalization of CDS composition

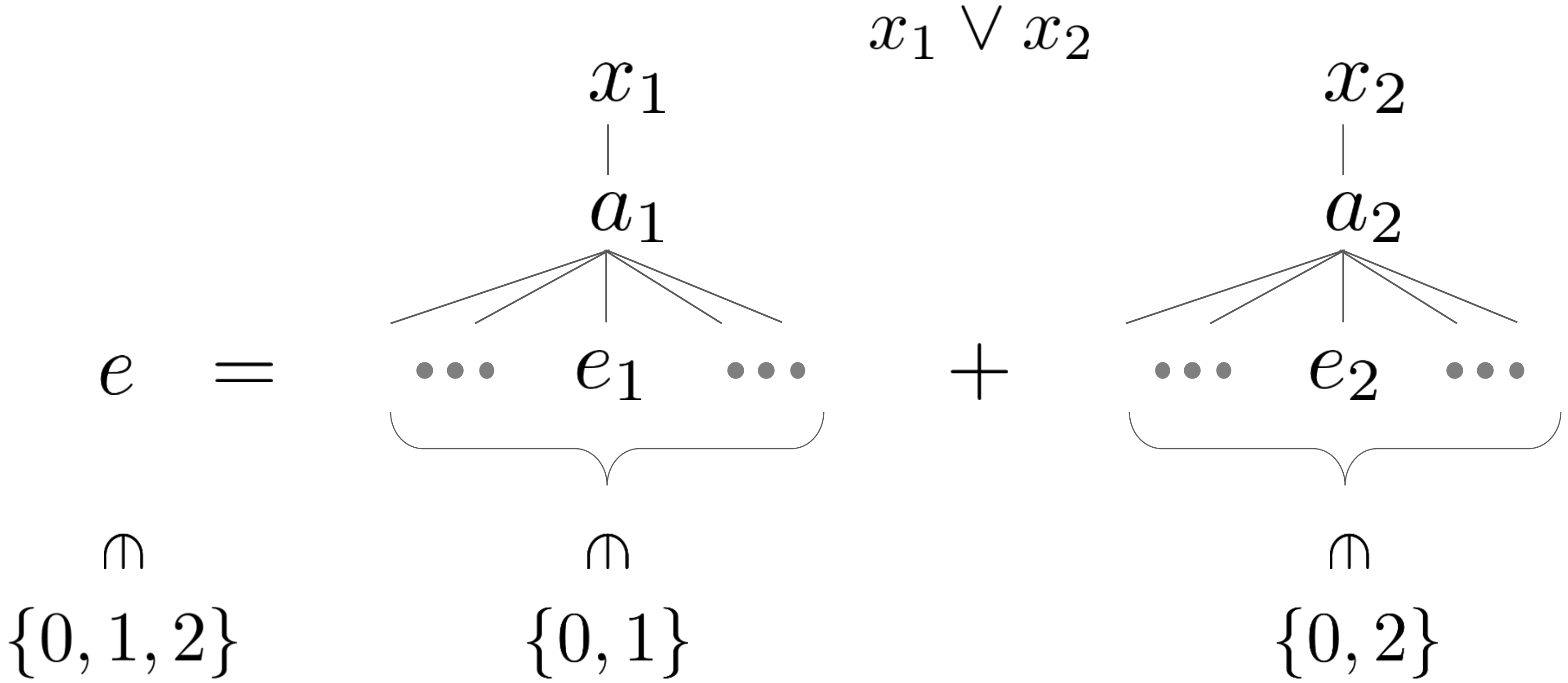
- Does CDS apply to $2\mu + 1$-round $(k_1, \ldots, k_\mu)$-special sound protocols?

Special Soundness	Number of Rounds	Expressibility	CDS works?
2-special sound	3	MSP	✓
$\boldsymbol{k}$-special sound	3	OR	×
2-special sound	$\boldsymbol{2\mu + 1}$	OR	×

- Neither of generalizations work below:
  - 2-special sound $\Longrightarrow$ k-special sound
  - 3-round $\Longrightarrow$ $2\mu + 1$-round

CDS fails- Case 1 (k-special soundness)

- e.g. 3-special soundness (Stern's Protocol)



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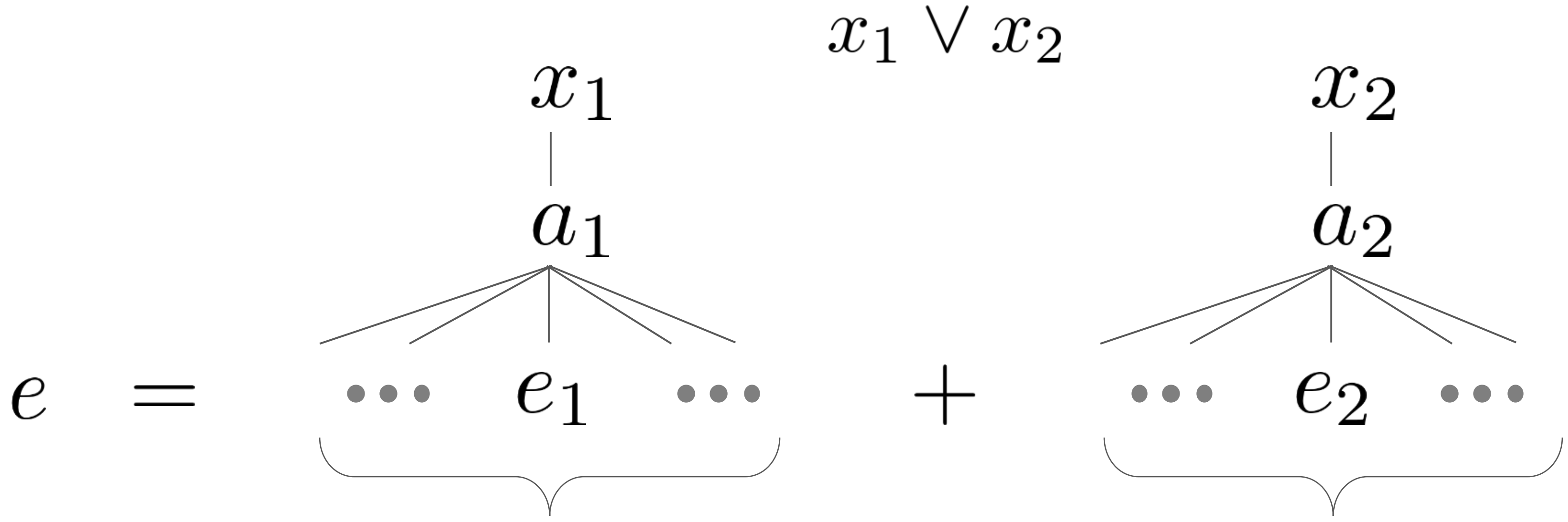
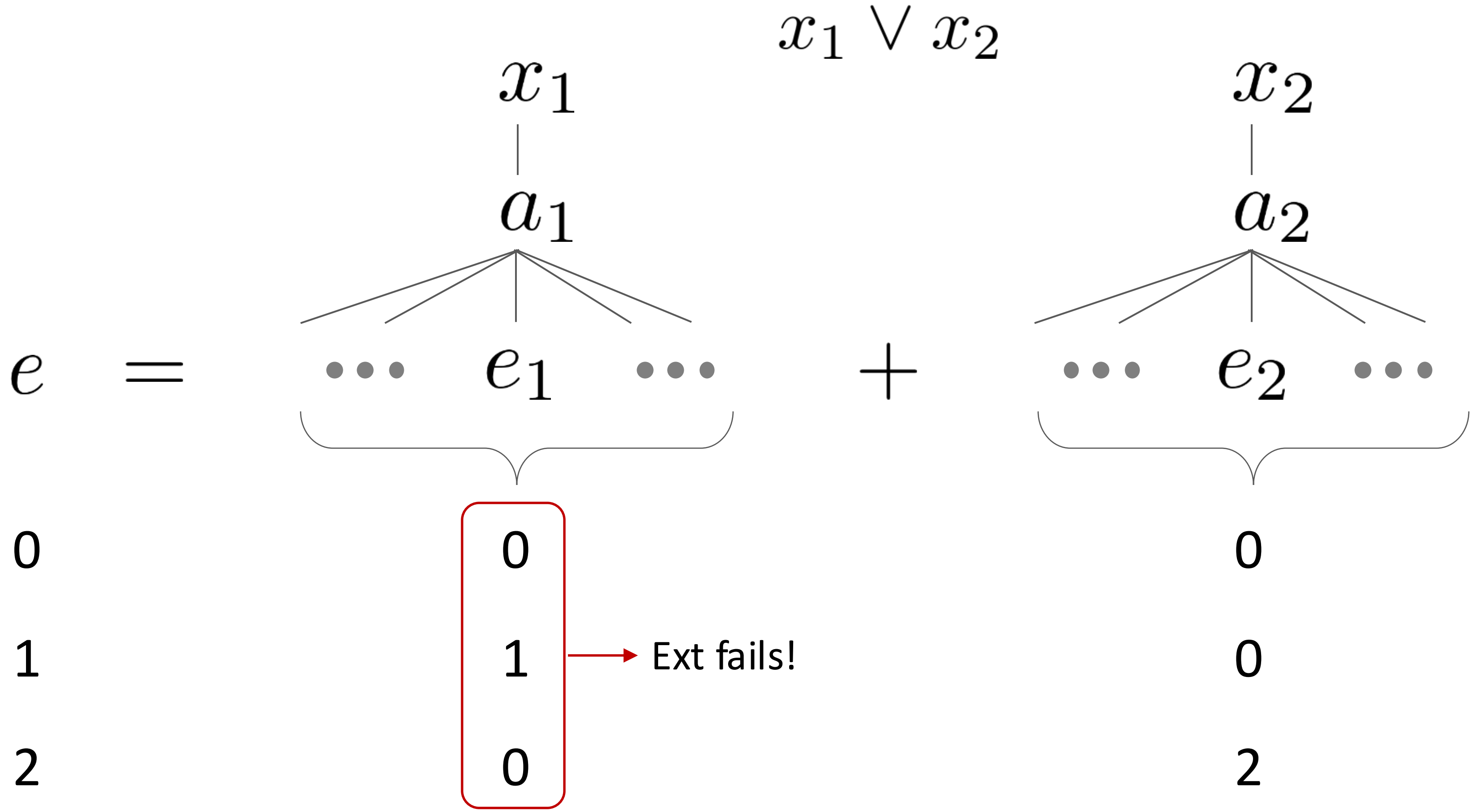


e	e_1	e_2
0	0	0
1	1	0
2	0	2

CDS fails- Case 1 (k-special soundness)

- e.g. 3-special soundness (Stern's Protocol)



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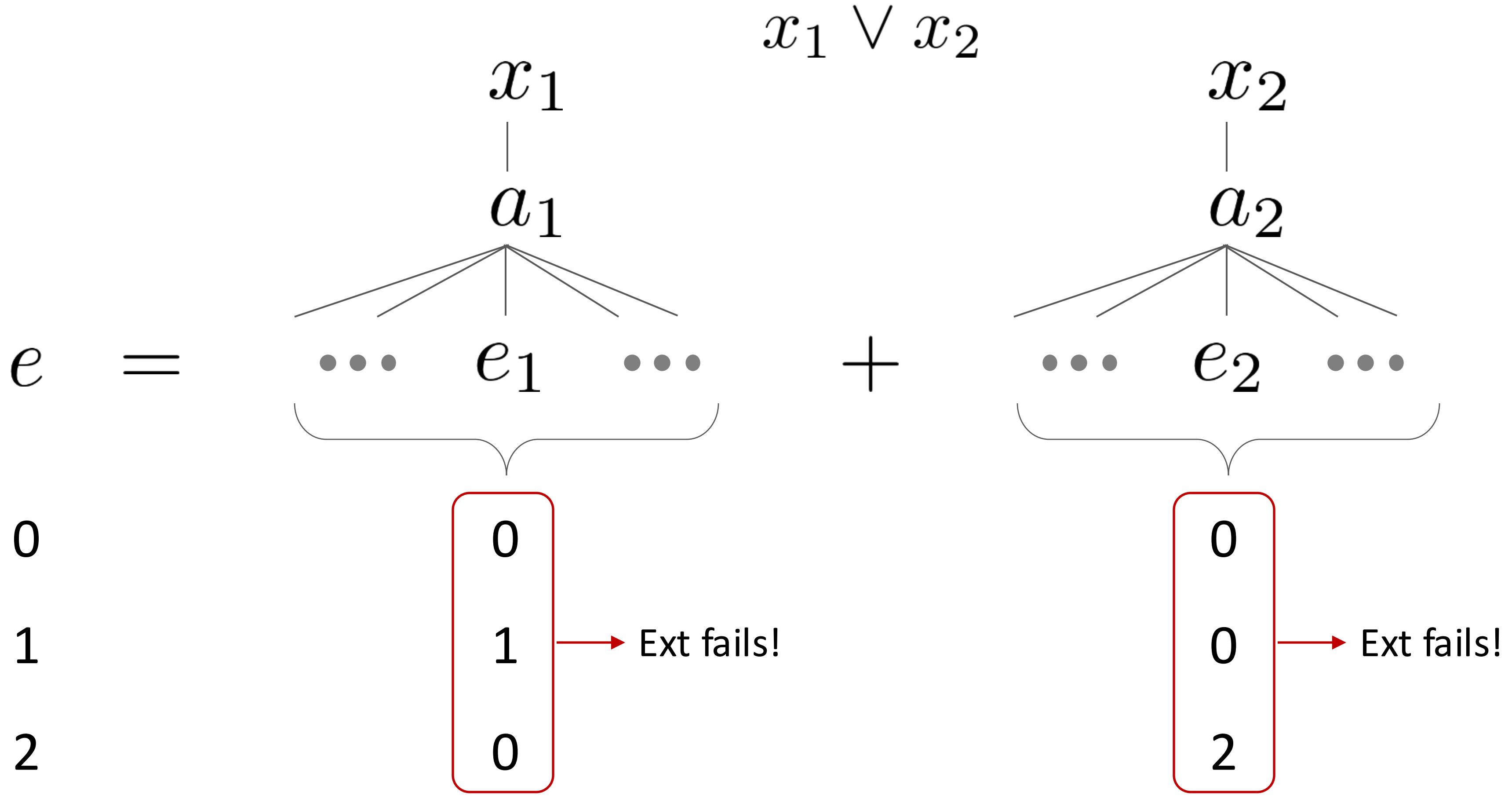
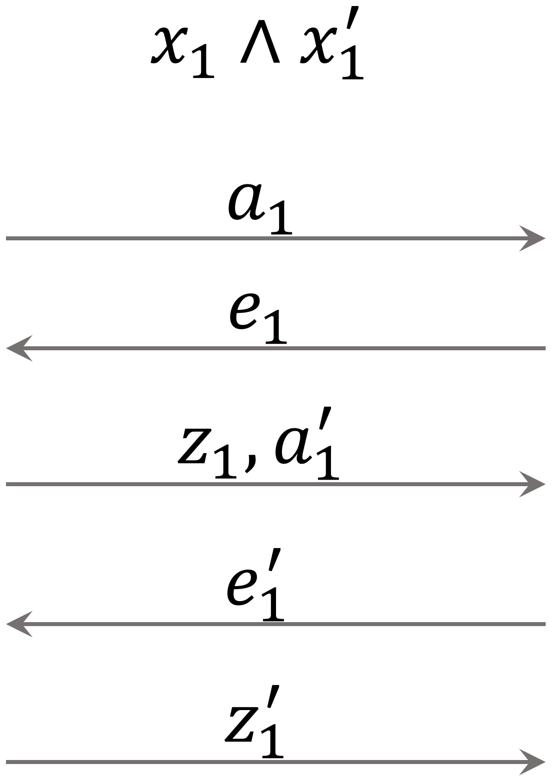


CDS fails- Case 2 ($2\mu + 1$-round)

- A 5-round protocol for relation $x_1 \wedge x_1'$:



CDS fails- Case 2 ($2\mu + 1$-round)

- A 5-round protocol for relation $x_1 \wedge x_1'$:

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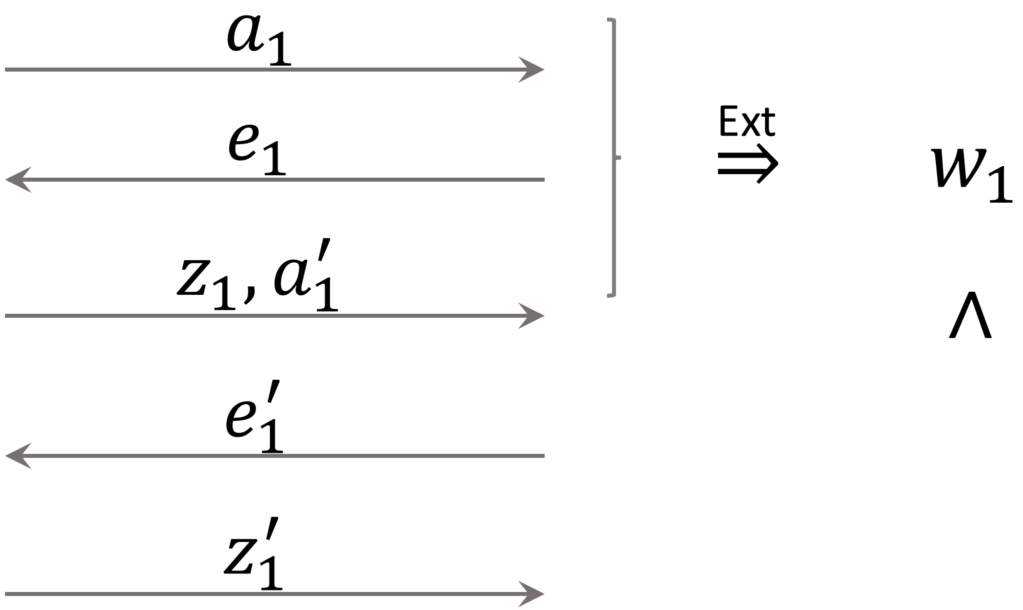
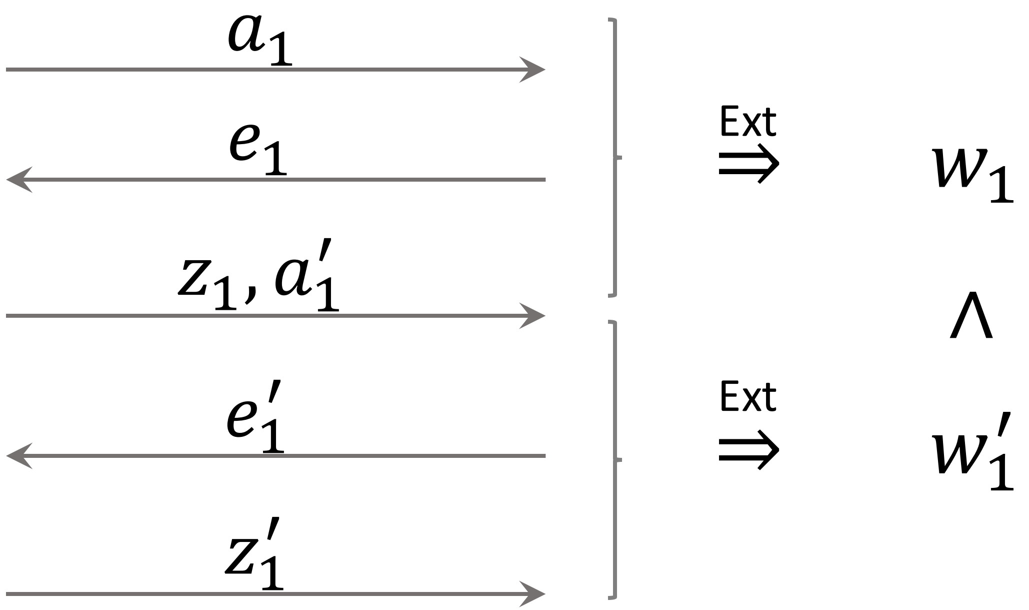


CDS fails- Case 2 ($2\mu + 1$-round)

- A 5-round protocol for relation $x_1 \wedge x_1'$:

$$x_1 \wedge x_1'$$



CDS fails- Case 2 ($2\mu + 1$-round)

- An OR-proof for $(x_1 \wedge x_1') \vee (x_2 \wedge x_2')$:

$x_1 \wedge x_1'$		$x_2 \wedge x_2'$		
a_1		a_2		
e_{1a}	$+$	e_2	$=$	e_a
z_1, a_1'		z_2, a_2'		
e_{1a}'	$+$	e_2'	$=$	e_a'
z_1' …		z_2'		

CDS fails- Case 2 ($2\mu + 1$-round)

- An OR-proof for $(x_1 \wedge x_1') \vee (x_2 \wedge x_2')$:

$x_1 \wedge x_1'$		$x_2 \wedge x_2'$		
a_1		a_2		
e_{1a} e_{1b}	$+$	e_2	$=$	e_a e_b
z_1, a_1'		z_2, a_2'		
e_{1a}'	$+$	e_2'	$=$	e_a'
z_1' …		z_2'		

CDS fails- Case 2 ($2\mu + 1$-round)

- An OR-proof for $(x_1 \wedge x_1') \vee (x_2 \wedge x_2')$:

$x_1 \wedge x_1'$ $\qquad\qquad$ $x_2 \wedge x_2'$

a_1 $\qquad\qquad$ a_2

$e_{1a} \quad e_{1b} \quad + \quad e_2 \quad = \quad e_a \quad e_b$

z_1, a_1' $\qquad\qquad$ z_2, a_2'

$e_{1a}' \quad e_{1b}' \quad + \quad e_2' \quad = \quad e_a' \quad e_b'$

z_1' $\quad \ldots \quad$ z_2'

CDS fails- Case 2 ($2\mu + 1$-round)

- An OR-proof for $(x_1 \wedge x_1') \vee (x_2 \wedge x_2')$:

$x_1 \wedge x_1'$ | $x_2 \wedge x_2'$

a_1 | a_2

w_1 ← Ext ← $e_{1a}\ \ e_{1b}$ $+$ e_2 $=$ $e_a\ \ e_b$

$\wedge$ | z_1, a_1' | z_2, a_2'

w_1' ← Ext ← $e_{1a}'\ \ e_{1b}'$ $+$ e_2' $=$ $e_a'\ \ e_b'$

z_1' … | z_2'

Extraction succeeds

CDS fails- Case 2 ($2\mu + 1$-round)

- Suppose prover has w_1, w_1', w_2, w_2':

$x_1 \wedge x_1'$ $\quad\quad$ $x_2 \wedge x_2'$

a_1 $\quad\quad$ a_2

w_1 ← Ext $\quad$ $e_{1a} \quad e_{1b}$ $\quad + \quad$ e_2 $\quad = \quad$ $e_a \quad e_b$

$\wedge$ $\quad$ z_1, a_1' $\quad\quad$ z_2, a_2'

e_1' $\quad + \quad$ e_{2a}' $\quad = \quad$ e_a'

z_1' $\quad$ … $\quad$ z_2'

CDS fails- Case 2 ($2\mu + 1$-round)

- Suppose prover has w_1, w_1', w_2, w_2':

$x_1 \wedge x_1'$ $\qquad$ $x_2 \wedge x_2'$

a_1 $\qquad$ a_2

w_1 ← Ext $\boxed{e_{1a} \quad e_{1b}}$ $+$ e_2 $=$ $e_a \quad e_b$

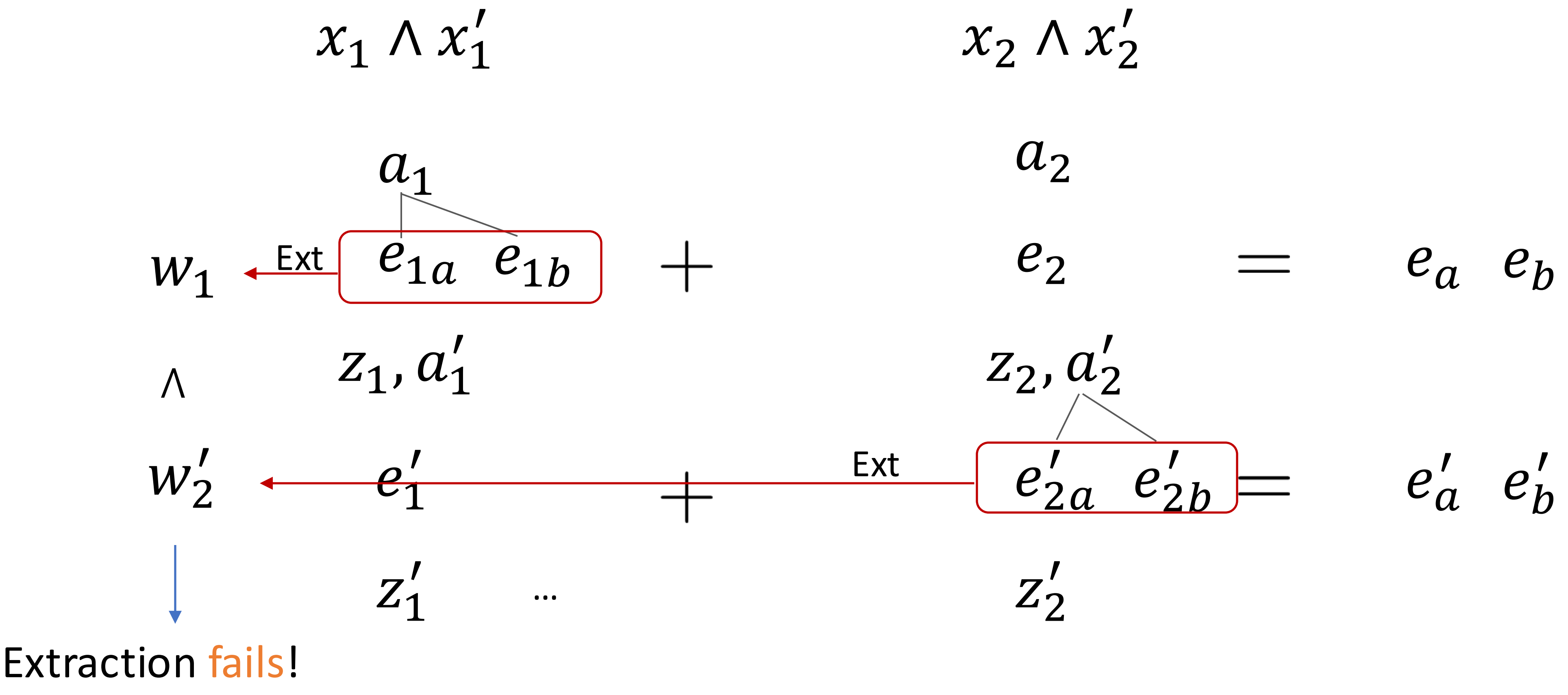
$\wedge$ $\qquad$ z_1, a_1' $\qquad$ z_2, a_2'

e_1' $+$ $e_{2a}' \quad e_{2b}'$ $=$ $e_a' \quad e_b'$

z_1' $\quad \dots \quad$ z_2'

CDS fails- Case 2 ($2\mu + 1$-round)

- Suppose prover has w_1, w_1', w_2, w_2':



Our Result

- Our result:

Composition method	Special Soundness	Number of Rounds	Expressibility
CDS	2-special sound	3	MSP
Ours	**$(k_1, \dots, k_\mu)$-special sound**	$2\mu + 1$	**mNC[1]**

- Observation:
  - CDS fails because the prover is given too much freedom on choosing challenges
- Our approach:
  - Let the prover commit to simulated challenges

Use of dual-mode commitments

- We use dual-mode commitments.
  - Plain model
  - Binding mode: perfect binding, comp. hiding
  - Hiding mode: perfect hiding, comp. binding, equivocable
  - Mode indistinguishable

- Prove: If simulated, Com works in binding mode
- And Coms are indistinguishable among all positions

Our Composition

Relation: $(x_1 \vee x_2; w_1)$

Ingredients: PCIPs + SS + dual-mode Com. (e.g. 5-round OR-proof)

$x_1; w_1$	x_2	
$a_1 \leftarrow A(x_1; r_1)$	$e_2, e_2' \leftarrow \$$	
$(c_1, d_1) \leftarrow \mathrm{Com}(0\|\|0)$	$(a_2, z_2, z_2') \leftarrow \mathrm{Sim}(x_2, e_2, e_2')$	$\{a_1, a_2, c_1, c_2\}$ →
	$(c_2, d_2) \leftarrow \mathrm{Com}(e_2\|\|e_2')$	

$e \leftarrow \$$ ←

$e_1 = e - e_2$

$z_1 \leftarrow Z(x_1, r_1, e_1, w_1)$

$\{e_1, e_2, z_1, z_2\}$ →

$e' \leftarrow \$$ ←

$e_1' = e' - e_2'$

$d_1' \leftarrow \mathrm{Equiv}(c_1, e_1\|\|e_1', 0\|\|0)$

$z_1' \leftarrow Z(x_1, r_1, e_1', e_1, w_1)$

$\{e_1', e_2', z_1', z_2', d_1', d_2\}$ →

Our Composition

Relation: $(x_1 \vee x_2; w_1)$

Ingredients: PCIPs + SS + dual-mode Com. (e.g. 5-round OR-proof)

$x_1; w_1$	x_2
	$e_2, e_2' \leftarrow \$$
$a_1 \leftarrow A(x_1; r_1)$	$(a_2, z_2, z_2') \leftarrow \mathrm{Sim}(x_2, e_2, e_2')$
$(c_1, d_1) \leftarrow \mathrm{Com}(0\|\|0)$	$(c_2, d_2) \leftarrow \mathrm{Com}(e_2\|\|e_2')$

Prove: at least one of them are **Binding**

$z_1 \leftarrow Z(x_1, r_1, e$

$e_1' = e' - e_2'$

$d_1' \leftarrow \mathrm{Equiv}(c_1, e_1\|\|e_1', 0\|\|0)$

$z_1' \leftarrow Z(x_1, r_1, e_1', e_1, w_1)$

$\{a_1, a_2, c_1, c_2\}$ →

← $e \leftarrow \$$

$\{e_1, e_2, z_1, z_2\}$ →

← $e' \leftarrow \$$

$\{e_1', e_2', z_1', z_2', d_1', d_2\}$ →

Complexity of adversarial structure

- Subproof gets more complicated beyond simple OR
- For simple OR:
  - Prove: "at least one of the Coms are binding"
- For general access structure Γ:
  - Prove: "Binding Coms are in a structure Γ'"
  - What is Γ'? Complexity of Γ'?

Complexity of adversarial structure

- We call Γ' the adversarial access structure.
- Negative:
  - Assuming NP $\subsetneq$ P/poly, $\exists\Gamma \in$ MSP s.t. Γ' blowup in circuit size.
- Positive:
  - If Γ is computed by a **read-once** formula (e.g. mNC^1), then Γ' can be computed in polynomial time.
  - Examples in mNC^1 :
    - t-out-of-n Threshold
    - Thresholds of thresholds
    - Ranked Choice Voting

Challenge Space vs Share Size

- Issue: Share size often exceeds max length of challenge space

- Solution [CDS94]: Parallel Repetition
  - Challenge space $C \Rightarrow C^t$
  - Works for 2-special sound sigma-protocols
  - 2-special soundness preserves in parallel repetition

- Fact: k-special soundness does not preserve in parallel repetition
  - k-special soundness $\xRightarrow{\text{t fold}} (k-1)^t + 1$-special soundness

Challenge Space vs Share Size

- Our relaxation: **statistical** k-special soundness
  - Observation: Number of bad transcripts are negligible

- Definition: $\mathrm{Ext}(\mathrm{transcript}) \to w$ fails only for negl. probability κ
  - κ depends on the choices of challenges
  - $(k_1, \ldots, k_\mu)$-special soundness $\overset{\mathrm{t\ fold}}{\Longrightarrow}$ **statistical** $(k_1, \ldots, k_\mu)$-special soundness.

Remark: essentially the same notion is introduced in [AAB+24] (ePrint:2024/311) in this conference

Conclusion

- Generalize CDS composition for multi-round PCIP
  - Plain model
  - Simulation precomputable
  - Soundness & round preserving
- Complexity of adversarial structure
  - mNC^1
- Challenge Space vs Share Size
  - statistical $(k_1, \ldots, k_\mu)$-special soundness

Thanks!