Towards Breaking the Half-Barrier of Local Leakage-resilient Shamir's Secret Sharing

Hai H. Nguyen

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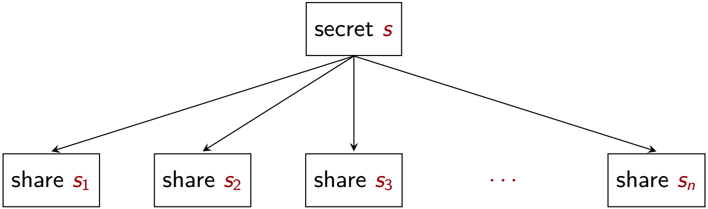
Threshold Secret Sharing [Shamir, Blakley]

Sharing:



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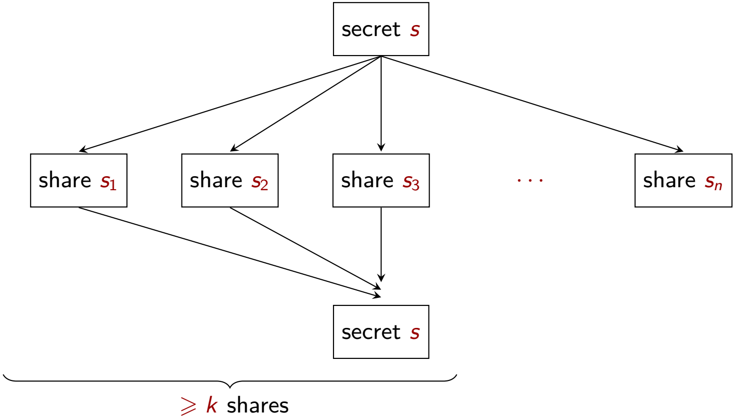

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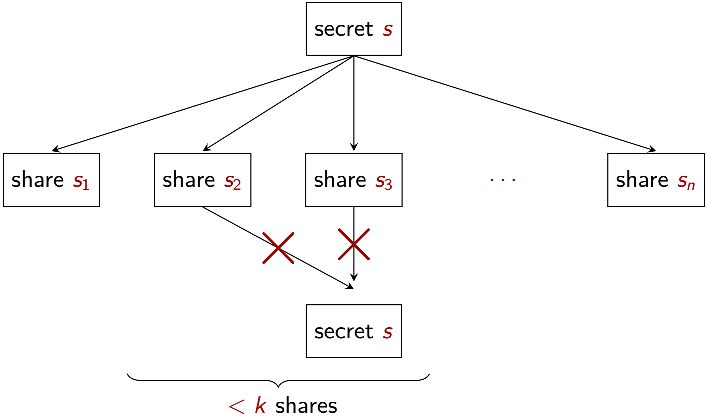

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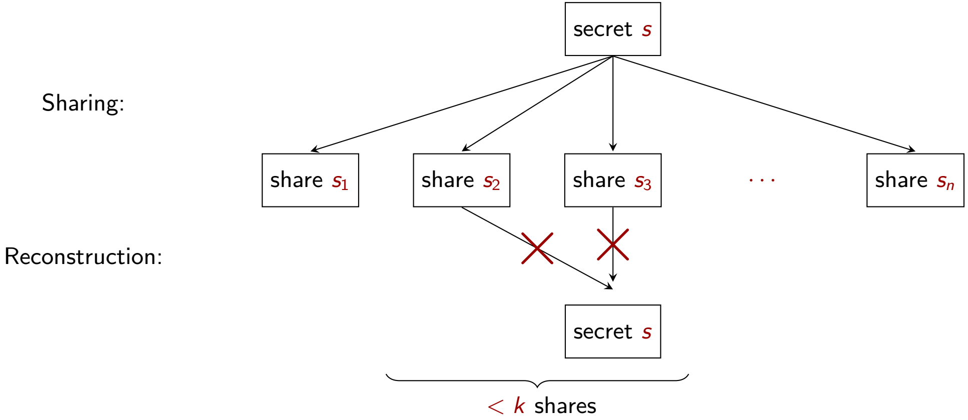
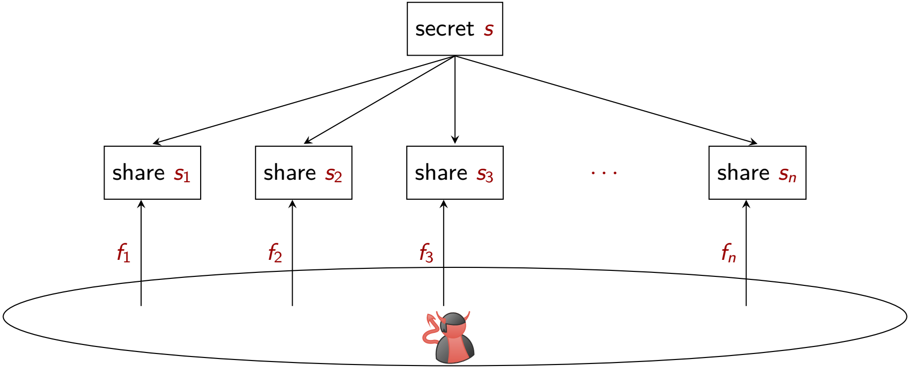


Concern: Side-channel attacks

- "All-or-nothing" no longer true
- Revealing partial information from every share

Local Leakage-resilient Secret Sharing

[Benhamouda-Degwekar-Ishai-Rabin-18, Goyal-Kumar-18]



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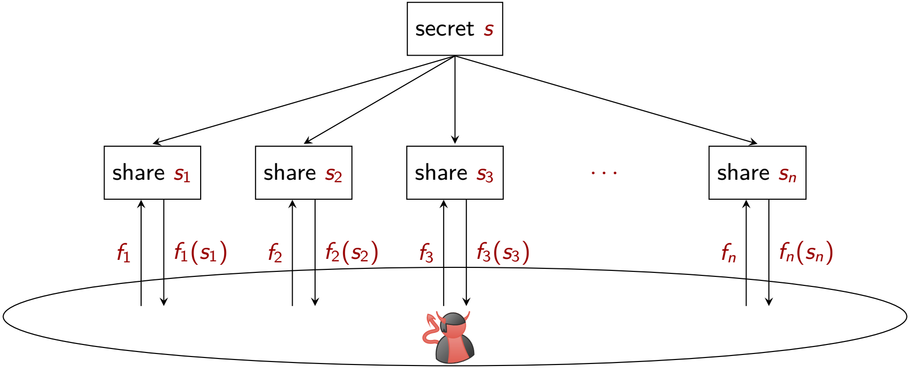

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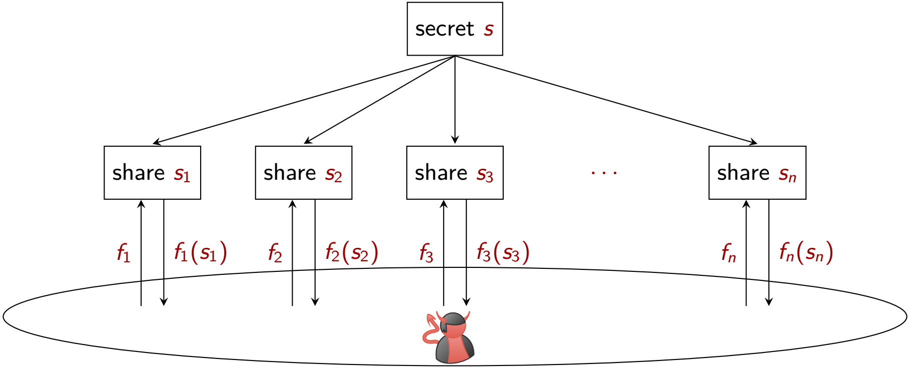

Example: Quadratic Residue Leakage

$$f_1 = f_2 = \ldots = f_n = \text{QR}, \text{ where } \text{QR}(x) = \begin{cases} 1 & \text{if } x = a^2 \text{ for some } a \in F_p, \\ 0 & \text{otherwise.} \end{cases}$$

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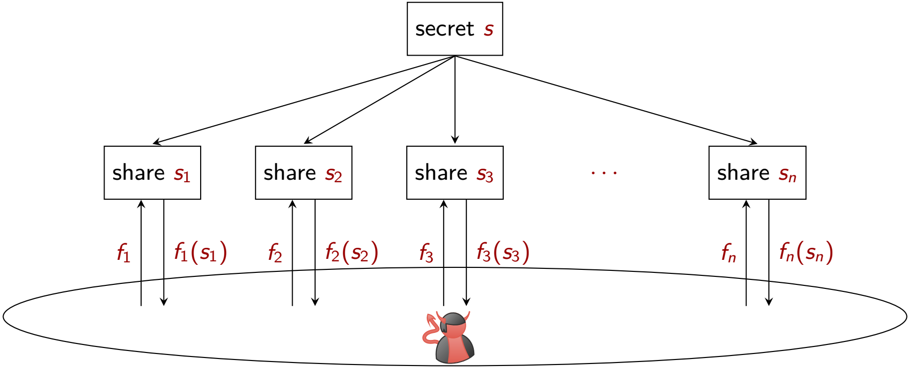

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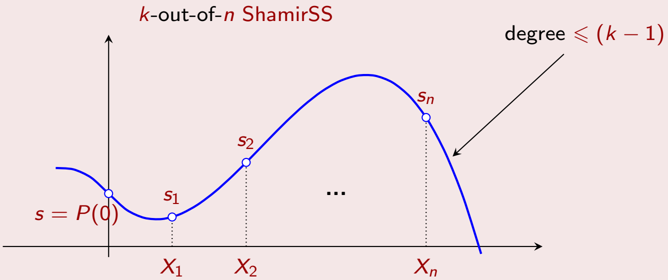
ε-leakage resilience: $\Delta(\ \boldsymbol{f}(\text{share}(s)),\ \boldsymbol{f}(\text{share}(s'))\ ) \leqslant \varepsilon$ for all s, s'.

Local Leakage-resilient Shamir's Secret Sharing



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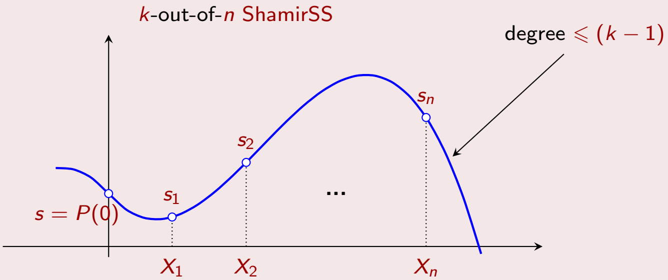

Applications: a useful primitive connected to many other fields

- Repairing Reed-Solomon codes
  [Guruswami Wootters'16, Tamo Ye Barg'17, Guruswami Rawat'17, ...]
- Secure multiparty computation protocol resilient to local leakage attacks
  [Benhamouda Degwekar Ishai Rabin'18, ...]
- Modular building block for other primitives (e.g., non-malleable secret-sharing)
  [Goyal Kumar'18, Srinivasan Vasudevan'19, ...]

Prior Work and Our Contribution: Leakage Resilience

Goal: The smaller k/n, the better. Typical parameters for MPC applications are $1/2$ and $1/3$.

Paper	Local leakage family	Fractional threshold k/n	Techniques
BDIR'18	all	0.907	linear Fourier (known half barrier: QR leakage)
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- Requires sufficiently large field, while others require n large (no matter what p is)
- Worst-case leakage remains open
- Extends to any MDS code-based secret sharing scheme

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Consider k-out-of-n Shamir's secret sharing over a prime field F_p.

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Remarks

- Techniques: exponential sums, particularly Weil's bounds

Technical Highlights

Our New Analytical Proxy

Our New Proxy

$$\Delta(\ \boldsymbol{f}(\text{share}(0)),\ \boldsymbol{f}(\text{share}(s))\ ) \leqslant \sum_{\boldsymbol{\ell} \in \{0,1\}^n} \sum_{i=1}^{n} \left\| \tilde{f}_{i,\ell_i} \right\|_{U^{d+1}}$$

- share(s): set of all possible (random) shares of secret s
- Leakage function: $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$, where $f_i \colon F_p \to \{0,1\}$
- Leakage distribution on s, denoted $\boldsymbol{f}(\text{share}(s))$:
  - samples $(s_1, s_2, \ldots, s_n) \leftarrow \text{share}(s)$
  - outputs $(f_1(s_1), f_2(s_2), \ldots, f_n(s_n))$
- Balanced leakage functions: $\tilde{f}_{i,\ell_i} = \mathbb{1}_{f_i^{-1}(\ell_i)} - \mathbb{1}_{-s+f_i^{-1}(\ell_i)}$

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Tools: Higher-order Fourier Analysis

- Gowers norms
- Generalized von Neumann inequality

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Implication

Suffices to bound the Gower's norms of balanced leakage functions.

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Linear Fourier Analysis

- Developed at least a few centuries ago
- Studies how a function correlates with a "linear phase": $x \mapsto \exp(2\pi i \zeta x)$
- Counts simple linear patterns: 3-term arithmetic progressions (Roth's theorem) $\mathbb{E}_{x,y}[\mathbb{1}_A(x)\mathbb{1}_A(x+y)\mathbb{1}_A(x+2y)]$

Higher-order Fourier Analysis

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n-Linear Form

Let $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)$ be linear functions over t variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_t)$ and $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$, where $\psi_i \colon F^t \to F$, $f_i \colon F \to [-1, 1]$. Define

$$\Lambda_\Psi(f_1, f_2, \ldots, f_n) = \mathbb{E}_{\boldsymbol{x} \in F^t}[f_1(\psi_1(\boldsymbol{x})) \cdot f_2(\psi_2(\boldsymbol{x})) \cdots f_n(\psi_n(\boldsymbol{x}))]$$

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- 3-term AP: $\Lambda_\Psi(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A)$, where $\psi_1(x, y) = x$, $\psi_2(x, y) = x + y$, $\psi_3(x, y) = x + 2y$.
- 4-term AP: $\Lambda_\Psi(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A)$, where additionally $\psi_4(x, y) = x + 3y$.

Main Ideas

Reduction to bounding linear forms

$$\Delta(\ \boldsymbol{f}(\mathrm{share}(0)),\ \boldsymbol{f}(\mathrm{share}(s))\ ) \leqslant \sum_{\boldsymbol{\ell}} \sum_{i}^{n} \Lambda_{\Psi}(\tilde{f}_{i,\ell_1}, \tilde{f}_{i,\ell_2}, \ldots, \tilde{f}_{i,\ell_n}).$$

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Theorem (Generalized von Neumann Inequality [GreenTao'10])

Let $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)$ be a system of linear functions with Cauchy-Schwarz complexity d. Let $g_i : F_p \to [-1, 1]$ for every $i \in [n]$. Provided $p \geqslant d$, it holds that

$$\Lambda_{\Psi}(g_1, g_2, \ldots, g_n) \leqslant \min_{1 \leqslant i \leqslant n} \|g_i\|_{U^{d+1}}.$$

One of the key ingredients in the proof of the breakthrough result: "The primes contain arbitrarily long arithmetic progressions."

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Applying this theorem extensively to all leakage values $\boldsymbol{\ell}$ and indices i yields

$$\Delta(\ \boldsymbol{f}(\mathrm{share}(0)),\ \boldsymbol{f}(\mathrm{share}(s))\ ) \leqslant \sum_{\boldsymbol{\ell}} \sum_{i=1}^{n} \left\| \tilde{f}_{i,\ell_i} \right\|_{U^{d+1}}.$$

Illustrative Example

Consider $n = 4$ parties, threshold $k = 4$ over prime field F_7 with evaluation places $\{1, 2, 3, 4\}$.

(Random) shares of secret 0

$$\text{share}(0) = \langle G_0 \rangle, \text{ where } G_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 2 \\ 1 & 1 & 6 & 1 \end{pmatrix}$$

$s_1 = x + y + z,\ s_2 = 2x + 4y + z,\ s_3 = 3x + 2y + 6z,\ s_4 = 4x + 2y + z$ for uniformly random x, y, z

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Probability of leakage being 1

$$\begin{aligned} \Pr[\boldsymbol{f}(\text{share}(0)) = \mathbf{1}] &= \mathbb{E}_{\boldsymbol{x},y,z}[\mathbb{1}_A(x+y+z)\mathbb{1}_A(2x+4y+z)\mathbb{1}_A(3x+2y+6z)\mathbb{1}_A(4x+2y+z)] \\ &= \Lambda_\Psi(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) \end{aligned}$$

$$\Pr[\boldsymbol{f}(\text{share}(s)) = \mathbf{1}] = \Lambda_\Psi(\mathbb{1}_{s+A}, \mathbb{1}_{s+A}, \mathbb{1}_{s+A}, \mathbb{1}_{s+A}) \quad \text{since share}(s) = (s, s, \ldots, s) + \text{share}(0)$$

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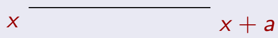
Not a linear form, but bounded by 4 linear forms:

$$\Lambda_\Psi(\mathbb{1}_A - \mathbb{1}_{s+A}, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) + \Lambda_\Psi(\mathbb{1}, \mathbb{1}_A - \mathbb{1}_{s+A}, \mathbb{1}_A, \mathbb{1}_A) + \Lambda_\Psi(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A - \mathbb{1}_{s+A}, \mathbb{1}_A) + \Lambda_\Psi(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A - \mathbb{1}_{s+A})$$

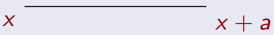
Gowers Norms

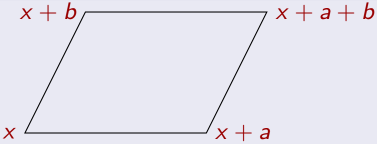


$$\|f\|_{U^1} = \mathbb{E}_{x,a}[f(x)f(x+a)]$$

Breaking the Half-barrier for QR Leakage & Almost all 1-bit Leakages

Gowers Norms





$\|f\|_{U^1} = \mathbb{E}_{x,a}[f(x)f(x+a)]$

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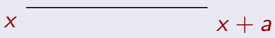
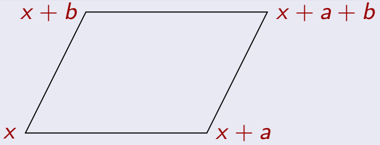
Gowers Norms





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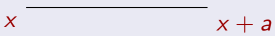
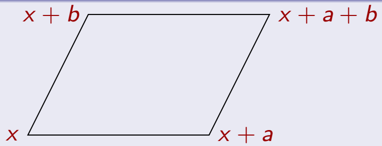
Remark

Bounding the Gowers norms of an arbitrary function is challenging.

Gowers Norms





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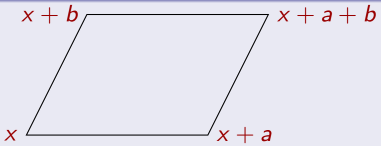
Balanced quadratic leakage functions

$$\|\mathbb{1}_{\mathrm{QR}} - \mathbb{1}_{s+\mathrm{QR}}\|_{U^d} \leqslant \frac{1}{p^{\Theta(c_d)}} \text{ for all } s.$$

Technique: multiplicative character sums

Gowers Norms



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Random balanced leakage function

$$\|\mathbb{1}_A - \mathbb{1}_{s+A}\|_{U^d} = O_d\left(\frac{1}{p}\right) \text{ for all } s.$$

Technique: standard probabilistic methods

Summary and Open Problems

Takeaway

1. Develop a new analytic framework using higher-order Fourier analysis
   - cn-out-of-n Shamir secret sharing is leakage-resilient against almost all 1-bit local leakage
2. Present an explicit 2-bit leakage attack that determines the secret when $k = \Theta(\sqrt{n})$, $p = \Theta(n)$

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1. Leakage resilience
   - Breaking the half threshold for the worst-case leakage
   - What if p is not large enough, says $p = \Theta(n)$?
   - Multiple-bit leakages
   - Does randomizing the evaluation places help?
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   - 1-bit leakage attack
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Thank you!