Scalable Multiparty Computation from Non-linear Secret Sharing

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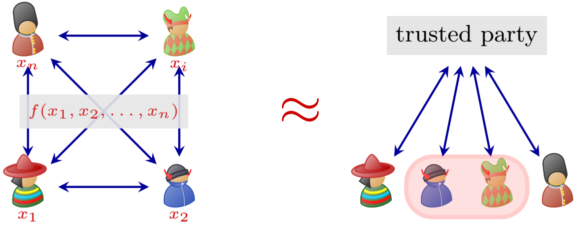
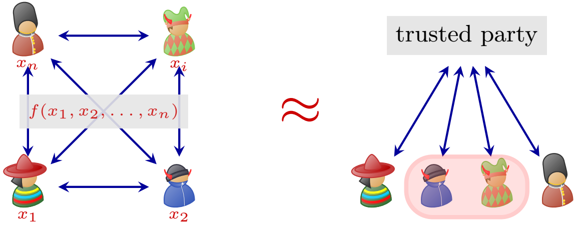
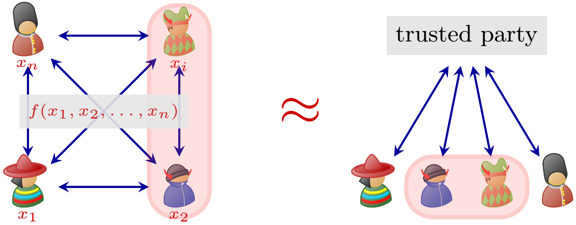


x_n
x_i
$f(x_1, x_2, \ldots, x_n)$
x_1
x_2
$\approx$
trusted party



This Work

- honest majority
- information-theoretic plain model
- semi-honest adversary



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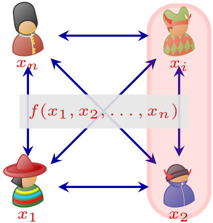
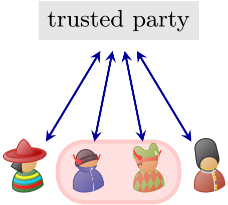


$\approx$



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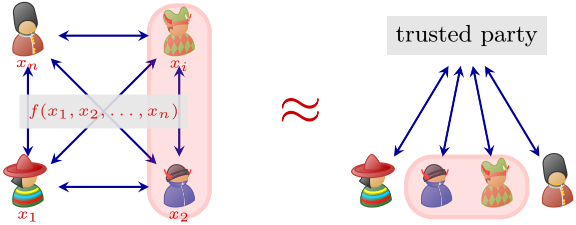

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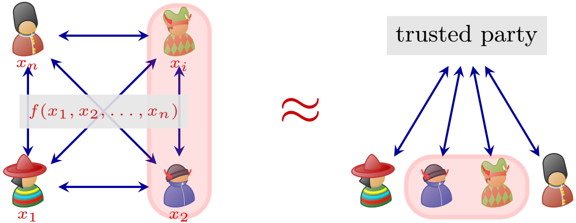

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- For an arithmetic circuit C over F, can we achieve overall computation complexity $|C|$ field operations?
  - Optimal since insecure evaluation requires the same complexity
- Scalable as the overall complexity does not grow with n.

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- Share transformation [Goyal-Polychroniadou-Song'21, Goyal-Polychroniadou-Song'22]
  - Only achieve communication complexity $|C|$ field element
  - Computation complexity is still $n \cdot |C|$ field operation

Can we build scalable MPC protocol in computation for general circuit?

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Assuming F is an exponentially large prime field,

For any general circuit C over F, there is a scalable MPC protocol among n parties

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- Not desirable due to high (concrete/asymptotic) cost of computation
  - Yao's Garbling vs. Arithmetic Garbling: [Applebaum-Ishai-Kushilevitz'11]

Application of MPC over large prime field

Delegating computation of resource-intensive cryptographic tasks:

- SNARK proof generation [Ozdemir-Boneh'22, Garg-Goel-Jain-Policharla-Sekar'23, Chiesa-Lehmkuhl-Mishra-Zhang'23]
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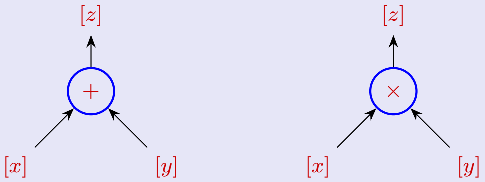
- $\log|F| \approx 256$
- Our protocol can plausibly ($n < \log|F|$) be applied to such scenarios with $100 \sim 200$ parties.

Technical Highlight

Existing Framework

Emulating the circuit evaluation gate by gate by secret sharing



Tricks required for Scalable MPC

- Packed secret sharing [Franklin-Yung'92]
  - This work: "Unpacked" secret sharing
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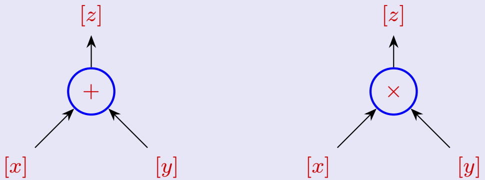

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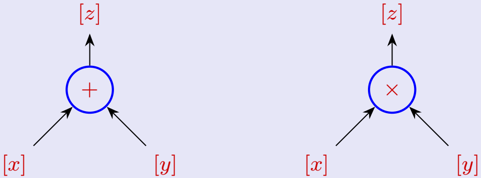

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- Degree-reduction after each multiplication gate, given double sharing $[r]_t$ and $[r]_{2t}$ of r
  - Reconstruct $[x]_t \cdot [y]_t - [r]_{2t}$
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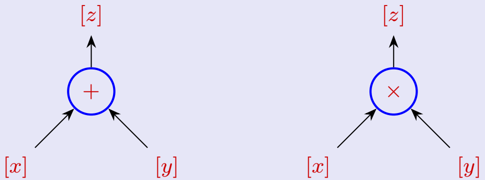

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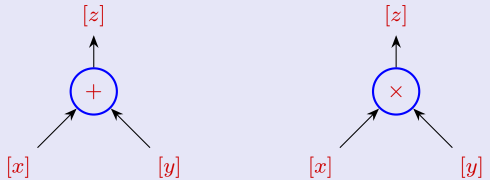

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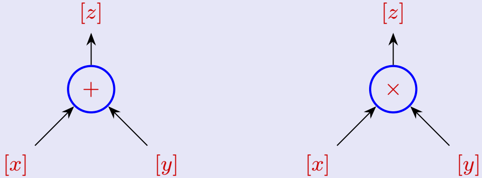

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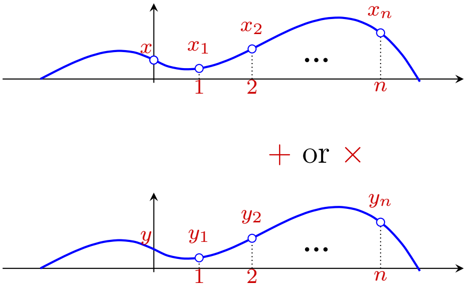

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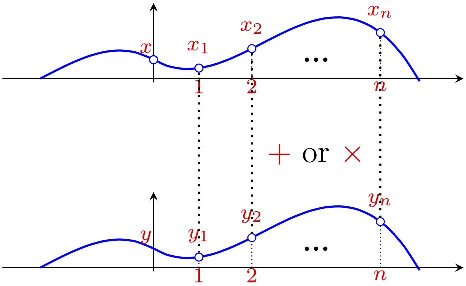

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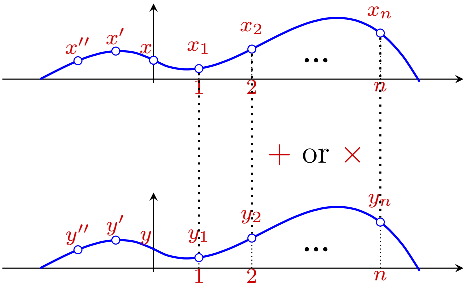

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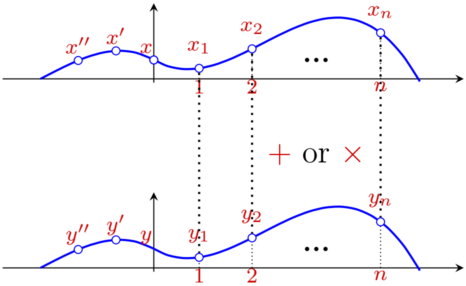

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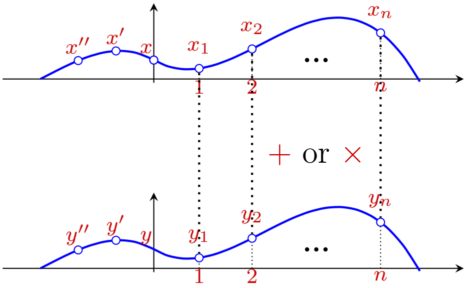

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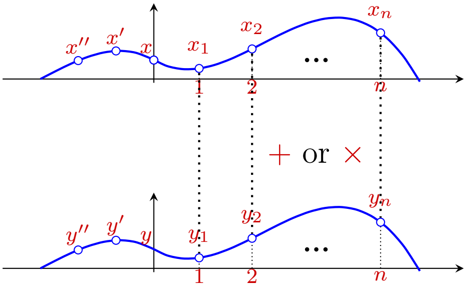

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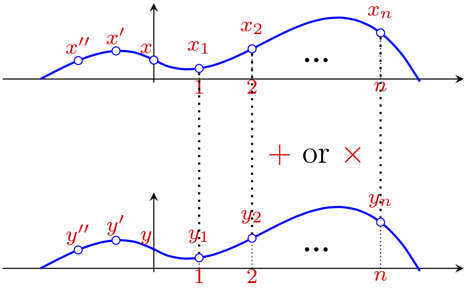

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How can we achieve this?

Chinese-remainder-Theorem based Secret Sharing

- recently introduced by [GJMSWZ'23] to build (weighted) mpc protocols
  - compatible with the existing framework; gate emulation + degree reduction

Remarks

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Long secret S	$S \mod p_1$	$S \mod p_2$	$\cdots$	$S \mod p_n$

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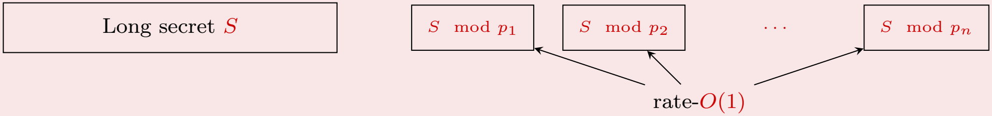

- Each share can be much smaller than the secret (e.g., $p_1 = 2$, $p_2 = 3$, ...)
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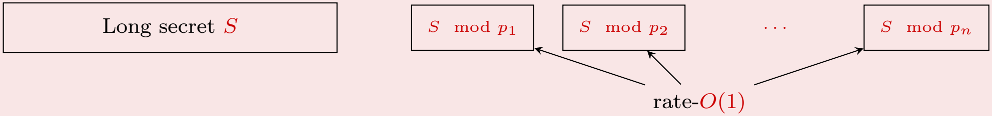

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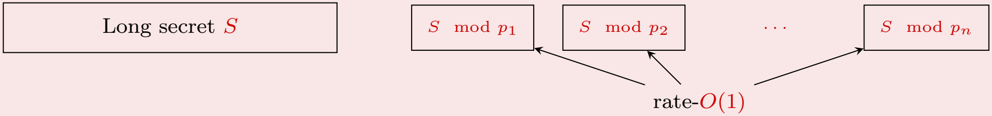

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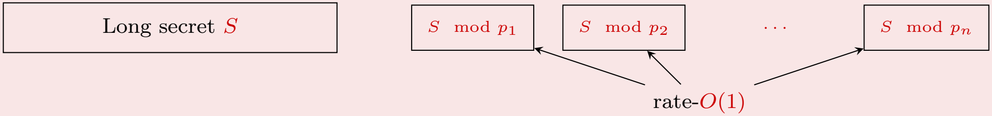
  - compatible with the existing framework; gate emulation + degree reduction
- A secret $s \in F_p$ is re-randomized as an integer $S = s + \alpha \cdot p$.



- Each share can be much smaller than the secret (e.g., $p_1 = 2$, $p_2 = 3$, ...)
- Pick $p_1, p_2, \ldots, p_n$ appropriately to make it rate-$O(1)$.

Remarks

- Secret length $\log F$ has to be $O(n)$
- Have to measure overall complexity at a bit level
- Already achieve online overhead $O(1)$ assuming we have $[r]_t$ and $[r]_{2t}$

Batch Randomness Generation

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VanderMonde Randomness Extraction Damgard-Nielson'07

Each party generates a pair $[r_i]_t$, $[r_i]_{2t}$ and

$$V_{n-t,n} \cdot \begin{pmatrix} [r_1]_t \\ [r_2]_t \\ \vdots \\ [r_n]_t \end{pmatrix} \qquad V_{n-t,n} \cdot \begin{pmatrix} [r_1]_{2t} \\ [r_2]_{2t} \\ \vdots \\ [r_n]_{2t} \end{pmatrix}, \qquad \text{e.g.,} \quad V_{n-t,n} = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1^2 & 2^2 & 3^2 & \dots & n^2 \\ \\ 1^{n-t} & 2^{n-t} & 3^{n-t} & \dots & n^{n-t} \end{pmatrix}$$

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- Extract $n-t$ pairs out of n pairs
- $V_{n-t,n}$ is super-invertible (any $n-t$ by $n-t$ submatrix is invertible)
- No matter which t parties are corrupted, the extracted masks are uniformly random.

Key Technical Barrier

For CRT secret sharing, how do we prove the security of these extracted masks?

$$V_{n-t,n} \cdot \begin{pmatrix} [r_1]_{2t} \\ [r_2]_{2t} \\ \vdots \\ [r_n]_{2t} \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_{n-t} \end{pmatrix} \approx V_{n-t,n} \cdot \begin{pmatrix} [r_1]_{2t} \\ [r_2]_{2t} \\ \vdots \\ [r_n]_{2t} \end{pmatrix} + \begin{pmatrix} D'_1 \\ D'_2 \\ \vdots \\ D'_{n-t} \end{pmatrix}$$

These are distributions over integers! Arguing statistical distance for distributions over integers is not easy.

High-dimensional Smudging Lemma

$$V_{n-t,n} \cdot \begin{pmatrix} [r_1]_{2t} \\ [r_2]_{2t} \\ \vdots \\ [r_n]_{2t} \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_{n-t} \end{pmatrix} \quad \text{and} \quad V_{n-t,n} \cdot \begin{pmatrix} [r_1]_{2t} \\ [r_2]_{2t} \\ \vdots \\ [r_n]_{2t} \end{pmatrix} + \begin{pmatrix} D'_1 \\ D'_2 \\ \vdots \\ D'_{n-t} \end{pmatrix}$$

for

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are close as long as $D_i - D'_i$ are divisible by

$$\prod_{1 \leqslant i < j \leqslant n} (j - i)$$

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- Due to proof techniques

Summary

Scalable MPC for general circuit over large prime field F:

- $|C| \cdot \log |F|$-bit communication/computation complexity
- Based on CRT-secret sharing
- "unpacked" secret sharing to achieve non-amortized rate-$O(1)$
- high-dimensional smudging lemma: randomness extraction over integers
- require $\log F = \widetilde{O}(n^2)$ — Open problem: can we prove the security for $\log F = \widetilde{O}(n)$?

Thanks!

Questions?