New Approaches for Estimating the Bias of Differential-Linear Distinguishers

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Overview

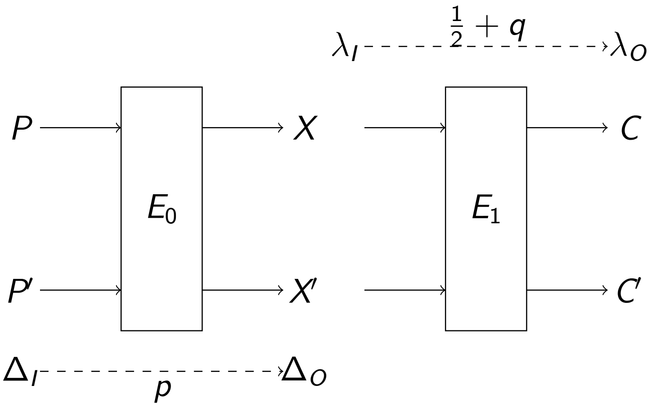
Symmetric Cryptanalysis

- Differential cryptanalysis
  - proposed by Biham and Shamir at CRYPTO 1990
  - broke DES at CRYPTO 1992

- Linear cryptanalysis
  - proposed by Matsui in 1993, broke DES again
  - the first experimental cryptanalysis of DES at CRYPTO 1994

Differential-Linear Cryptanalysis

- A combination of differential and linear cryptanalysis
  - proposed by Langford and Hellman at CRYPTO 1994
  - a chosen plaintext two-stage technique of cryptanalysis



Differential-Linear Approximation

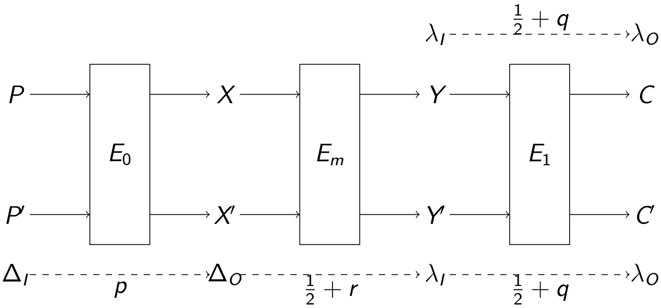
- Differential: $\Pr[\Delta_I \xrightarrow{p} \Delta_O] = p$

- Linear approximation: $\Pr[\lambda_I \xrightarrow{q} \lambda_O] = 1/2 + q$

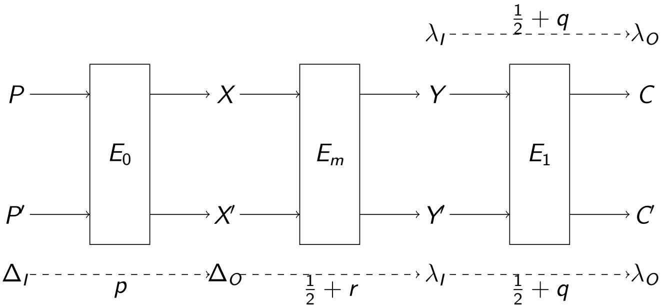
- Differential-linear approximation:
  $\Pr[C \cdot \lambda_O = C' \cdot \lambda_O | P \oplus P' = \Delta_I] = p(1/2 + 2q^2) + (1-p) \cdot 1/2 = 1/2 + 2pq^2$

Estimating the bias of a DL approximation in the middle

- Differential-Linear Connectivity Table (DLCT, EUROCRYPT 2019)
  - inspired by Boomerang Connectivity Table
  - more accurate than before
  - applications: ICEPOLE, DES, Serpent, Ascon



Estimating the bias of a DL approximation in the middle



The theoretical bias of a differential-linear approximation:

$$\mathcal{E}_{\Delta_I,\lambda_O} = 4p \cdot \overline{DLCT}_{E_m}(\Delta,\lambda) \cdot q^2 = 4prq^2$$

Differential-Linear Probability

Definition

For a t-round differential-linear approximation $(\Delta \xrightarrow{t\ round} \lambda)$, where Δ is the input difference, and λ is the output difference, the differential-linear probability (DLP) is defined by

$$\mathrm{DLP}(\Delta, \lambda) = \mathrm{Pr}[\Delta \xrightarrow{t\ round} \lambda] = \frac{|\{X|\lambda \cdot (f(X) \oplus f(X \oplus \Delta)) = 0\}|}{2^n}$$

The differential-linear bias is $\varepsilon = \mathrm{DLP}(\Delta, \lambda) - \frac{1}{2}$.

An Important Observation on DLP

$$\begin{aligned}\mathrm{DLP}(\Delta,\lambda) &= \frac{|\{X|\lambda\cdot(f(X)\oplus f(X\oplus\Delta))=0\}|}{2^n}\\ &= \frac{\sum_{\Delta_i\in\mathbb{F}_2^n,\lambda\cdot\Delta_i=0}|\{X|f(X)\oplus f(X\oplus\Delta)=\Delta_i\}|}{2^n}\\ &= \sum_{\Delta_i\in\mathbb{F}_2^n,\lambda\cdot\Delta_i=0}\overline{\mathrm{DDT}}_f(\Delta,\Delta_i)\end{aligned} \quad (1)$$

An Important Observation on DLP

- $\lambda \cdot \Delta_i = \lambda^{\{n-1\}}\Delta_i^{\{n-1\}} \oplus \lambda^{\{n-2\}}\Delta_i^{\{n-2\}} \oplus \cdots \lambda^{\{1\}}\Delta_i^{\{1\}} \oplus \lambda^{\{0\}}\Delta_i^{\{0\}}$

- If $\lambda^{\{j\}} = 0$, then $\lambda^{\{j\}}\Delta_i^{\{j\}} = 0$ always holds, which means that the value of this bit of Δ_i does not affect the value of $\lambda \cdot \Delta_i$.

- $\mathrm{DLP}(\Delta, \lambda) = \sum_{\Delta_i \in \mathbb{F}_2^n, \lambda \cdot \Delta_i = 0} \mathrm{DP}(\Delta, \Delta_i) = \sum_{\substack{0 \le j < 2^{hw(\lambda)} \\ \lambda \cdot \mathcal{T}_{t,j} = 0}} \mathrm{Pr}[\Delta \xrightarrow{t \text{ rounds}} \mathcal{T}_{t,j}]$

The Truncated Difference Distribution Table

Definition

Let f: $\{0,1\}^n \rightarrow \{0,1\}^n$ be a bijective vectorial boolean function. the TDT of f is a three-dimensional table whose first parameter $\Delta_I \in \{0,1\}^n$ is an input difference of f, and whose second parameter $\mathcal{M} \in \{0,1\}^n$ is the TD mask of a truncated output difference $\mathcal{T} \in \{*,0,1\}^n$ of f and whose third parameter is $\mathcal{Z} \in \{0,1\}^n$. Define the TDT entry $(\Delta_I, \mathcal{M}, \mathcal{Z})$ of f as

$$\mathrm{TDT}_f(\Delta_I, \mathcal{M}, \mathcal{Z}) = |\{X | \mathcal{M} \& (f(X) \oplus f(X \oplus \Delta_I)) = \mathcal{Z}\}|$$

where the TDT entry is equal to zero when $\mathcal{M} \& \mathcal{Z} \neq \mathcal{M}$.

The Truncated Difference Distribution Table

Proposition 1

The TDT is an extension of the DDT. There is a connection between DDT and TDT:

$$\mathrm{TDT}_f(\Delta_I, \mathcal{M}, \mathcal{Z}) = \sum_{\Delta : \mathcal{M} \& \Delta = \mathcal{Z}} \mathrm{DDT}_f(\Delta_I, \Delta)$$

Proposition 2

Let f: $\{0,1\}^n \rightarrow \{0,1\}^n$ be a bijective vectorial boolean function, Δ and λ denote an input difference and an output mask of f respectively.

$$\mathrm{DLP}(\Delta, \lambda) - \frac{1}{2} = \sum_{\substack{0 \leq \mathcal{Z} < 2^n \\ \lambda \cdot \mathcal{Z} = 0}} \mathrm{TDTP}(\Delta, \lambda, \mathcal{Z}) - \frac{1}{2}$$

Properties of the TDT

Property 1

$$\mathrm{TDT}_f(0, \mathcal{M}, \mathcal{Z}) = \begin{cases} 2^n, & \textit{if } \mathcal{Z} = 0 \\ 0, & \textit{if } \mathcal{Z} \neq 0 \end{cases}$$

Property 2

$$\mathrm{TDT}_f(\Delta_I, 0, \mathcal{Z}) = \begin{cases} 2^n, & \textit{if } \mathcal{Z} = 0 \\ 0, & \textit{if } \mathcal{Z} \neq 0 \end{cases}$$

Property 3

$$\mathrm{TDT}_f(\Delta_I, 2^n - 1, \mathcal{Z}) = \mathrm{DDT}_f(\Delta_I, \mathcal{Z})$$

Property 4

Given Δ_I and $\mathcal{M}$, there are at most $2^{hw(\mathcal{M})}$ non-zero entries in the TDT.

Estimation of the DLP based on TDT

The probability of a truncated differential characteristic

$$\Pr[\mathcal{T}_0 \xrightarrow{R} \mathcal{T}_1 \xrightarrow{R} \cdots \xrightarrow{R} \mathcal{T}_t] = \prod_{i=0}^{t-1} \prod_{|\mathcal{T}_i[j]|=1} \overline{\mathrm{TDT}}(\mathcal{T}_i[j], \mathcal{M}_i[j], \mathcal{Z}_i[j]) \quad (2)$$

The probability of a truncated differential

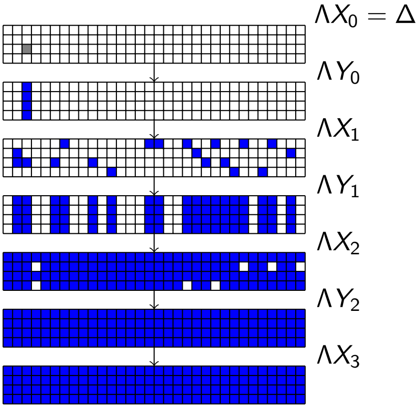
$$\Pr[\mathcal{T}_0 \xrightarrow{t\ rounds} \mathcal{T}_t] = \sum_{\mathcal{T}_1, \cdots, \mathcal{T}_{t-1}} \prod_{i=0}^{t-1} \prod_{|\mathcal{T}_i[j]|=1} \overline{\mathrm{TDT}}(\mathcal{T}_i[j], \mathcal{M}_i[j], \mathcal{Z}_i[j]) \quad (3)$$

Estimation of the DLP based on TDT

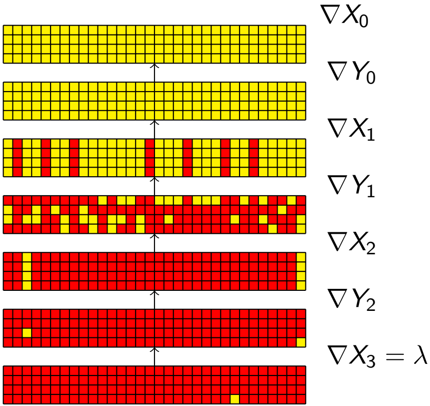
The relationship between DLP and TDT

$$\begin{aligned}\mathrm{DLP}(\Delta,\lambda) &= \sum_{\substack{0\le k<2^{hw(\lambda)}\\ \lambda\cdot\mathcal{T}_{t,k}=0}} \Pr[\Delta \xrightarrow{t\ rounds} \mathcal{T}_{t,k}] \\ &= \sum_{\substack{0\le k<2^{hw(\lambda)}\\ \lambda\cdot\mathcal{T}_{t,k}=0}} \sum_{\mathcal{T}_1,\cdots,\mathcal{T}_{t-1}} \prod_{i=0}^{t-1} \prod_{|\mathcal{T}_i[j]|=1} \overline{\mathrm{TDT}}(\mathcal{T}_i[j], \mathcal{M}_i[j], \mathcal{Z}_i[j])\end{aligned} \quad (4)$$

Computing the Differential-Linear Pattern



(a) The forward propagation of Δ: a blank cell indicates a bit difference always inactive; a gray cell indicates an active bit difference; a blue cell indicates a bit difference undetermined



(b) The backward propagation of λ: a yellow cell indicates a bit of which the bit difference need to be determined; a red cell indicates a bit of which the bit difference is arbitrary

Computing the Differential-Linear Pattern

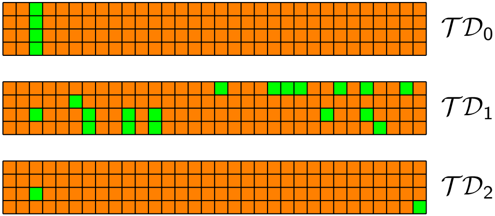


Figure: The DL pattern $(\mathcal{TD}_0, \mathcal{TD}_1, \mathcal{TD}_2)$ of 3-round Serpent: an orange cell indicates a bit of which the bit difference always be inactive or arbitrary, which is of no concern; a green cell indicates a bit difference need to be determined

- If $\wedge Y_i^{\{j\}} = 0$, $\mathcal{TD}_i^{\{j\}} = 0$.
- If $\nabla Y_i^{\{j\}} = 0$, $\mathcal{TD}_i^{\{j\}} = 0$.
- If $\wedge Y_i^{\{j\}} = 1$ and $\nabla Y_i^{\{j\}} = 1$, $\mathcal{TD}_i^{\{j\}} = 1$.

Estimate of the DL Bias when E_m Consists of Multiple Rounds

- Phase 1: Computing the differential-linear bias of E_m using the breadth-first method

- Phase 2: Extending a differential with high-probability in the differential part E_0 and a linear approximation with high-bias in the linear part E_1

- Phase 3: Computing the overall bias of the differential-linear bias of E

Computing the differential-linear bias of E_m

- For the $\mathcal{S}$ layer in the 0-th round and for each $0 \leq j < |A_0|$

$$\Pr[\Delta \xrightarrow{1\ round} A_{0,j}] = \prod_{k:\mathcal{TD}_0[k]\neq 0} \overline{\mathrm{TDT}}(\Delta[k], \mathcal{TD}_0[k], A_{0,j}[k])$$

- For the $\mathcal{S}$ layer in the i-th round ($1 \leq i < R_m$), and for each $0 \leq j < |A_i|$

$$\begin{aligned}\Pr[\Delta \xrightarrow{i\ rounds} A_{i,j}] &= \sum_{t=0}^{|B_{i-1}|-1} \mathrm{TDP}_{i-1,t} \cdot \Pr[B_{i-1,t} \xrightarrow{\mathcal{S}} A_{i,j}] \\ &= \sum_{t=0}^{|B_{i-1}|-1} \mathrm{TDP}_{i-1,t} \cdot \left(\prod_{k:\mathcal{TD}_i[k]\neq 0} \overline{\mathrm{TDT}}(B_{i-1,t}[k], \mathcal{TD}_i[k], A_{i,j}[k]) \right)\end{aligned}$$

- Finally, $\mathrm{DLP}(\Delta, \lambda) = \sum_{j=0}^{|B_{R_m-1}|-1} \mathrm{TDP}_{R_m-1,j} \cdot \pi(\lambda \cdot B_{R_m-1,j})$.
  where $\pi(x) = 1$ if $x = 0$ and $\pi(x) = 0$ otherwise. Finally, the bias of E_m is $\mathrm{DLP}(\Delta, \lambda) - \frac{1}{2}$.

The complexity

- The computational complexity:

$$|A_0| + |A_{R_m-1}| + \sum_{i=1}^{R_m-1} |A_{i-1}| \cdot |A_i| = 2^{hw(\mathcal{TD}_0)} + 2^{hw(\mathcal{TD}_{R_m-1})} + \sum_{i=1}^{R_m-1} 2^{hw(\mathcal{TD}_{i-1} \| \mathcal{TD}_i)}$$

- The memory complexity:

$$\max_{1 \leq i < R_m} (|A_{i-1}| + |A_i| + |\mathrm{TDP}_{i-1}| + |\mathrm{TDP}_i|) = \max_{1 \leq i < R_m} (2 \times (2^{hw(\mathcal{TD}_{i-1})} + 2^{hw(\mathcal{TD}_i)}))$$

Estimate of the DL Bias when E_m Consists of One Rounds

- Phase 1: Computing the probability of a truncated differential of E_0 using the depth-first method

- Phase 2: Searching a linear approximation with high-bias for the linear part E_1

- Phase 3: Using DLCT to connect the strong truncated differential and the strong biased linear approximation

Computing the probability of a truncated differential of E_0

Procedure Round-0

Begin the program.

Let $P_{TD} = 0$.

For each candidate for $\mathcal{Z}_0$ with fixed $\mathcal{TD}_0$, do the following:

Let $p_0 = \overline{\mathrm{TDT}}(\Delta X_0, \mathcal{TD}_0, \mathcal{Z}_0)$.

If $p_0 \geq \overline{TS}$, then call *Procedure Round-1*.

Exit the program.

Procedure Round-i $(1 \leq i < R_0 - 1)$

For each candidate for $\mathcal{Z}_i$ with fixed $\mathcal{TD}_i$, do the following:

Let $\Delta X_1 = \mathcal{L}(\mathcal{Z}_0)$ and $p_i = \overline{\mathrm{TDT}}(\Delta X_i, \mathcal{TD}_i, \mathcal{Z}_i)$.

If $\prod_{k=0}^{i} p_k \geq \overline{TS}$, then call *Procedure Round-*$(i+1)$.

Reture to the upper procedure.

Computing the probability of a truncated differential of E_0

Procedure Round-$(R_0 - 1)$

For each candidate for $\mathcal{Z}_{R_0-1}$ with fixed $\mathcal{TD}_{R_0-1}$, do the following:

Let $\Delta X_{R_0-1} = \mathcal{L}(\mathcal{Z}_{R_0-2})$.

Let $p_{R_0-1} = \overline{\mathrm{TDT}}(\Delta X_{R_0-1}, \mathcal{TD}_{R_0-1}, \mathcal{Z}_{R_0-1})$.

If $p = \prod_{k=0}^{R_0-1} p_k \geq \overline{TS}$, then a linear transformation is performed, i.e., $\Delta X_{R_0} = \mathcal{L}(\mathcal{Z}_{R_0-1})$.

Let $\mathcal{Z}_{R_0} = \lambda \& \mathcal{T}$. If $\Delta X_{R_0} = \mathcal{Z}_{R_0}$, then $P_{TD} = P_{TD} + p$.

Reture to the upper procedure.

Applications

- Authenticated encryption
  - Ascon
  - KNOT

- Bit-wise block cipher
  - Serpent

- Byte-wise block cipher
  - AES
  - CLEFIA

Conclusion: Ascon

Cipher	Rounds	Experimental value	Theoretical estimate				
			DLCT [1]	DATF [2]	HATF [3]	Method in Sect.4.2	Method in Sect.4.3
Ascon	4/12	2^{-2}	2^{-5}	$2^{-2.365}$	$2^{-2.09}$	$\mathbf{2^{-2}}$	
	5/12	2^{-10}					$\mathbf{2^{-10.1}}$
	6/12‡						$\mathbf{2^{-22.43}}$

- 4-round DL distinguisher: the same as the experimental result
- 5-round DL distinguisher: extrmely close to the experimental result
- 6-round DL distinguisher: first introduced

Conclusion: Serpent

Cipher	Rounds	Experimental value	Theoretical estimate			
			DLCT [1]	DATF [2]	HATF [3]	Method in Sect.4.2
Serpent	**3/32**†	$\mathbf{2^{-1.415}}$				$\mathbf{2^{-1.415}}$
	4/32	$2^{-13.75}$	$2^{-13.68}$	$2^{-13.736}$		$\mathbf{2^{-13.696}}$
	4/32†	$\mathbf{2^{-5.30}}$				$\mathbf{2^{-5.415}}$
	5/32	$2^{-17.75}$		$2^{-17.736}$		$\mathbf{2^{-17.696}}$
	5/32†	$\mathbf{2^{-11.44}}$				$\mathbf{2^{-11.415}}$
	6/32†					$\mathbf{2^{-19.61}}$
	7/32†					$\mathbf{2^{-29.45}}$
	8/32†					$\mathbf{2^{-39.45}}$
	9/32		$2^{-57.68}$	$2^{-57.736}$		$\mathbf{2^{-57.696}}$
	9/32†					$\mathbf{2^{-52}}$
	9/32†					$\mathbf{2^{-55.33}}$

Conclusion: Serpent

- revisiting 4-round and 5-round DL distinguisher
- searching for the DL distinguisher up to 9 rounds
- ignoring the key recovery, a 9-round DL distinguisher with bias of 2^{-52}
- considering the key recovery, two better 9-round DL distinguishers with bias of $2^{-55.33}$

Conclusion: KNOT256

Cipher	Rounds	Experimental value	Theoretical estimate			
			DLCT [1]	DATF [2]	HATF [3]	Method in Sect.4.2
KNOT-256	9/52	$\mathbf{2^{-1.20}}$				$\mathbf{2^{-1.20}}$
	10/52	$\mathbf{2^{-3.27}}$				$\mathbf{2^{-3.66}}$
	11/52	$\mathbf{2^{-4.31}}$				$\mathbf{2^{-6.38}}$
	12/52	$\mathbf{2^{-9.91}}$				$\mathbf{2^{-9.27}}$
	13/52	$\mathbf{2^{-14.04}}$				$\mathbf{2^{-12.27}}$
	14/52					$\mathbf{2^{-16.23}}$
	15/52					$\mathbf{2^{-23.31}}$
	16/52					$\mathbf{2^{-30.52}}$

Conclusion: KNOT256

- focusing on the initialization phase

- searching for the DL distinguishers up to 16 rounds

- 16-round DL distinguisher: $2^{-30.52}$

Conclusion: AES, CLEFIA

Cipher	Rounds	Experimental value	Theoretical estimate			
			DLCT [1]	DATF [2]	HATF [3]	Method in Sect.4.2
AES	2/10	$\mathbf{2^{-1}}$				$\mathbf{2^{-1}}$
	3/10	$\mathbf{2^{-8.66}}$				$\mathbf{2^{-8.66}}$
	4/10					$\mathbf{2^{-27.85}}$
	5/10					$\mathbf{2^{-51.85}}$
CLEFIA	4/18	$\mathbf{2^{-1}}$				$\mathbf{2^{-1}}$
	5/18	$\mathbf{2^{-2.415}}$				$\mathbf{2^{-2.415}}$
	6/18	$\mathbf{2^{-6.81}}$				$\mathbf{2^{-6.80}}$
	7/18	$\mathbf{2^{-11.81}}$				$\mathbf{2^{-11.80}}$
	8/18					$\mathbf{2^{-32.70}}$
	9/18					$\mathbf{2^{-54.37}}$

Conclusion: AES, CLEFIA

- 3/4/5-round AES's DL distinguishers: $2^{-8.66}/2^{-27.85}/2^{-51.85}$

- searching for CLEFIA's DL distinguishers up to 9 round

- 9-round CLEFIA DL distinguisher: $2^{-54.37}$

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