Polytopes in the Fiat-Shamir with Aborts Paradigm

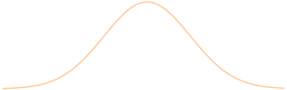
Crypto 2024

Henry Bambury [1,2], **Hugo Beguinet** [1,3], Thomas Ricosset [3], Éric Sageloli [1,3,4]

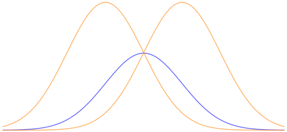
[1]DIENS, Inria Team CASCADE [2]DGA [3]Thales [4]École polytechnique

21st of August 2024

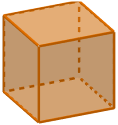
Rejection Sampling: A Brief History of Distributions



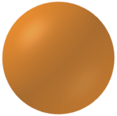
[Lyu12]



[DDLL13, CCD^{+}23]



[Lyu09, DKL^{+}21]



[CCD^{+}23]

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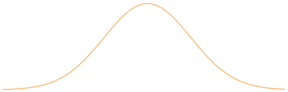

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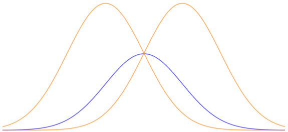

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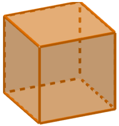

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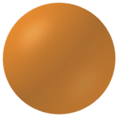

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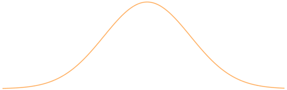

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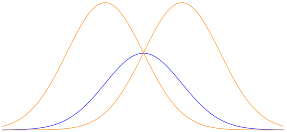

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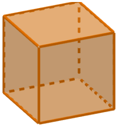

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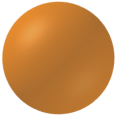
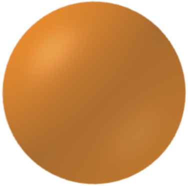


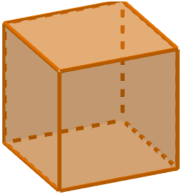
[CCD^{+}23]

Focus on uniform distributions.

Lattice-based FSwA Signatures: Haetae and Dilithium



$\longleftrightarrow$



Signature size	✔ ✔
Verification key	✔
Sampler	✘

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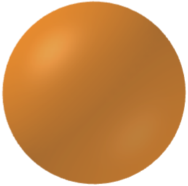

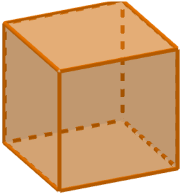

Signature size	✔ ✔	Signature size	✘ ✘
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Sampler	✘	Sampler	✔✔

Is there a shape embracing the best of both?

Outline

I. Intro: Fiat-Shamir and Rejection Sampling

II. The Polytope-based Framework

III. Choosing a Polytope $\mathcal{H}$

IV. In Application

Outline

Fiat-Shamir (with Aborts) on Lattice Assumptions.

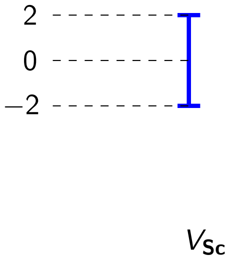
Notation: $V_{\mathbf{x}}$ the support of the distribution from which $\mathbf{x}$ is taken.

Signer		**Verifier**
$\mathsf{sk} = \mathbf{s}, \mu$		$\mathsf{vk} = \mathbf{As}$
$\mathbf{y} \leftarrow \mathcal{C}$		
$\mathbf{w} = \mathbf{Ay}$		
$c = \mathtt{H}(\mathbf{w}, \mu)$		
$\mathbf{z} = \mathbf{y} + \mathbf{S}c$		
$\boxed{\text{if } \mathbf{z} \in V_{\mathbf{z}}}$	$\xrightarrow{c, \mathbf{z}}$	$c == \mathtt{H}(\mathbf{Az} - \mathsf{vk} \cdot c)$

Goal: obtaining the shape of $V_{\mathbf{z}}$.

Rejection Sampling: Motivation

1D Example:



Remark:

- $V_{\mathbf{z}}$, $V_{\mathbf{y}}$ and $V_{\mathbf{Sc}}$ are all public.
- $\mathbf{z} = \mathbf{y} + \mathbf{Sc}$

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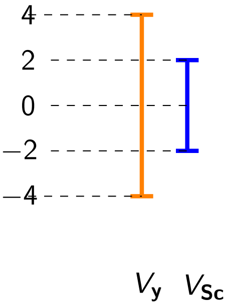

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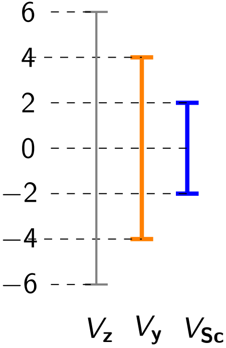

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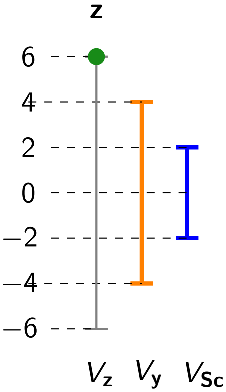

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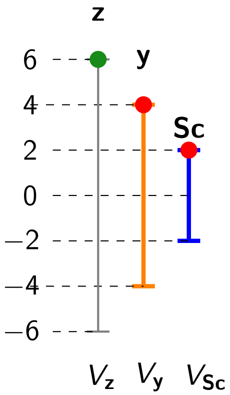

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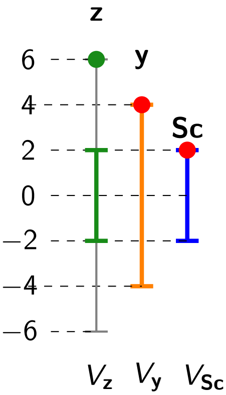

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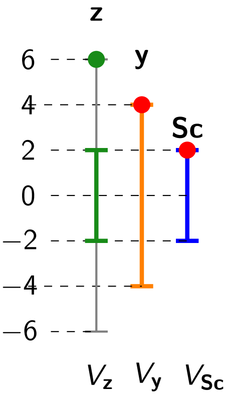

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$\mathbf{z}$ should reveal **no information** on $\mathbf{y}$ and $\mathbf{Sc}$.

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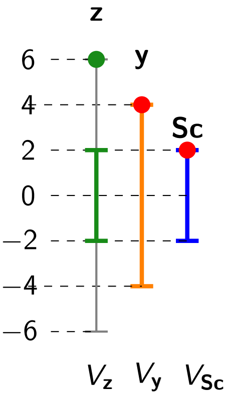

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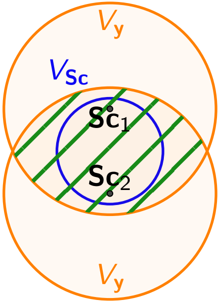
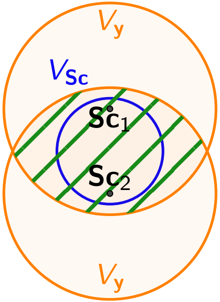
z should reveal **no information** on **y** and **Sc**.

How should $V_{\mathbf{z}}$ be?



- Possible $\mathbf{z}$ are in the green area.

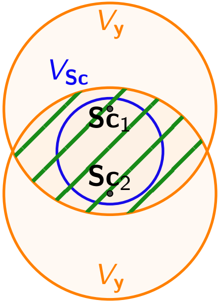
From V_z to ...



- Possible z are in the green area.
- z avoids information leakage if and only if:

$$V_z \subseteq \bigcap_{c \in V_{Sc}} (V_y + c).$$

From $V_{\mathbf{z}}$ to ...



- Possible $\mathbf{z}$ are in the green area.
- $\mathbf{z}$ avoids information leakage if and only if:

$$V_{\mathbf{z}} \subseteq \bigcap_{\mathbf{c} \in V_{\mathbf{Sc}}} (V_{\mathbf{y}} + \mathbf{c}).$$

The bigger $V_{\mathbf{z}}$ is, the lower the signature size becomes (at equal rejection rate):

$$V_{\mathbf{z}} = \bigcap_{\mathbf{c} \in V_{\mathbf{Sc}}} (V_{\mathbf{y}} + \mathbf{c}).$$

Roadmap

I. Intro: Fiat-Shamir and Rejection Sampling

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Polytope intersection: a useful tool

Theorem ($\mathcal{P}$-ception: Intersection of polytopes)

Let $\mathcal{P}$ be a symmetric inscriptible and circumscriptible polytope. Let $r, R \in \mathbb{R}_{>0}$ such that $R > r$ and $\mathcal{P}_r := r \cdot \mathcal{P}$. Then:

$$\bigcap_{\mathbf{c} \in \mathcal{P}_r} \mathcal{P}_R + \mathbf{c} = \mathcal{P}_{R-r}.$$

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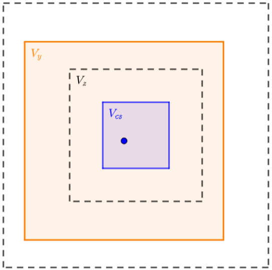

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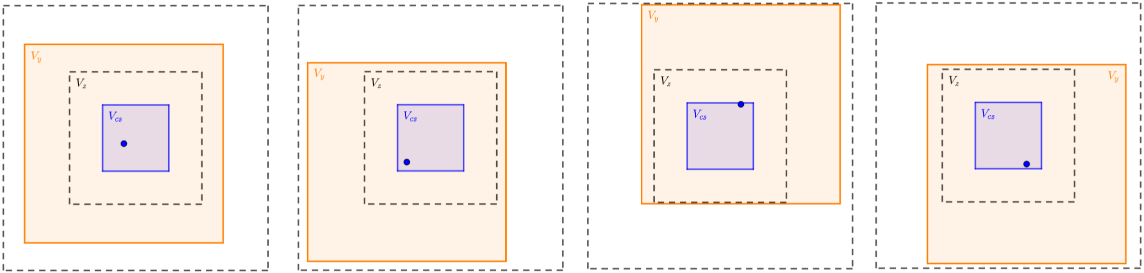
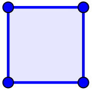


$\mathcal{P}$-ception case 1: Restriction to integral points



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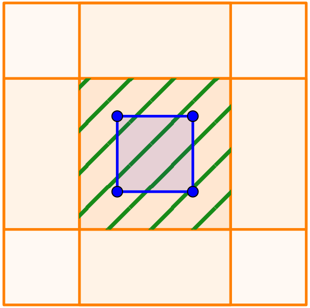

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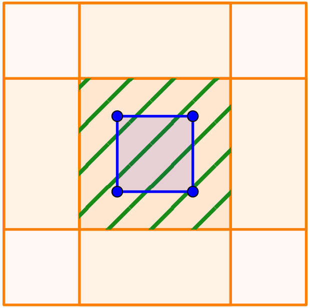


- Same result using only the vertices of $V_{\mathbf{Sc}}$.
- Yes and?

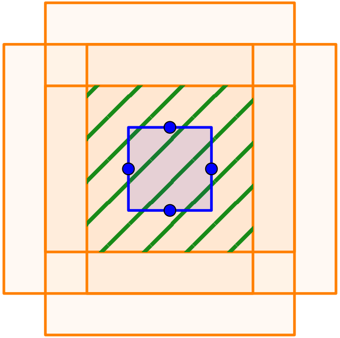
Wait, is it not better to work directly on integers?

Theorem ($\mathcal{P}$-ception: Generalization 1)

If $\mathcal{P}_r$ is an integral polytope, then:

$$\bigcap_{\mathbf{c} \in \mathcal{P}_r \cap \mathbb{Z}^n} \mathcal{P}_R \cap \mathbb{Z}^n + \mathbf{c} = \mathcal{P}_{R-r} \cap \mathbb{Z}^n.$$

$\mathcal{P}$-ception case 2: Different shape for $V_{\mathbf{Sc}}$



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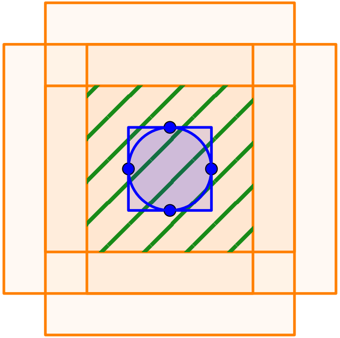

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In practice $V_{\mathbf{Sc}}$ is not a square but a sphere...

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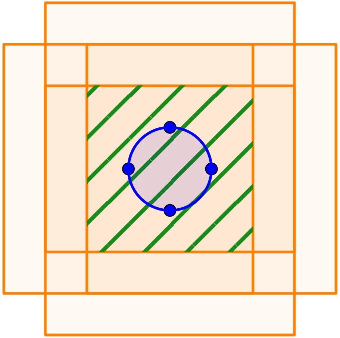

- Same result using only one point on each facet of $V_{\mathbf{Sc}}$.
- Again, yes and?

> In practice $V_{\mathbf{Sc}}$ is not a square but a sphere...

Theorem ($\mathcal{P}$-ception: Generalization 2)

If $\mathcal{S}$ is the inscribed sphere of $\mathcal{P}_r$, then:

$$\bigcap_{\boxed{\mathbf{c} \in \mathcal{S}}} \mathcal{P}_R + \mathbf{c} = \mathcal{P}_{R-r}.$$

Roadmap

Polytope Choice: Cutting a Rare Gem

What we want for $\mathcal{P}$:

- Verifies simple assumptions
- Integral vertices
- Efficiently samplable
- Small ratio

Definition (Ratio ρ)

Given the circumradius R of $\mathcal{P}$ and its in radius r:

$$\rho := \frac{R}{r}$$

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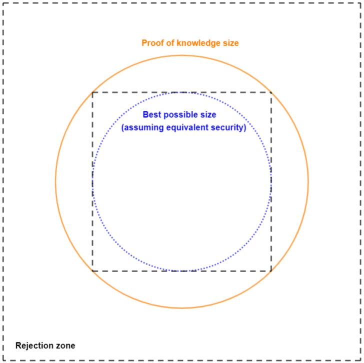
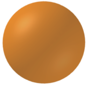
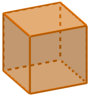
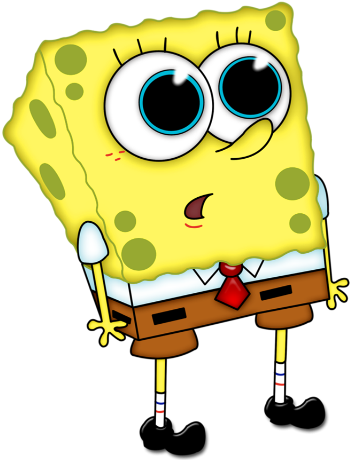


Where Should It Be?

	Signature	Verification Key	Sampling Method	Bimodal	Ratio
	✔✔	✔	✘✘	✔	1
					
	✘✘	✔	✔✔	✘	$\sqrt{n}$

Interlude: High-dimensional Balls



The Hypercube:

$$\mathcal{B}_\infty(R) = \{\mathbf{x} \in \mathbb{R}^n : \forall i, |x_i| \le R\}.$$

- Volume: $(2R)^n$.
- Radius ratio: $\sqrt{n}$.
- Mass concentrates: at the corners.

Interlude: High-dimensional Balls

The Cross-polytope[1]:

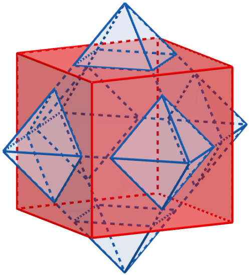
$$\mathcal{B}_1(R\sqrt{n}) = \{\mathbf{x} \in \mathbb{R}^n : \sum |x_i| \leq R\sqrt{n}\}.$$

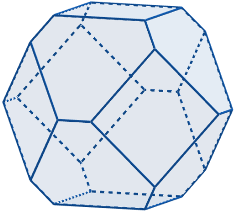
- Volume: $\frac{(2\sqrt{n}R)^n}{n!}$.
- Radius ratio: $\sqrt{n}$.
- Mass concentrates: at the center.

[1] also called Hyperoctahedron, Orthoplex, or Cocube.

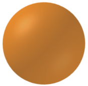
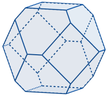
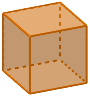
The Polytope $\mathcal{H}$

$$\mathcal{H}_r^n = \mathcal{B}_\infty^n(r) \cap \mathcal{B}_1^n(r\sqrt{n})$$





Recap Table

	Signature	Verification Key	Sampling Method	Bimodal	Ratio
	✔✔	✔	✘✘	✔	1
			✔	✘	$\sqrt[4]{n}$
	✘✘	✔	✔✔	✘	$\sqrt{n}$

Roadmap

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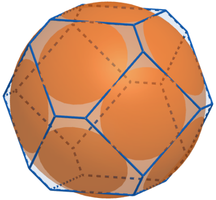
Reject More for Better Performances



$$\mathcal{C}^n_{\theta,r} = \mathcal{H}^n_r \cap \mathcal{B}_2(\theta \cdot r) \text{ with } \theta \approx 1.5$$

- Low rejection rate.
- Ratio: from $n^{1/4}$ to θ.
- θ decreases as (n, r) grows.
- **Warning**: not a polytope anymore.

A new Fiat-Shamir with Aborts Signature Scheme: PATRONUS

- **Signature sizes:** (in bytes)

Security target (bits)	120	180	260
HAETAE	1,463	2,337	2,908
PATRONUS (this work)	**2,070**	**2,575**	**3,721**
DILITHIUM	2,420	3,293	4,595

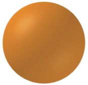
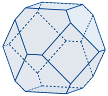
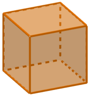
- **Verification key sizes:** (in bytes)

HAETAE	992	1,472	2,080
PATRONUS (this work)	**832**	**1,152**	**1,632**
DILITHIUM	1,312	1,952	2,592

- **Rejection rate:**

HAETAE	6	5	6
PATRONUS (this work)	**3**	**4.250**	**3**
DILITHIUM	4.250	5.1	3.850

Recap Table

	Signature	Verification Key	Sampling Method	Bimodal	Ratio
	✔✔	✔	✘✘	✔	1
	✔	✔✔	✔	✘	$\sqrt[4]{n} \to 1.5$
	✘✘	✔	✔✔	✘	$\sqrt{n}$

Takeaway

What you should remember:

- We propose a new framework for rejection sampling in polytopes.
- This allows for rigorous analysis of perfect rejection in Fiat-Shamir.
- Our polytope $\mathcal{H}$ uses L_1 and L_∞ balls to approach an optimal L_2 ball.
- It is easy to sample from $\mathcal{H}_\mathbb{Z}$.
- This leads to the signature scheme PATRONUS , an interesting tradeoff between DILITHIUM and HAETAE.

Roadmap

I. Intro: Fiat-Shamir and Rejection Sampling

II. The Polytope-based Framework

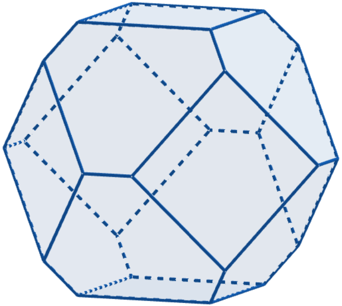
III. Choosing a Polytope $\mathcal{H}$

IV. In Application

V. Bonus: Open Questions and Perspectives

The End: Questions?

Thank you for listening!



Article: eprint.iacr.org/2024/411

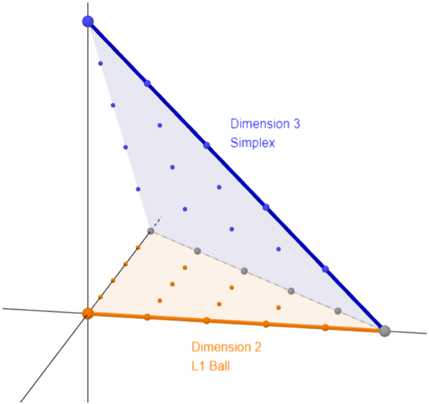
A useful projection

The following sets are isomorphic via a simple projection:

$$\mathcal{S}^{n+1}_{1,\mathbb{Z}^+}(r\sqrt{n}) = \{\mathbf{y} \in \mathbb{Z}^{n+1}_{\geq 0} : \|\mathbf{y}\|_1 = r\sqrt{n}\},$$

$$\mathcal{B}^{n}_{1,\mathbb{Z}^+}(r\sqrt{n}) = \{\mathbf{y} \in \mathbb{Z}^{n}_{\geq 0} : \|\mathbf{y}\|_1 \leq r\sqrt{n}\}.$$

Bonus trick: project away from the largest coordinate to lower $\mathbb{E}(\|\mathbf{y}\|_\infty)$.

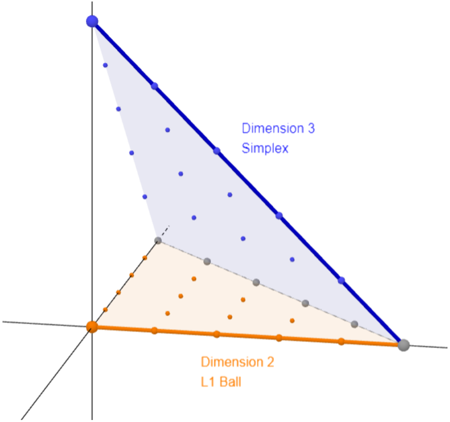


Making the Sampler Uniform and Isochronous

Mind the sides!

- Flip n coins for signs.
- Restart for each 0 coordinate, with probability $1/2$.

- Uniform: ✓
- IsoSignachronous: ✓
- Expected restarts: small if $n \ll r$.



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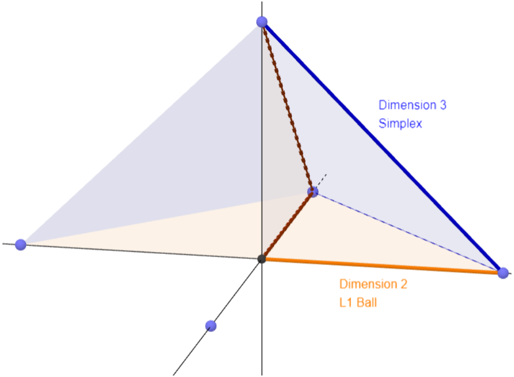

Can we get a Better Polytope?

Theorem (From [Kas77])

There exists a constant $1 < c < 32$ such that for each n, there exists an orthogonal $U \in \mathcal{O}_n(\mathbb{R})$ such that

$$\mathcal{B}_2^n(1) \subseteq \mathcal{B}_1^n(\sqrt{n}) \cap U\mathcal{B}_1^n(\sqrt{n}) \subseteq \mathcal{B}_2^n(c).$$

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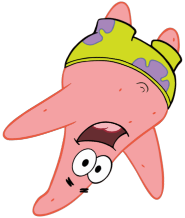
$\bigcap$



$=$

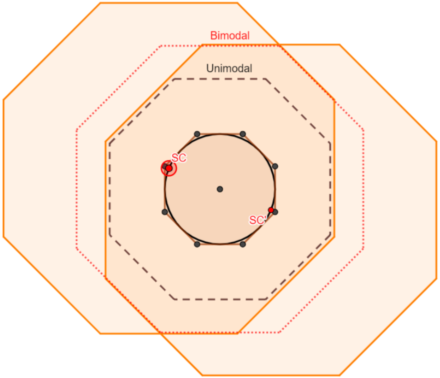
The Bimodal Situation

Objective: Use the trick by [DDLL13] for better sizes.

- We need to study

$$I = \bigcap_{\mathbf{sc} \in \mathcal{B}_2(r)} (\mathcal{P}_{R,\mathbf{sc}} \cup \mathcal{P}_{R,-\mathbf{sc}})$$

- No improvement in the Hypercube case.
- For $\mathcal{H}$, no obvious improvement after dim 4 as the largest $\mathcal{H}$ in I is $\mathcal{H}_{R-r}$.
- For $\mathcal{C}$, less unlikely.



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