On the practical CPA^D security of "exact" and threshold FHE schemes and libraries

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France 2030 ANR Programs SecureCompute & TRUSTINCloudS
Horizon Europe Program

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   - Security model and CPA^D game
2. A CPA^D attack on “exact” FHE schemes
3. Impact on Threshold FHE
4. Countermeasures for “exact” and Threshold Schemes
5. Conclusion and Key takeaways

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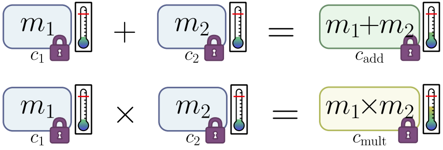
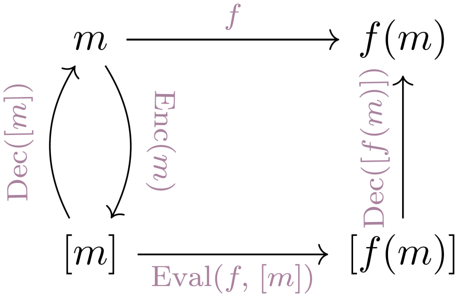
Homomorphic Encryption

What are we talking about?





Ensure confidentiality *during* calculations

Typically, for Cloud Computing

Homomorphic Encryption

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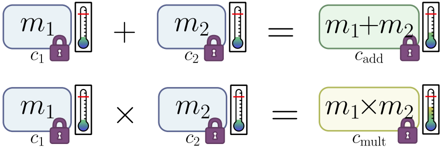

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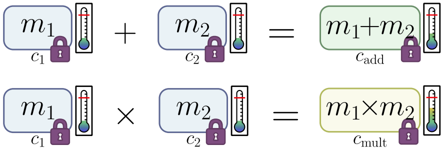

$$c = \left(a \ , \ b = m + \langle a, s \rangle + e \right)$$

FHE relies on **LWE**

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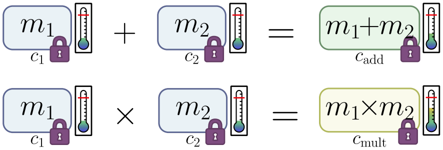
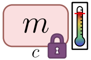


$$c = \left(a \;,\; b = m + \langle a, s \rangle + e \right)$$

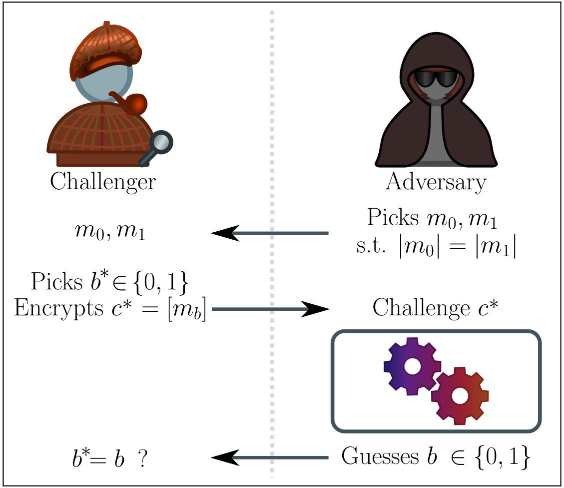
FHE relies on **LWE**

To ensure security, **noise is added during encryption**

but **it increases** at each homomorphic operation,

and may lead...  to **incorrect decryptions**!

CPA, CCA and CPA^D security game



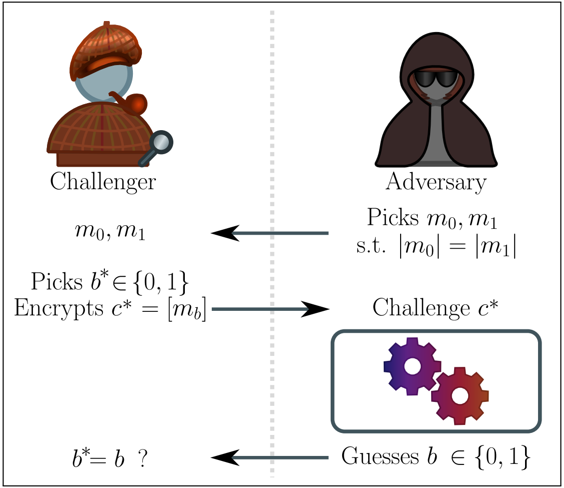
CPA, CCA and CPA^D security game

CPA Security Game

Chosen Plaintext Attack

- ➢ Encryption oracle

  ✔ FHE



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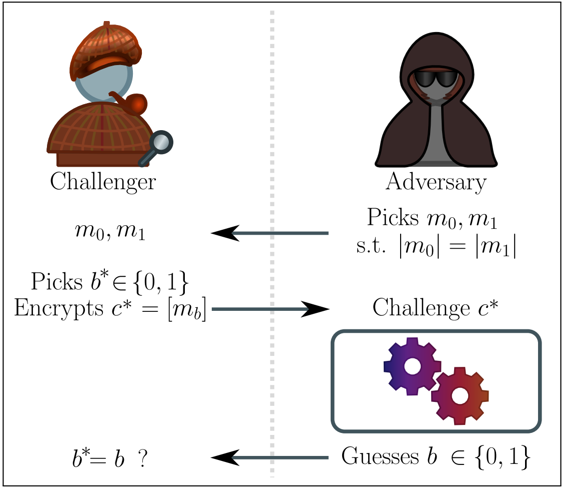
CCA Security Game

Chosen Ciphertext Attack

- ➢ Encryption oracle
- ➢ Decryption oracle

✘ FHE



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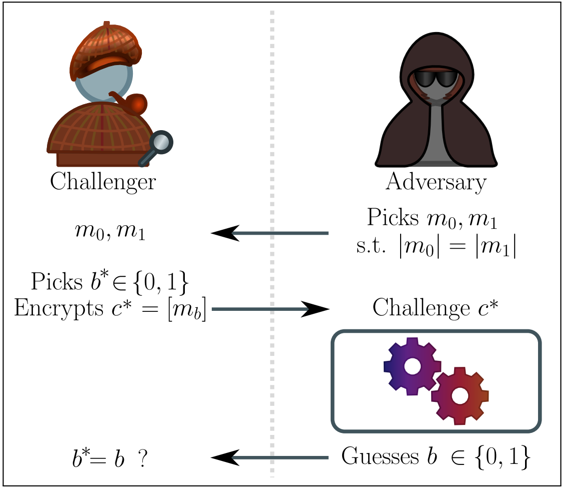

CPA^D Security Game

Chosen Plaintext Attack with "Decryption oracle"

➢ Encryption oracle

➢ Evaluation oracle

➢ Limited Decryption oracle on well-formed ctxt

? FHE

Li & Micciancio. *On the security of homomorphic encryption on approximate numbers*. EUROCRYPT'21

CPA = CPA^D?

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- CPA^D = CPA + Limited Decryption Oracle
- The adversary seems to know all the output of the decryption oracle!

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Li & Micciancio attack on **CKKS Approximate LWE Scheme** - EUROCRYPT'21

$$\begin{aligned}
&\text{CKKS.Encrypt}(m): &&\text{return} && (c_0, c_1) &= (m - a \cdot s + e,\ a), \qquad \text{with } a \xleftarrow{\$} \mathbb{Z}_q,\ e \xleftarrow{\$} \chi \\
&\text{CKKS.Decrypt}((c_0, c_1)): &&\text{return} && c_0 + c_1 \cdot s &= m - a \cdot s + e + a \cdot s \\
& && && &= m + \boxed{e} \\
& && && &\simeq m
\end{aligned}$$

CPA $=$ CPA^D?

- CPA^D = CPA + Limited Decryption Oracle
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&&&&& = m + \boxed{e} \\
&&&&& \simeq m
\end{aligned}$$

Compare to usual "Exact" LWE Schemes

$$\begin{aligned}
&\text{Encrypt}(m): && \text{return } (c_0, c_1) && = (Bm - a\cdot s + e,\ a), \quad \text{with } a \xleftarrow{\$} \mathbb{Z}_q,\ e \xleftarrow{\$} \chi \\
&\text{Decrypt}((c_0, c_1)): && \text{return } \lceil (c_0 + c_1 \cdot s)/B \rfloor && = \lceil (Bm - a\cdot s + e + a\cdot s)/B \rfloor \\
&&&&& = m
\end{aligned}$$

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Find noiseless ciphertexts

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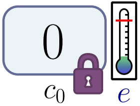
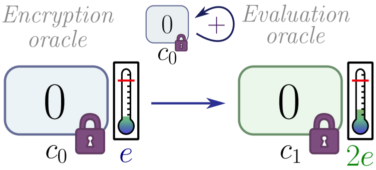
Encryption oracle



Find noiseless ciphertexts



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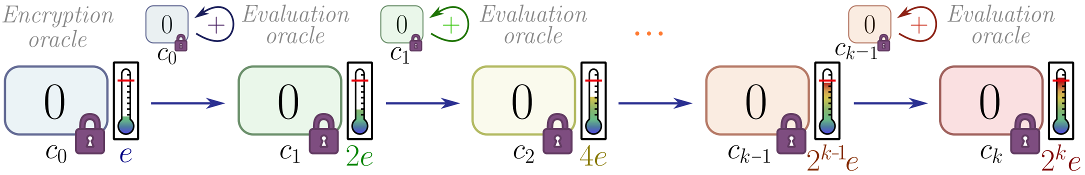

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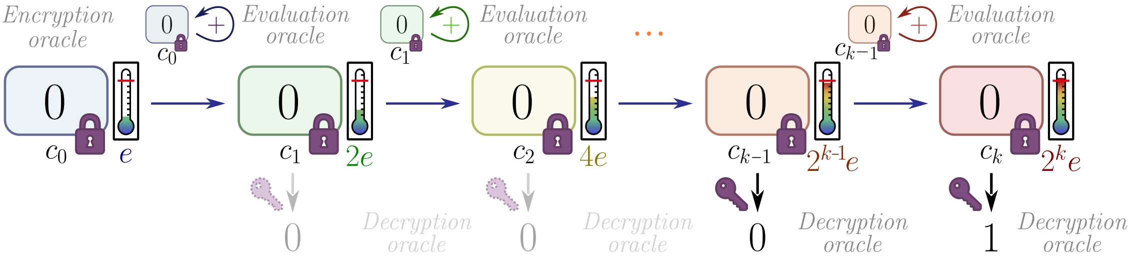
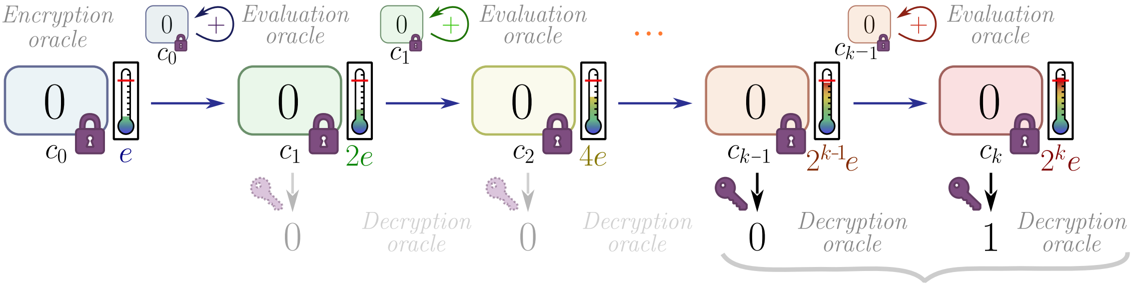


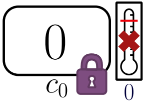
Find noiseless ciphertexts



$c_k = c_{k-1} + c_{k-1}$ until
$= (2^k a, \langle 2^k a, s\rangle + 2^k e)$

either $\mathsf{Dec}(c_{k-1})=0$ & $\mathsf{Dec}(c_k)=1$

or $\mathsf{Dec}(c_k)=0$ & $2^k \geq q/2t$



A CPAD Key recovery attack - Dichotomic search for $|e|$ (1/3)

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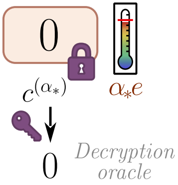
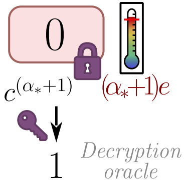
$c^{(\alpha_*)}$ encryption of 0 with noise $\alpha_* e$



$c^{(\alpha_*+1)}$ encryption of 0 with noise $(\alpha_* + 1)e$



A CPAD Key recovery attack - Dichotomic search for $|e|$ (1/3)

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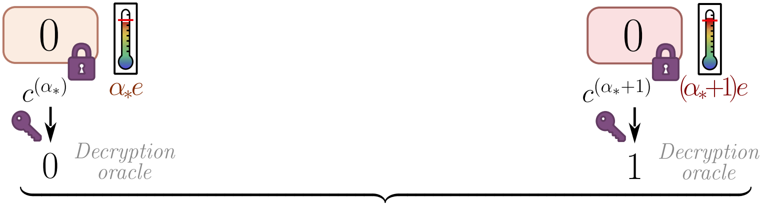

with $\frac{q}{2t(\alpha_*+1)} \leqslant |e| < \frac{q}{2t\alpha_*}$

➢ $|e|$ is uniquely determined when $\left\lceil \frac{q}{2t(\alpha_*+1)} \right\rceil = \left\lfloor \frac{q}{2t\alpha_*} \right\rfloor$

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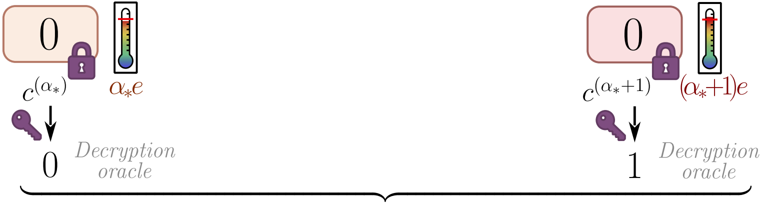

with $\frac{q}{2t(\alpha_*+1)} \leqslant |e| < \frac{q}{2t\alpha_*}$

- $|e|$ is uniquely determined when $\left\lceil \frac{q}{2t(\alpha_*+1)} \right\rceil = \left\lfloor \frac{q}{2t\alpha_*} \right\rfloor$ $\rightarrow$ Occurs when $|e| < \sqrt{\frac{q}{2t}}$

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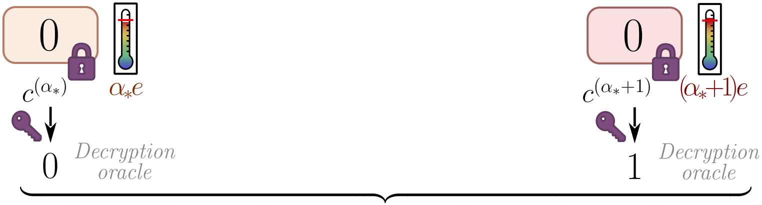

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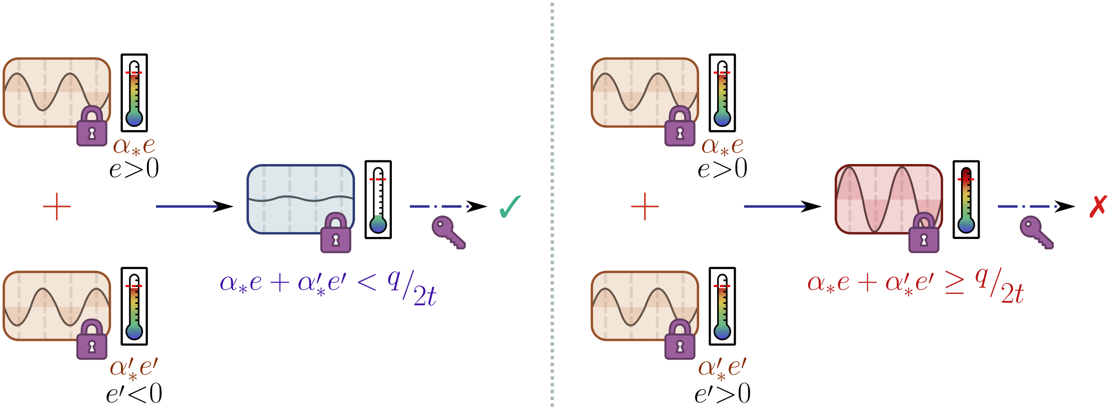
- Construct $c_k = c_{k-1} + c_{k-1}$ and $c^{(\alpha)} := \sum_k (\alpha_k = 1) c_k$, then $c^{(\alpha)} = (\alpha a, \langle \alpha a, s\rangle + \alpha e)$

Identify ciphertexts with same noise sign.

$\hookrightarrow$ *Evaluation and decryption oracles*



A CPAD Key recovery attack - Dichotomic search for $|e|$ (3/3)

Solve two systems of n linear equations to recover the key

$$\begin{cases} b_1 &= \langle a_1, s \rangle + |e_1| \\ b_2 &= \langle a_2, s \rangle + |e_2| \\ b_3 &= \langle a_3, s \rangle + |e_3| \\ b_4 &= \langle a_4, s \rangle + |e_4| \\ \cdots &= \cdots \\ b_n &= \langle a_n, s \rangle + |e_n| \end{cases} \qquad \begin{cases} b_1 &= \langle a_1, s \rangle - |e_1| \\ b_2 &= \langle a_2, s \rangle - |e_2| \\ b_3 &= \langle a_3, s \rangle - |e_3| \\ b_4 &= \langle a_4, s \rangle - |e_4| \\ \cdots &= \cdots \\ b_n &= \langle a_n, s \rangle - |e_n| \end{cases}$$

Try to decrypt fresh encryptions of 0 → the correct key always outputs 0

Win the CPA^D game by decrypting the challenge ciphertext c^*!

Some experimental results

Library	Scheme	Parameters					Proportion of ctxt with $\|e\|$ recovered	Proportion of noisefree ctxt	Time
		λ	n	$\log_2(q)$	σ	t			
SEAL	BFV	95	4096	109	3.2	1024	1	6250/232858	$2m50s$
	BFV	227	4096	58	3.2	1024	1	1481/860557	$1m20s$
	BGV	227	4096	58	3.2	1024	1	124/65405	$52s$
OpenFHE	BFV	128	8192	120	3.19	1024	1	69/48929	$19m30s$
	BFV	256	16384	120	3.19	1024	1	173/130535	$75m30s$
	BGV	128	8192	69	3.19	1024	1	59/32811	$18m30s$
	BGV	256	16384	71	3.19	1024	1	80/65559	$68m50s$
TFHElib	TFHE	97	630	32	2^{17}	2	1295/5427	0	$0.245s$
	TFHE	128	700	32	81604.378	2	1363/3678	0	$0.195s$
	TFHE	128	1024	32	81604.378	4	2070/5608	0	$0.412s$
	TFHE	128	1024	32	279.172	16	2021/2041	11/2041	$0.237s$
Lattigo	BFV	95	4096	109	3.2	65537	1	785/29241	$5m50s$
	BFV	98	2048	54	3.2	65537	1	69/24518	$46s$
	BFV	106	4096	101	3.2	65537	1	829/32260	$6m40s$
	BFV	106	8192	202	3.2	65537	1	457/23943	$52m00s$
	BFV	217	4096	60	3.2	65537	1	828/31934	$1m25s$

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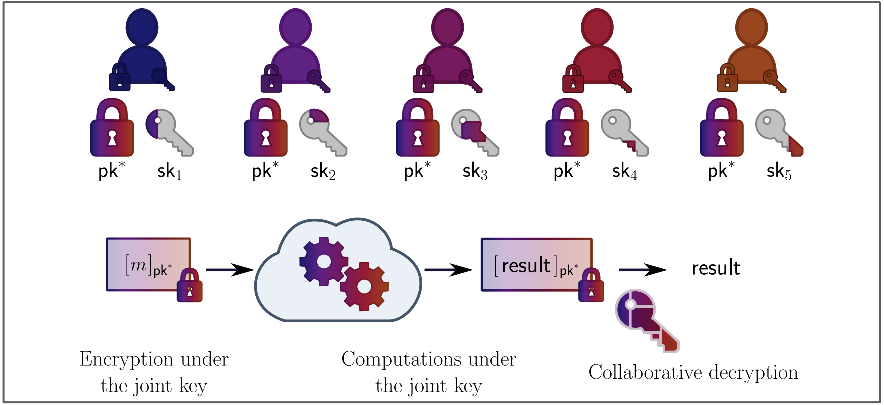
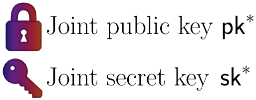
Impact on Threshold FHE





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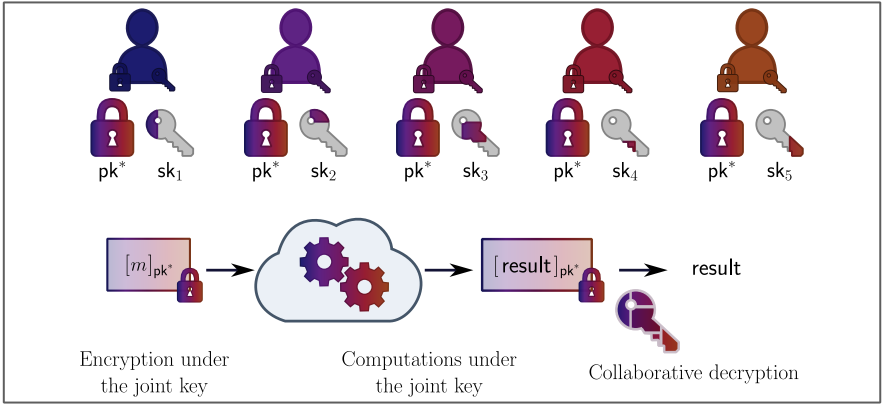
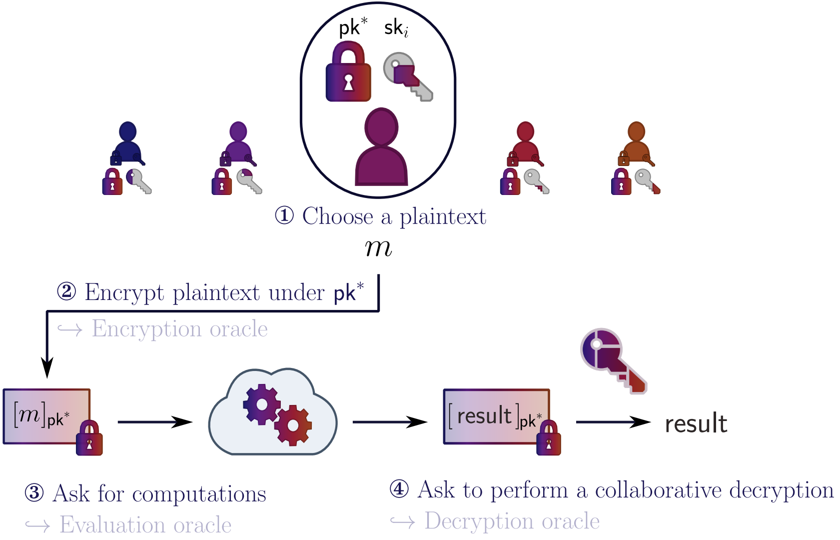


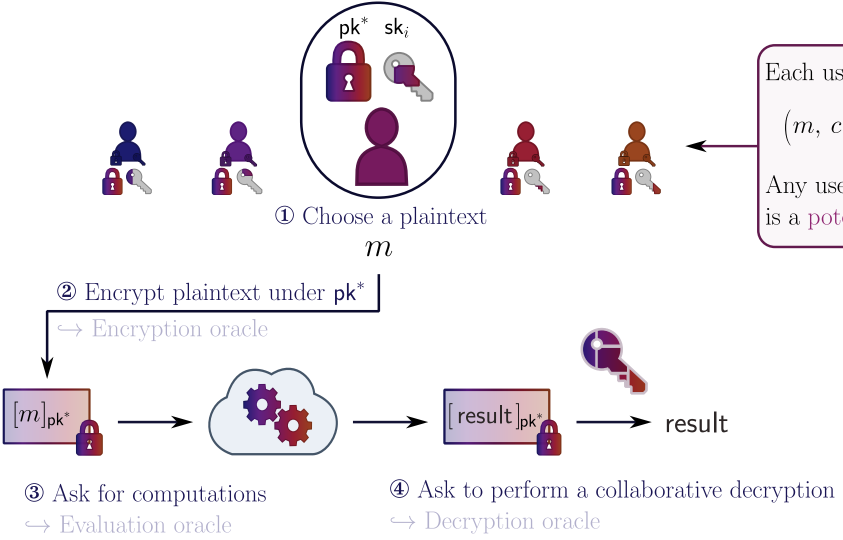
Joint public key pk^* — $\mathsf{pk}^* = \sum \mathsf{pk}_i$:

Joint secret key sk^* — $\mathsf{sk}^* = \sum \mathsf{sk}_i$:

Relationship between CPA^D and Threshold FHE



Relationship between CPA^D and Threshold FHE



Each user has

$$\big(m,\ c = \mathsf{Enc}_{\mathsf{pk}^*}(m),\ \mathsf{Dec}_{\mathsf{sk}^*}(c)\big)$$

Any user in a threshold scheme is a potential CPA^D attacker

Does the attack work against Threshold FHE schemes?

Algorithm 1: Collective Key Switch

Input: Ciphertext $ct = (c_0, c_1)$ of variance σ_{ct}^2
Private input: s_i, s'_i for each party P_i
Output: Key-switched ciphertext $ct' = (c'_0, c_1)$
Each *party* P_i
- Samples $e_i \leftarrow \chi_{CKS}(\sigma_{ct}^2)$
- Computes and Discloses $h_i = (s_i - s'_i) \cdot c_1 + e_i$

return $ct' = (c_0 + \sum_{P_i} h_i, c_1)$

Mouchet et al. *Multiparty Homomorphic Encryption from Ring-Learning-with-Errors*. PoPETs'21

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$$c_k^{(\mathrm{smg})} = (2^k a, \langle 2^k a, s\rangle + 2^k e + e_{\mathrm{smg}}) \quad \text{indistinguishable from} \quad c_k = (2^k a, \langle 2^k a, s\rangle + e_{\mathrm{smg}}),$$

$$\text{where } \sigma_{\mathrm{smg}} = \sigma_{\mathrm{ct}}\sqrt{K}2^{\frac{\lambda}{2}} \text{ and } \sigma_{\mathrm{ct}} = 2^k\sigma$$

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Countermeasures

- **Bootstrapping** ($\sim$ 50% cost)
  - **Bootstrap** after each homomorphic operation
  - Since bootstrapping resets the noise variance to a preset value, decryption errors cannot occur.
  - Choose FHE parameters such that bootstrapping errors occur with prob neg(λ).
- **Monitor & Block** ($\sim$ 35% cost)
  - Fix a noise deviation budget B.
  - Choose FHE parameters such that decryption error occur with prob neg(λ) at noise dev. B.
  - **Monitor** (worst-case) noise deviation during FHE execution.
  - **Block**: return $\perp$ when noise deviation $>$ B.
- **Monitor & Smudge** ($\sim$ 45% cost)
  - Prior to decryption, **flood/smudge** the ciphertext with a large λ-dependent and σ_{ct}-dependent variance.
  - Works for threshold scheme (and must not be optional)

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Concurrent works

- **Guo et al.**
  Key recovery attacks on approximate homomorphic encryption with non worst-case noise flooding countermeasures. Usenix Security 2024
  - CPA^D attack on CKKS, when smudging based on non worst-case noise estimation

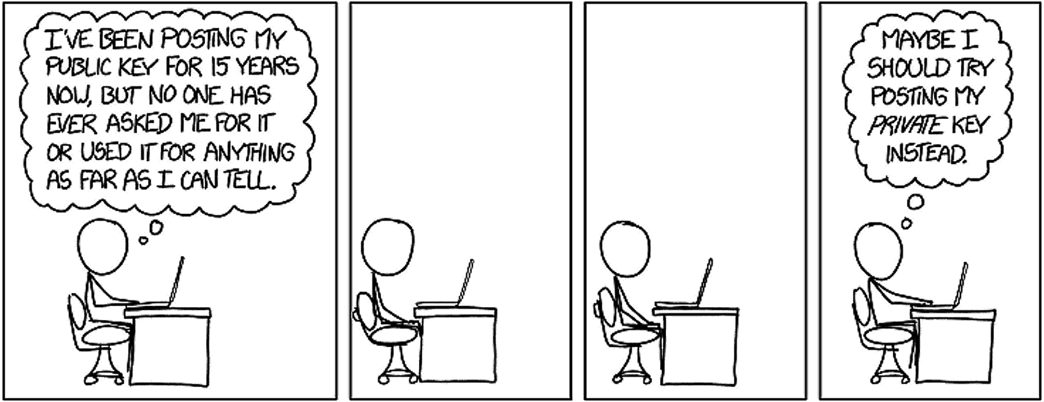
- **Cheon et al.**
  Attacks Against the IND-CPA^D Security of Exact FHE Schemes. IACR Eprint 2024/127
  - BGV/BFV CPA^D attack, migrate the noise polynomial in the plaintext domain
  - TFHE CPA^D attack, exploit bootstrapping error

- **Alexandru et al.**
  Application-aware approximate homomorphic encryption: configuring FHE for practical use. IACR Eprint 2024/203
  - Application-aware security: new *weaker* variant of CPA^D security
  - CPA^D security should be defined relative to a circuit class and a noise estimation strategy

Key Takeaways

- CPA^D is not just a theoretical threat, thus...
  CPA^D security must be carefully considered by all FHE schemes
- Simple CPA^D attacks can be implemented in most popular FHE libraries, but...
  Simple countermeasures can be devised, but have an impact on performance
- Recall that CPA^D is a natural security context in multi-user threshold FHE, so...
  Recall to have smudging appropriately implemented in your favorite threshold library

Thank you for your kind attention!



credit: xkcd.com

Generalization to RLWE

LWE

$$a \xleftarrow{\$} \mathbb{Z}_q,\, e \xleftarrow{\$} \chi(\mathbb{Z}_q)$$
$$c = (a, b := m + \langle a, s\rangle + e)$$

RLWE

$$A \xleftarrow{\$} \mathbb{Z}_q[X]/X^n{+}1,\, E \xleftarrow{\$} \chi(\mathbb{Z}_q[X]/X^n{+}1)$$
$$C = (A, B := M + A \cdot S + E)$$

➢ Just have to look at one coefficient of the RLWE polynomial: it is an LWE instance!

$$C = c_0 + \boxed{c_1} X + c_2 X + \cdots + c_{n-1} X^{n-1}$$

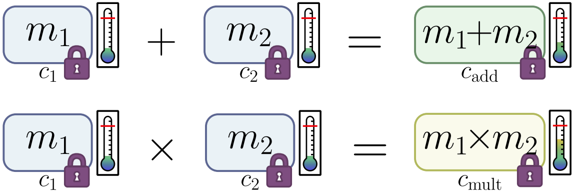
Bootstrapping

And boot...boot... What? Bootstrapping!

What is it and what for?

Bootstrapping (∼50% cost).

- Bootstrap after **each** homomorphic operation.
- Since bootstrapping resets the noise variance to a preset value, decryption errors cannot occur.
- Choose FHE parameters such that bootstrapping errors occur with prob neg(λ).



Noise grows with each homomorphic operations.
We need to regularly reduce the noise:
that's bootstrapping!

Monitor & Block

Monitor & Block (∼35% cost).

- Fix a noise deviation budget B.
- Choose FHE parameters such that decryption error occur with prob neg(λ) at noise dev. B.
- Monitor (worst-case) noise deviation during FHE execution.
- **Block** decryption when noise dev. $> B$.
- Scheme becomes "somewhat correct".

d	$\log_2(q)$	n	$\log_2(q)$	n	ratio
1	120	8192	131	8192	1,09
2	180	8192	181	8192	1,00
3	180	8192	237	16384	2,96
4	240	16384	289	16384	1,35
5	240	16384	341	16384	1,68
6	300	16384	392	16384	1,46
7	300	16384	444	16384	1,66
8	360	16384	516	32768	3,37
9	360	16384	570	32768	3,93
10	420	16384	624	32768	3,65

Illustration of the performance cost of the Monitor&Block countermeasure for OpenFHE/BFV.

Monitor & Smudge

Monitor & Smudge (∼45% cost).

- ➢ Prior to decryption, flood the ciphertext with a large λ-dependent and σ_{ct}-dependent variance.
- ➢ Works for threshold scheme
  $\hookrightarrow$ **must not be optional!**

d	$\log_2(q)$	n	$\log_2(q)$	n	ratio
1	120	8192	153	8192	1,28
2	180	8192	202	8192	1,12
3	180	8192	258	16384	3,22
4	240	16384	310	16384	1,45
5	240	16384	362	16384	1,79
6	300	16384	414	16384	1,55
7	300	16384	483	32768	3,99
8	360	16384	537	32768	3,70
9	360	16384	591	32768	4,30
10	420	16384	645	32768	3,95

Illustration of the performance cost of the Monitor&Smudge countermeasure for OpenFHE/BFV and K-out-of-K decryption, with $K = 5$.

Correctness and CPA^D

Correctness

A scheme is a correct/exact scheme if

$$\mathbf{P}\left(\mathsf{Dec}\left(\mathsf{Enc}(m,r)\right) \neq m\right) \leqslant \text{neg}(\lambda)$$

and

$$\mathbf{P}\left(\mathsf{Dec}\left(\mathsf{Eval}\left(f, \mathsf{Enc}(m_1, r_1), \ldots, \mathsf{Enc}(m_k, r_k)\right)\right) \neq f(m1, \ldots, m_k)\right) \leqslant \text{neg}(\lambda)$$

If the scheme is correct/exact, our attack is not applicable

Li & Micciancio, EUROCRYPT'21, **Lemma 1.**

"Any exact homomorphic encryption scheme $\mathscr{E}$ is IND-CPA secure if and only if it is IND-CPA^D secure."

CPA^D Security Game

Encryption scheme $\mathscr{E} = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Eval})$, plaintext domain $\mathscr{P}$ and security parameter λ. Adversary $\mathscr{A}$.

Game parameterized by $b^* \xleftarrow{\$} \{0, 1\}$ unknown to $\mathscr{A}$, and an initially empty state S of msg-msg-ctxt triplets:

- **Key generation.** Run $(\mathsf{ek}, \mathsf{dk}) \leftarrow \mathsf{KeyGen}(1^\lambda)$, and give ek to $\mathscr{A}$.
- **Encryption request.** $\mathscr{A}$ queries $(\texttt{test\_messages}, m_0, m_1)$, where $m_0, m_1 \in \mathscr{P}$.
  Compute $c = \mathsf{Enc}_{\mathsf{ek}}(m_{b^*})$, give c to $\mathscr{A}$ and do $S := [S; (m_0, m_1, c)]$.
- **Evaluation request.** $\mathscr{A}$ queries $(\texttt{eval}, f, l_1, \ldots, l_K)$.
  Compute $m'_0 = f(S[l_1].m_0, \ldots, S[l_K].m_0)$, $m'_1 = f(S[l_1].m_1, \ldots, S[l_K].m_1)$, and $c' = \mathsf{Eval}(f, S[l_1].c, \ldots, S[l_K].c)$. Update S as follows: $S := [S; (m'_0, m'_1, c')]$
- **Decryption request.** $\mathscr{A}$ queries $(\texttt{ciphertext}, l)$.
  If $S[l].m_0 \neq S[l].m_1$, return $\perp$. Otherwise return $\mathsf{Dec}_{\mathsf{dk}}(S[l].c)$.
- **Guessing stage.** $\mathscr{A}$ outputs $(\texttt{guess}, b)$.
  If $b = b^*$, $\mathscr{A}$ wins the game, otherwise $\mathscr{A}$ looses it.