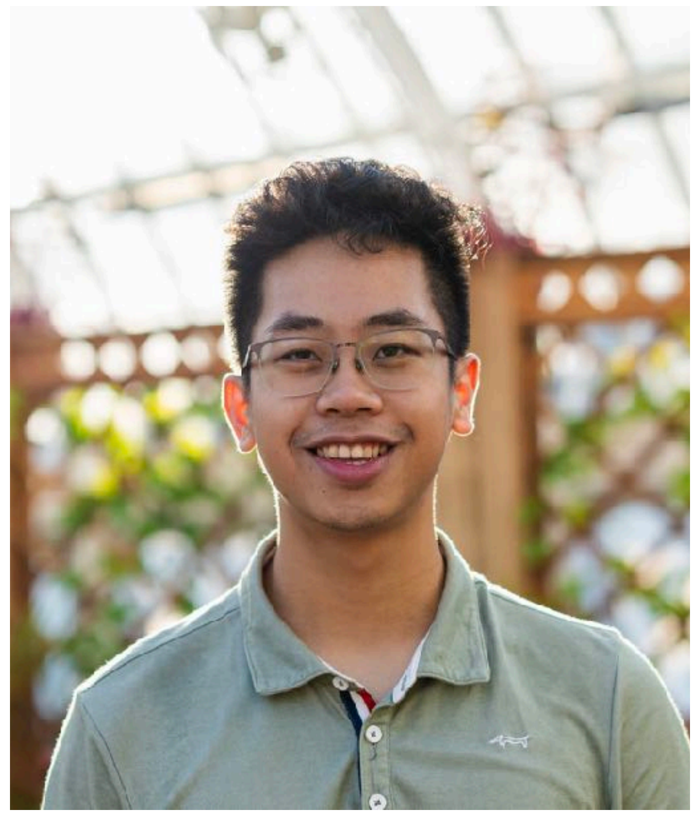
Lossy Cryptography from Code-Based Assumptions

Quang Dao

Aayush Jain



Crypto 2024

Post-Quantum Cryptography

Post-Quantum Cryptography

Exciting time for standardization & industry adoption!

CRYSTALS

Cryptographic Suite for Algebraic Lattices

FALCON

SPHINCS$^+$

Stateless hash-based signatures

Defending against future threats: Cloudflare goes post-quantum

10/03/2022

Quantum Resistance and the Signal Protocol

ehrenkret on 19 Sep 2023

February 21, 2024

iMessage with PQ3: The new state of the art in quantum-secure messaging at scale

How Google is preparing for a post-quantum world

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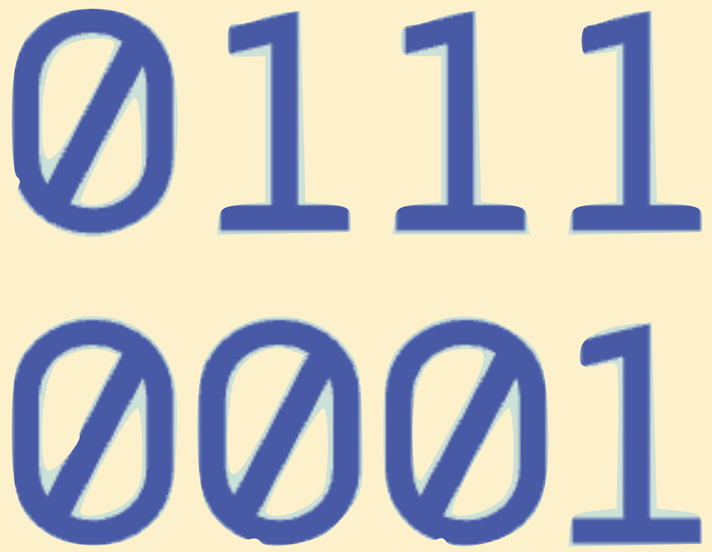
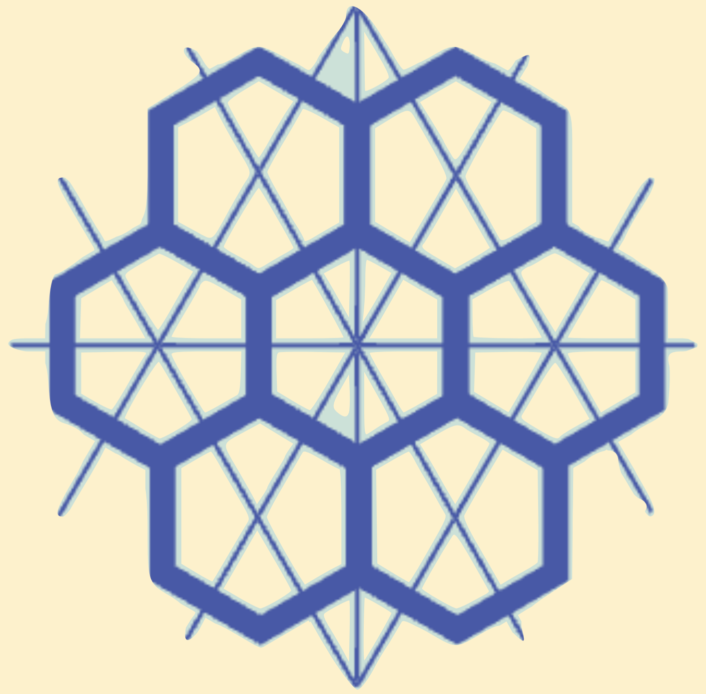
Problem: Not enough ways to construct post-quantum primitives!

The Current State of Post-Quantum Crypto

Code-based

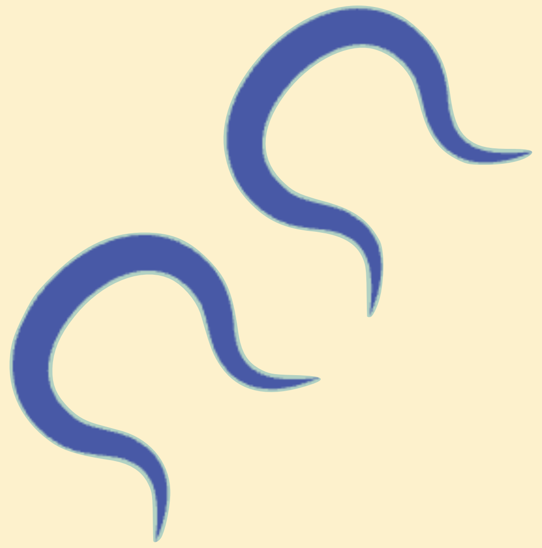


Lattice-based



Multivariate-based

Isogeny-based

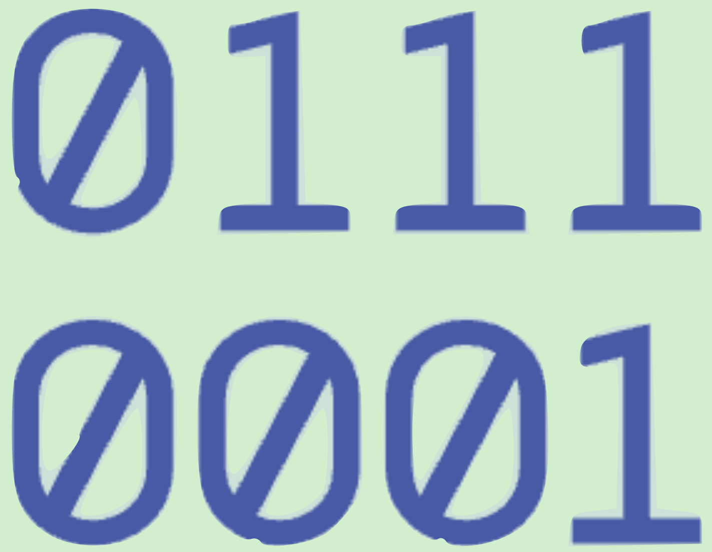


Images from Quanta Magazine

The Current State of Post-Quantum Crypto

One-way Functions

Code-based



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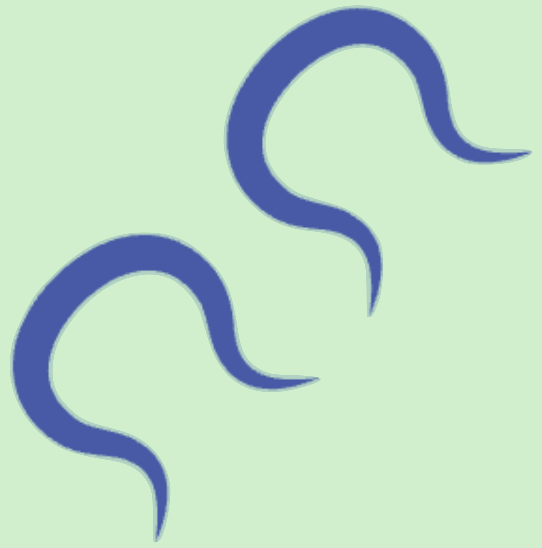

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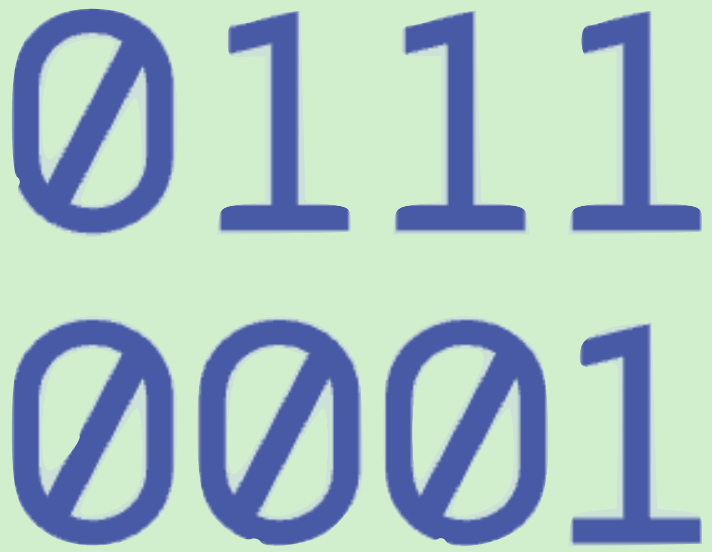
The Current State of Post-Quantum Crypto

Public-Key Encryption

Code-based

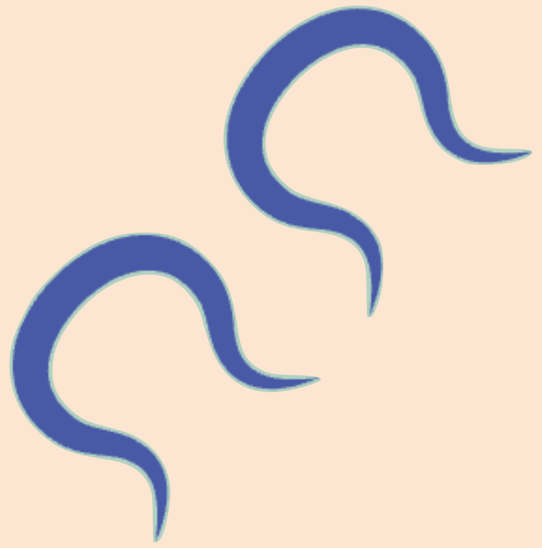


Lattice-based

Multivariate-based

many proposals, most are broken

Isogeny-based



SIDH broken, but still CSIDH, new proposals, etc.

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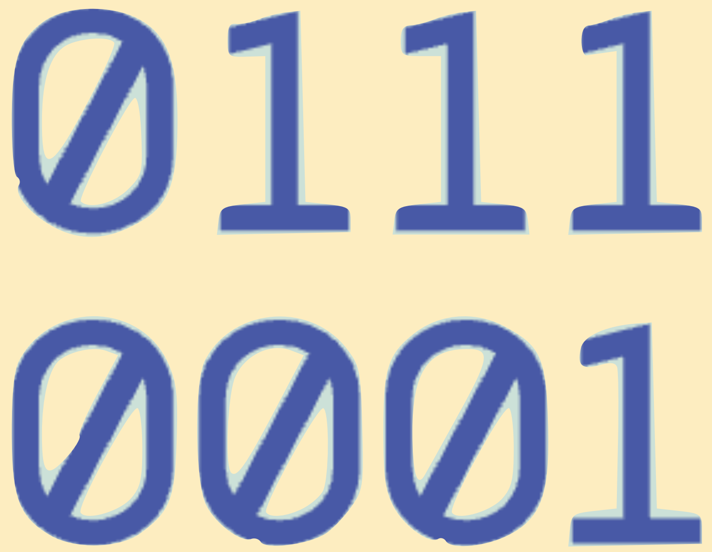
The Current State of Post-Quantum Crypto

Identity-Based Encryption

Code-based

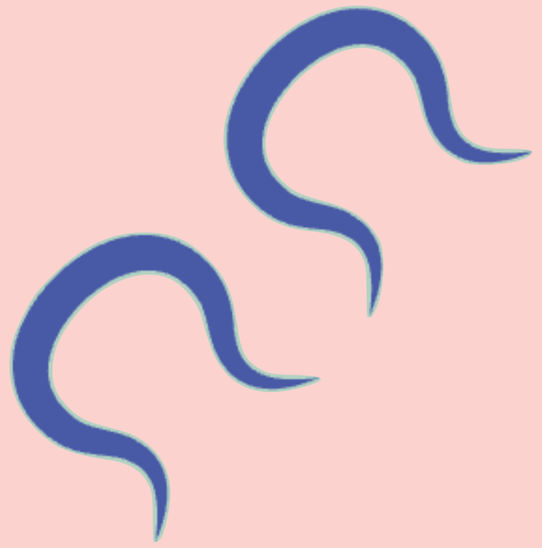


* Only quasi-polynomially secure

Lattice-based

Multivariate-based

Isogeny-based



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The Current State of Post-Quantum Crypto

Proof Systems (BARGs, SNARGs), Advanced Encryption (ABE, FE, FHE), …

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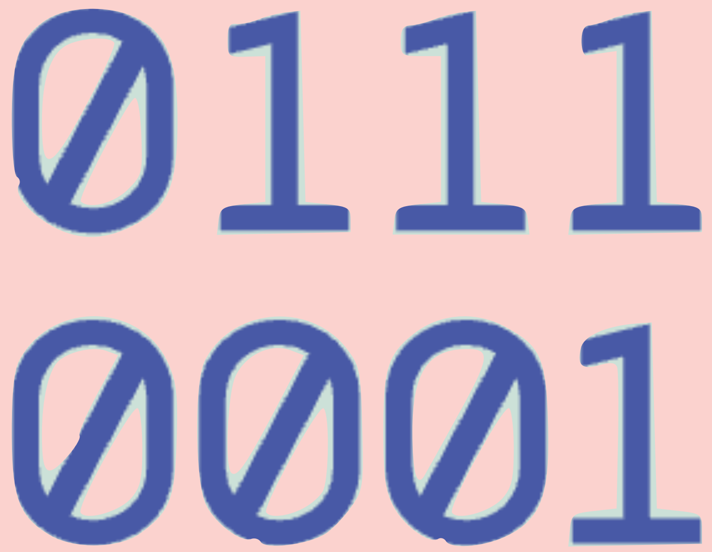

Lattice-based

Multivariate-based

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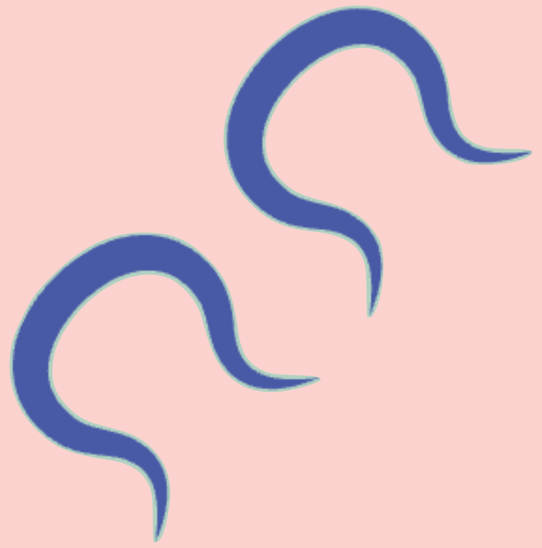

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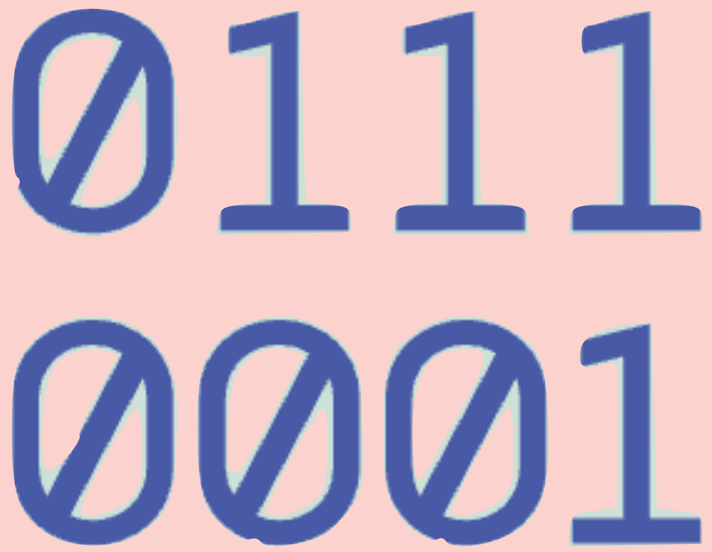
The Current State of Post-Quantum Crypto

Program Obfuscation

Code-based



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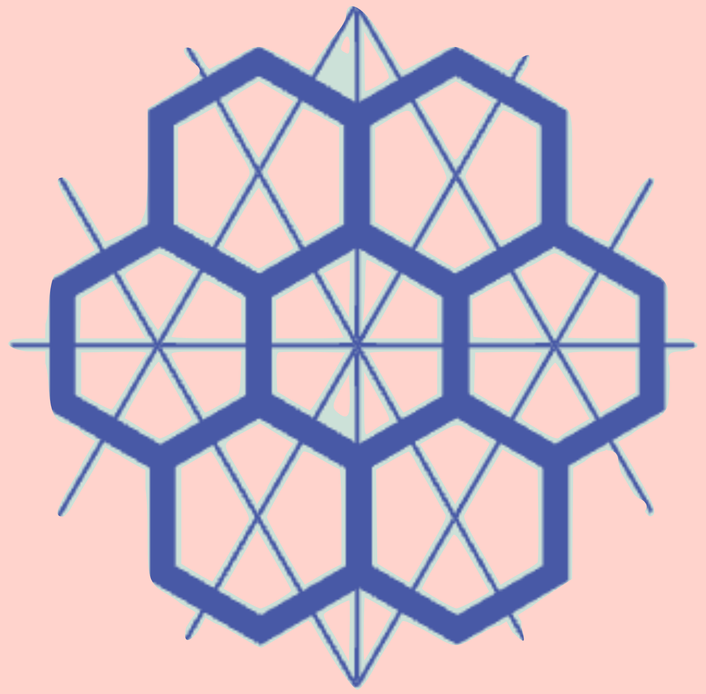
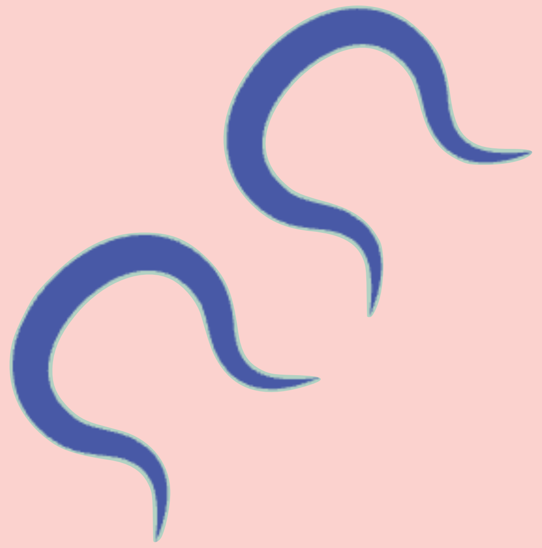


Multivariate-based

Isogeny-based



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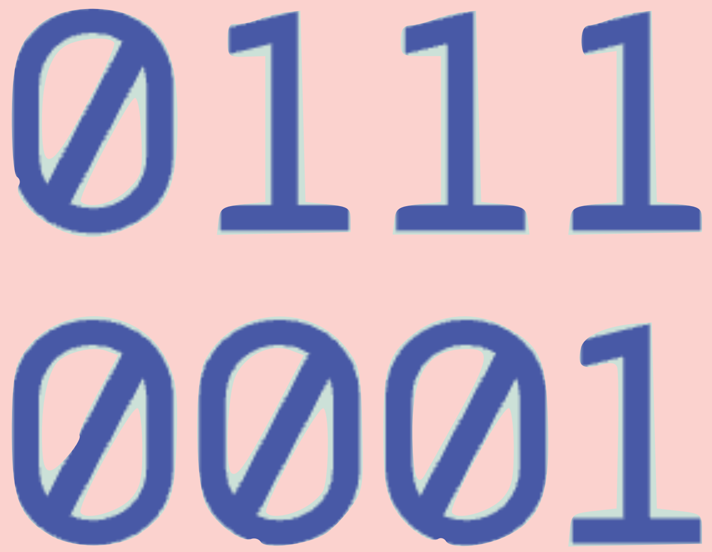
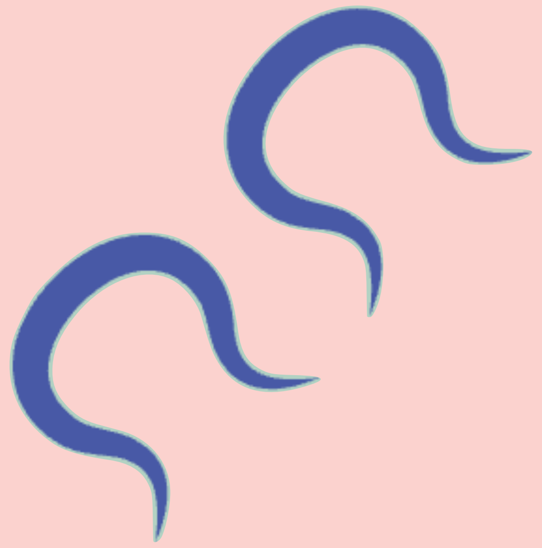
	

Multivariate-based	Isogeny-based
	

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The Current State of Post-Quantum Crypto

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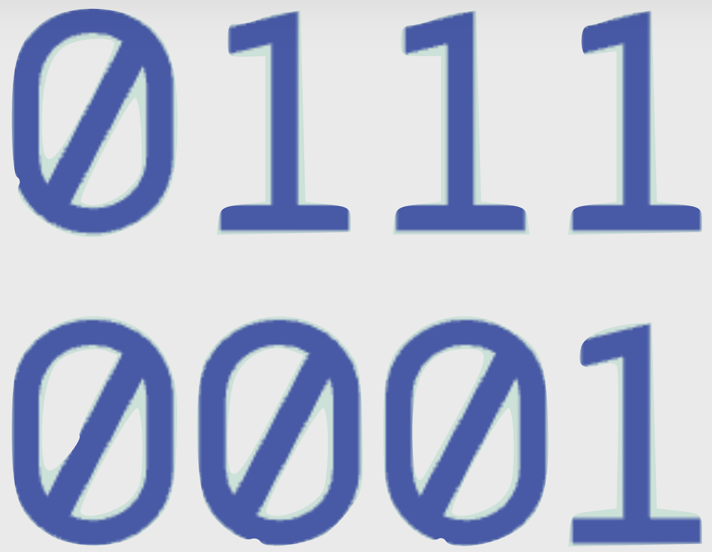
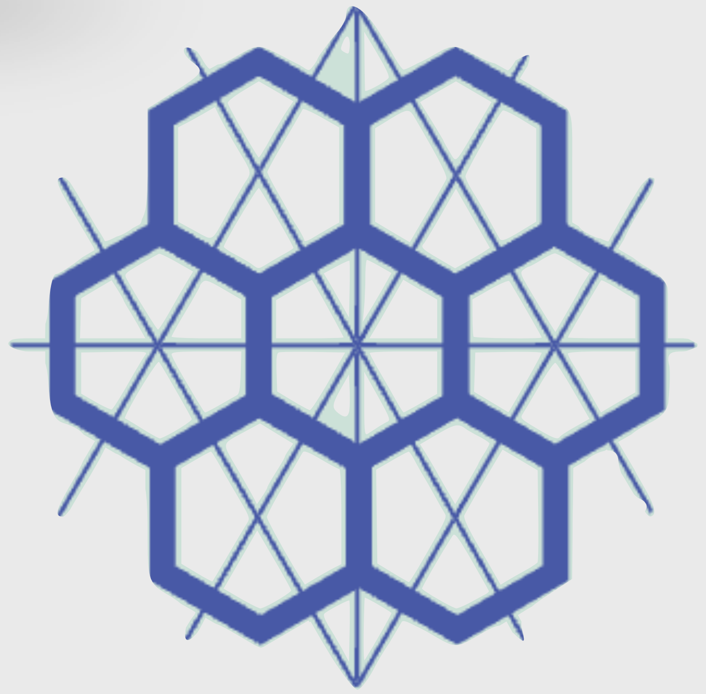
Why should we care about diversity of assumptions?



ttice-based



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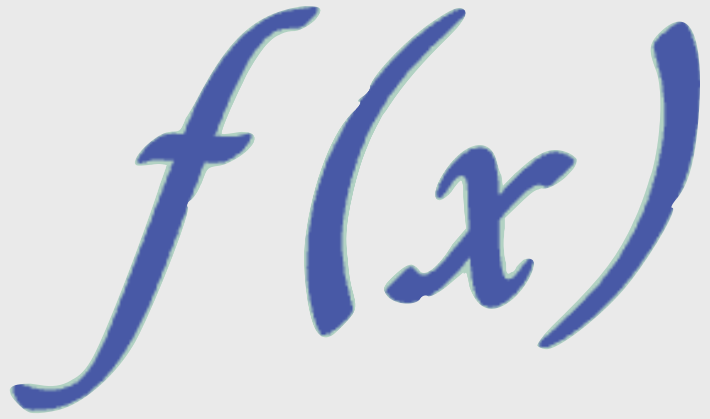

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Images from Quanta Magazine

The Current State of Post-Quantum Crypto

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ttice-based

1. **Hedge against advances in cryptanalysis**
   - Continual attempts to break lattices
2. **Different assumptions give different algebraic structures:**
   - Enable new feasibility results
   - Improved practical performance
3. **Cross-pollination with other areas:**
   - Coding theory, number theory, algebraic geometry, etc.

Images from Quanta Magazine

Code-Based vs. Lattice-Based Cryptography

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$$\left(\overbrace{A \;,\; s \cdot A}^{\text{Linear algebra}} + \overbrace{e}^{\text{Noise}} \right) \approx_c \left(A \;,\; u \right)$$

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- Little understanding of worst-case hardness
- Huge gap in cryptographic constructions

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Can we build more advanced primitives from code-based assumptions?

Our Result

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A new code-based assumption: Dense-Sparse LPN

- Variant of Learning Parity with Noise (LPN) with ***structured*** matrix distribution
- Initial cryptanalysis shows resistance to known attacks (linear tests, etc.)

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We construct lossy trapdoor functions (LTDFs) from Dense-Sparse LPN

- Simple-to-state primitive with many applications such as CCA-secure PKE, etc.
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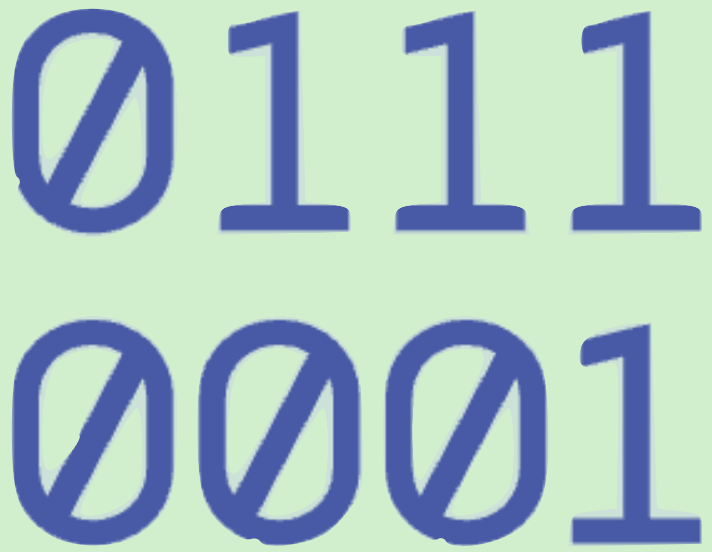
Why a new assumption?

- Overcome a barrier in noise management for LPN
- Circumvent a **new** attack against Sparse LPN (in relevant parameter regime)

Our Result

Lossy Trapdoor Functions

Code-based



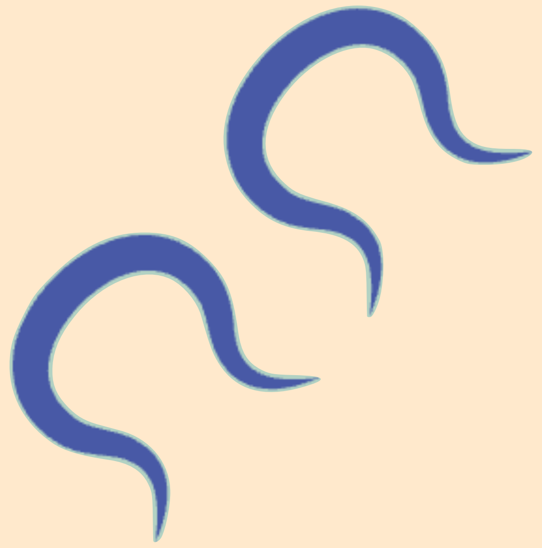
Ours!

Lattice-based

[PW08]

Multivariate-based

Isogeny-based



only achieve constant # bits of lossiness, not sufficient for many applications

Images from Quanta Magazine

Talk Outline

1. LTDF Template from Noisy Learning Problems

   (and why it fails from LPN)

2. Introducing Dense-Sparse LPN

3. Cryptanalysis & Open Questions

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Lossy Trapdoor Functions [PW08]

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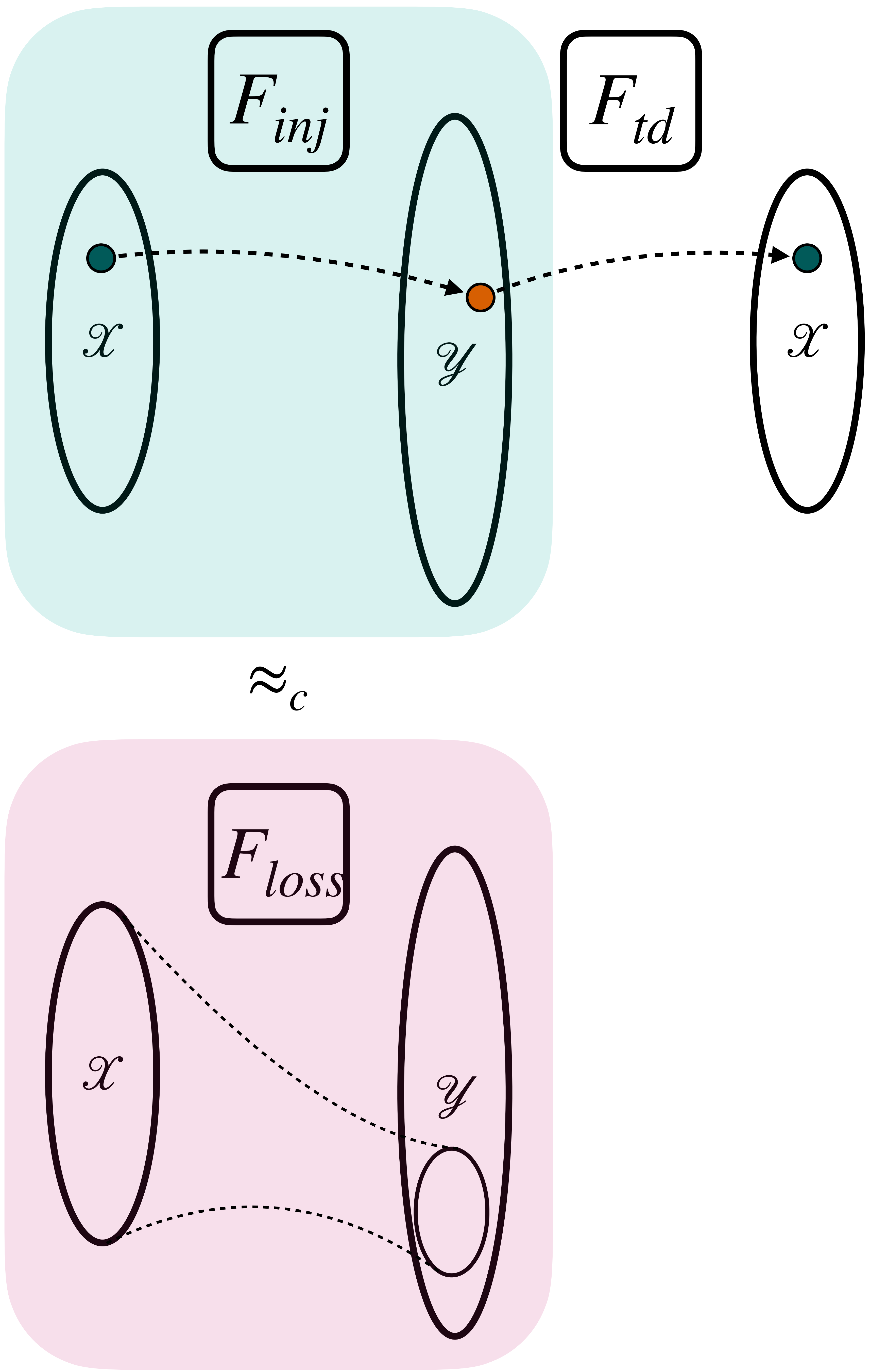
$\mathsf{Setup}(1^\lambda, \mathsf{inj}) \to (\mathsf{ek}, \mathsf{td}) \quad \approx_c \quad \mathsf{Setup}(1^\lambda, \mathsf{loss}) \to \mathsf{ek}$

$\mathsf{Eval}(\mathsf{ek}, x) \to y \qquad\qquad \mathsf{Invert}(\mathsf{td}, y) \to x$



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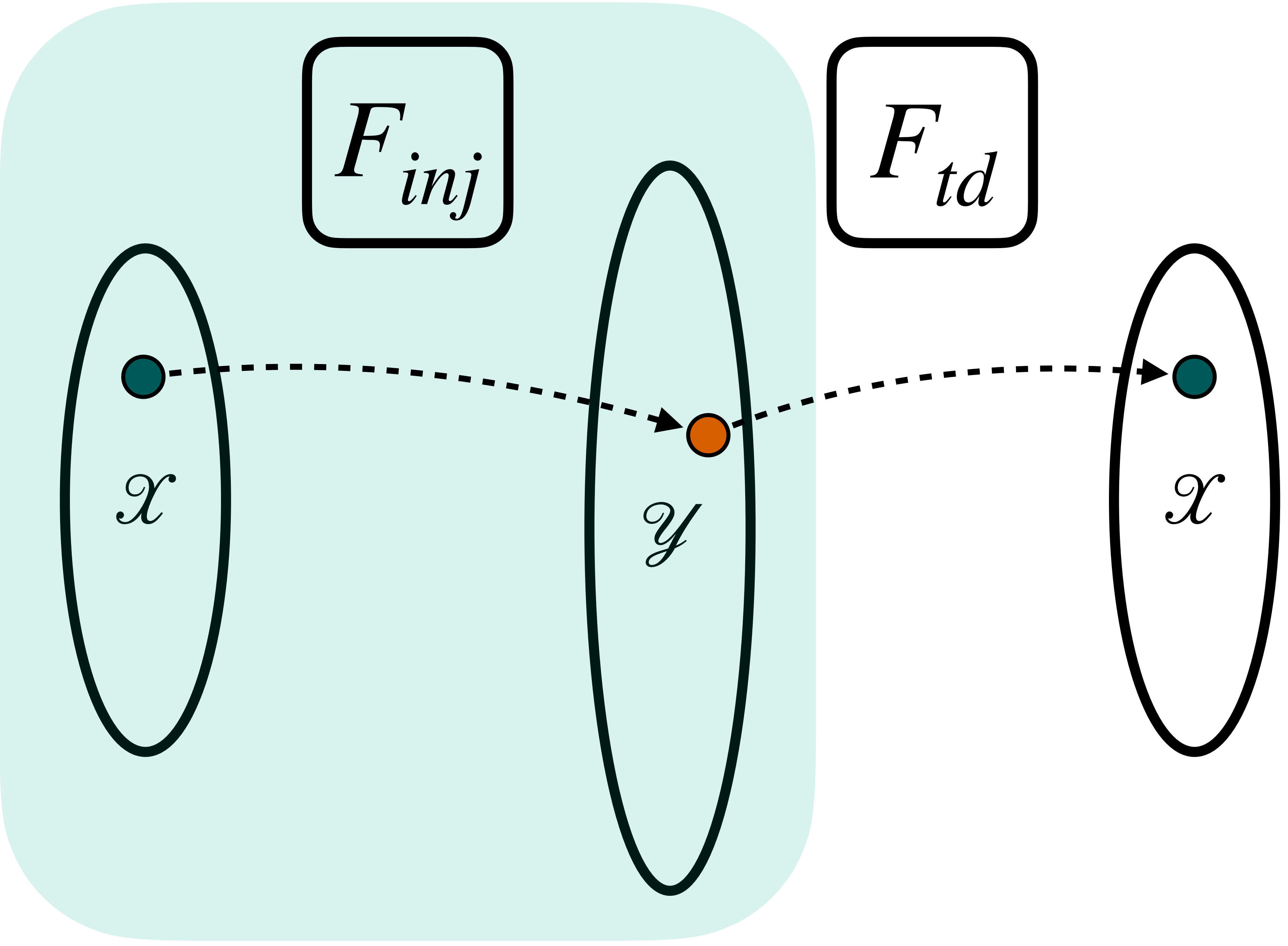
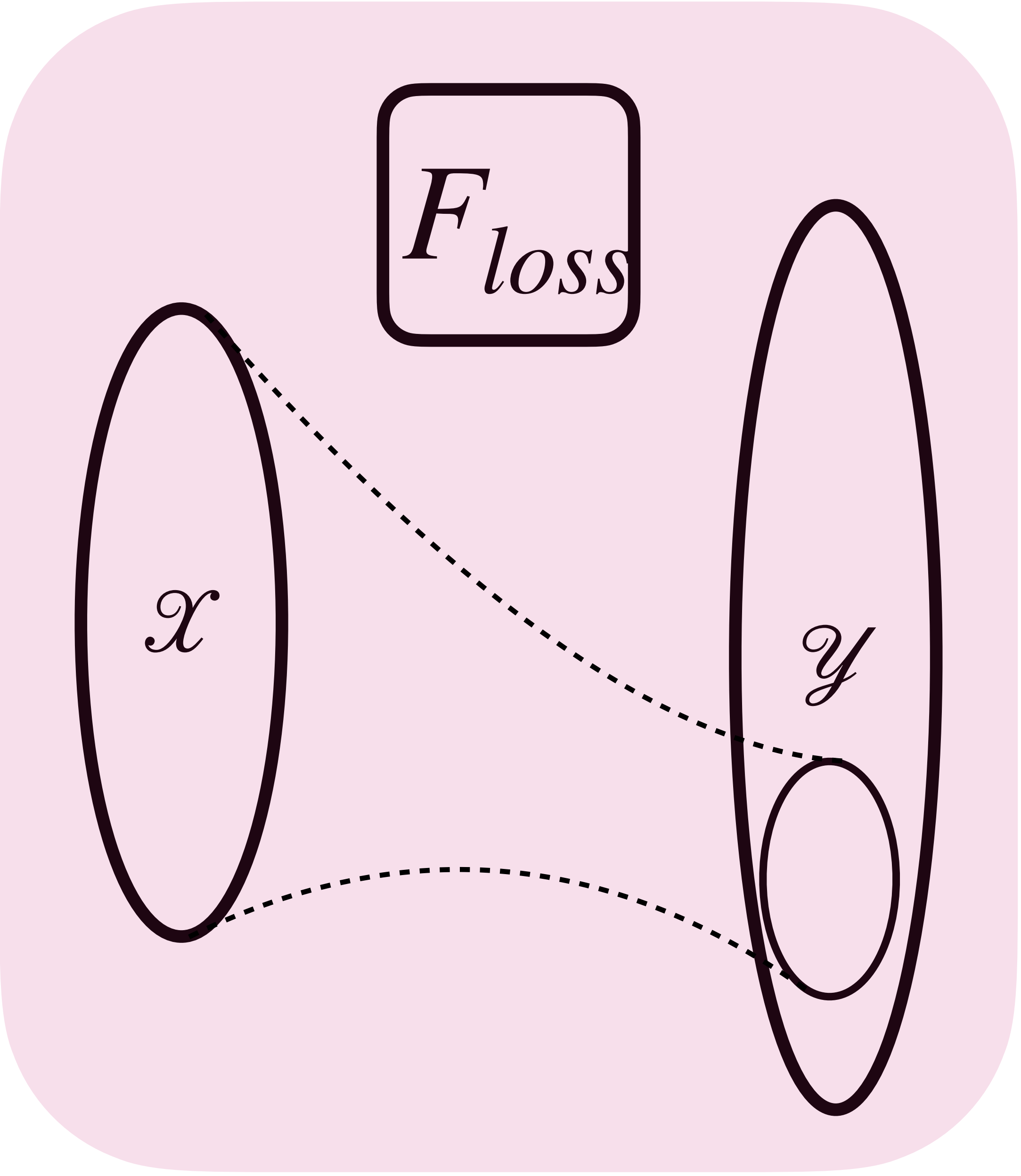


$\approx_c$



Applications of LTDFs:

CCA-secure encryption

Deterministic encryption

Collision-resistant hash functions

Analyzing OAEP

Selective opening security

Incompressible encodings

Computational extractors

Point function obfuscation

Pseudo-entropy functions

…and more!

Noisy Learning Problems: LWE vs. LPN

$$\left(\boxed{A \xleftarrow{\$} \mathcal{R}^{n\times m}} \;,\; \boxed{S \xleftarrow{\$} \mathcal{R}^{\ell\times n}} \cdot \boxed{A} + \boxed{E \xleftarrow{\$} \chi^{\ell\times m}} \right) \approx_c \left(\boxed{A \xleftarrow{\$} \mathcal{R}^{n\times m}} \;,\; \boxed{U \xleftarrow{\$} \mathcal{R}^{\ell\times m}} \right)$$

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Noisy Learning Problems: LWE vs. LPN

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<u>Learning with Errors:</u> [Regev05]

$\mathscr{R} = \mathbb{Z}/q\mathbb{Z}, \chi =$ Discrete Gaussian(α)

Entries of E are **<u>small</u>**

<u>Learning Parity with Noise:</u> [BKFL94]

$\mathscr{R} = \mathbb{F}_q$ (usually $q = 2$), $\chi =$ Bernoulli(ϵ)

Entries of E are **<u>mostly zero</u>**

LTDF Template from Noisy Learning Problem

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Lossy Mode: $\mathsf{ek} = \big(A, B := S \cdot A + E\big)$ $\approx_c$ **Injective:** $\begin{cases} \mathsf{ek} = \big(A, B := S \cdot A + E + C\big) \\ \mathsf{td} = S \end{cases}$

suitable code

LTDF Template from Noisy Learning Problem

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Function: $F : \text{Supp}(\chi)^m \to \mathbb{F}_2^{n+\ell}$ ($\text{Supp}(\chi)$ = support of error distribution)

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Evaluation: $F\big((A, B), x\big) = \big(A \cdot x, B \cdot x\big) \; = \; (A \cdot x, S \cdot A \cdot x + E \cdot x)$

Proof sketch: in lossy mode, second argument = first argument + noise

$\Longrightarrow$ requires $x \mapsto A \cdot x$ be ***compressing***, and $E \cdot x$ remains ***low noise***

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Inversion: $F^{-1}\big(S, (y_1, y_2)\,\big) = \mathsf{Decode}_C(y_2 - S \cdot y_1) \;=\; \mathsf{Decode}_C(C \cdot x + E \cdot x)$

Proof sketch: in injective mode, recover x via ***decoding*** from noise using C

LTDF Template from Noisy Learning Problem

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Hash Function: $\mathbb{F}_2^m \supsetneq \{t\text{-sparse}\} \ni x \quad \mapsto \quad A \cdot x \in \mathbb{F}_2^n$

Compression: $\binom{m}{t} \approx \left(\frac{m}{t}\right)^t > 2^n \quad \overset{\text{requires}}{\Longrightarrow} \quad m = n^{1+\Omega(1)}, \quad t = \Omega\left(\frac{n}{\log n}\right)$

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Accumulated Noise: $E \xleftarrow{\$} \mathrm{Ber}(\epsilon)^{\ell \times m} \quad \Longrightarrow \quad E \cdot x \xleftarrow{\$} \mathrm{Ber}(\delta)^{\ell}$

Noise growth: $\delta \approx \epsilon t \leq O(1) \quad \overset{\text{requires}}{\Longrightarrow} \quad \epsilon = O\left(\frac{\log n}{n}\right)$

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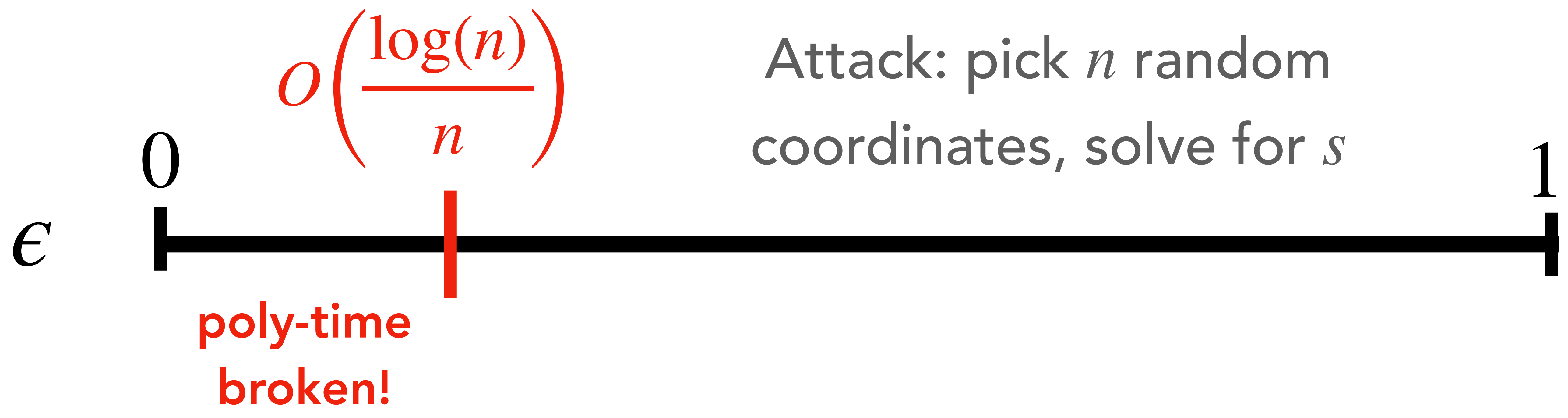
LPN Security:



Attack: pick n random coordinates, solve for s

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Accur

Can we achieve compression with better parameters?

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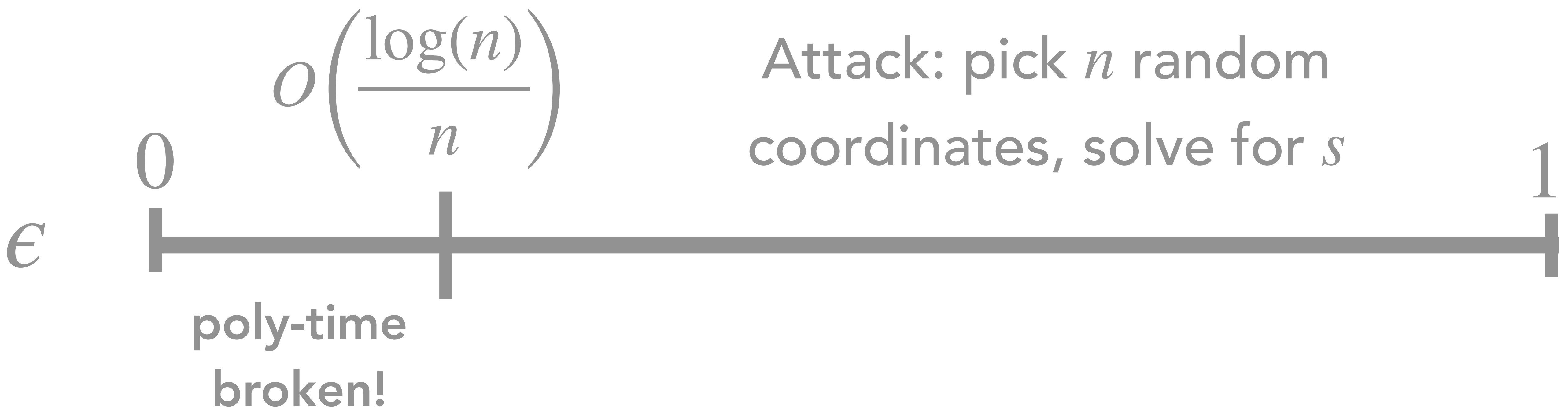

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Accur

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Yes, by restricting the hash range

(via changing distribution of A)!

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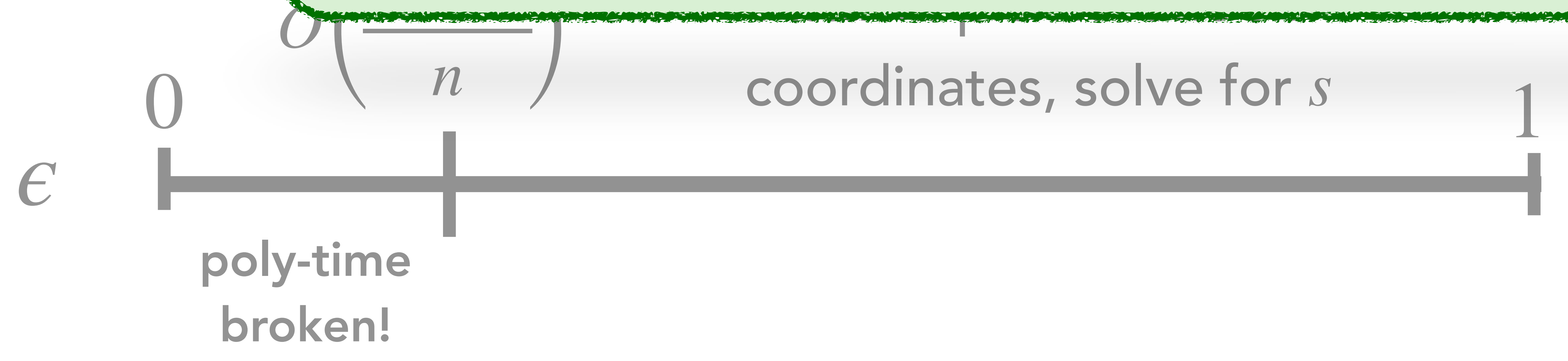

Sparse Learning Parity with Noise

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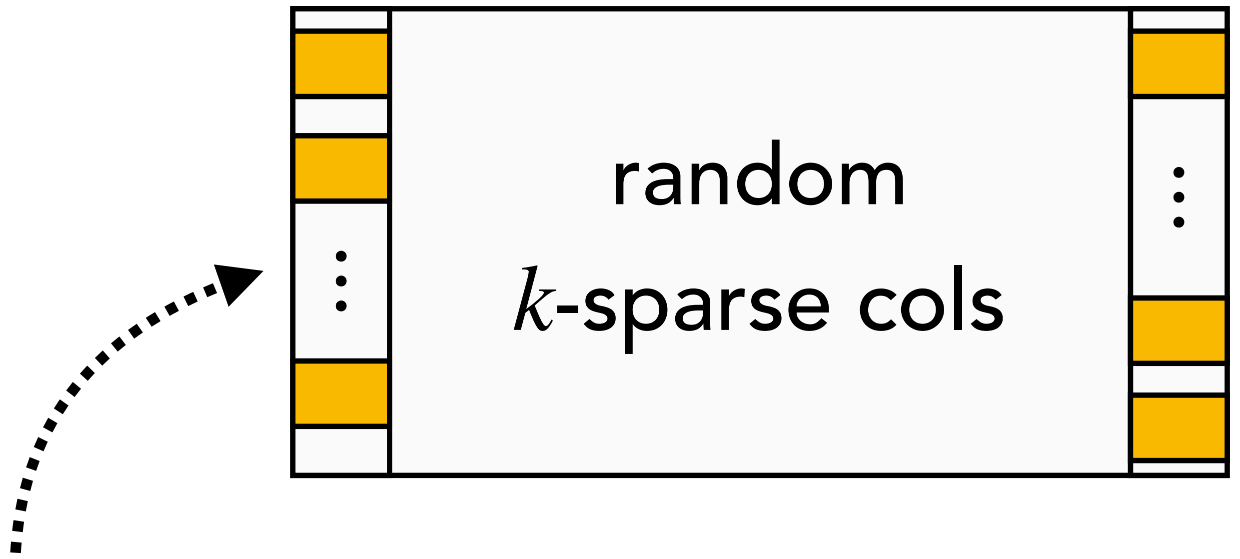
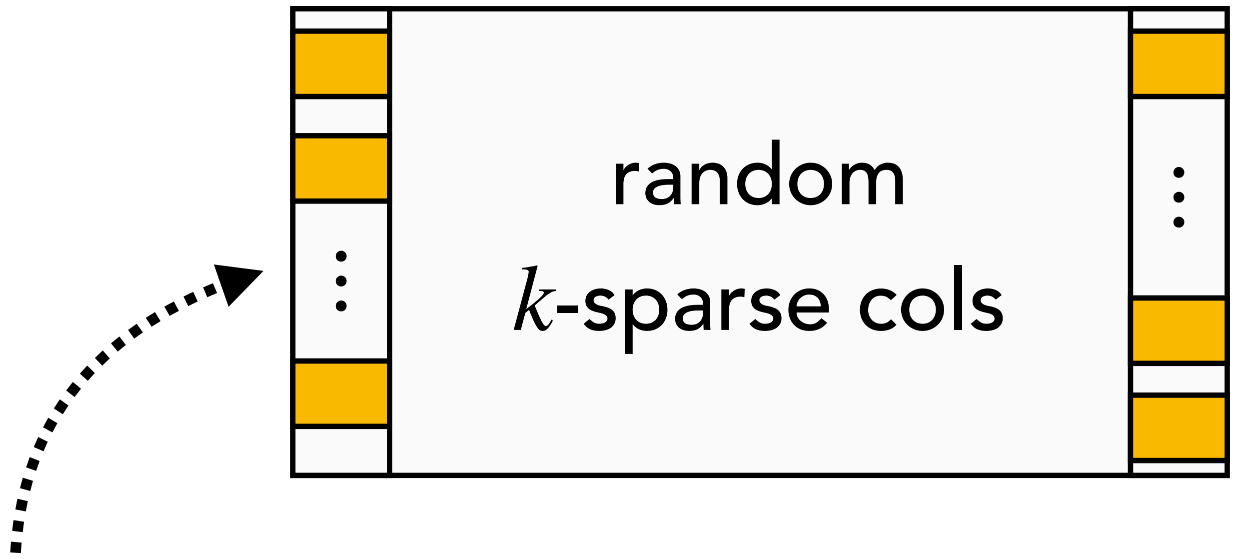


$$(A, S \cdot A + E) \approx_c (A, U)$$

$$\left(\boxed{\begin{array}{c} A \xleftarrow{\$} \mathbb{F}_2^{n \times m} \\ k\text{-sparse cols} \end{array}} \,,\; \boxed{S \xleftarrow{\$} \mathbb{F}_2^{\ell \times n}} \cdot \boxed{\quad A \quad} + \boxed{E \xleftarrow{\$} \mathsf{Ber}(\epsilon)^{\ell \times m}} \right) \approx_c \left(\boxed{\begin{array}{c} A \xleftarrow{\$} \mathbb{F}_2^{n \times m} \\ k\text{-sparse cols} \end{array}} \,,\; \boxed{U \xleftarrow{\$} \mathbb{F}_2^{\ell \times m}} \right)$$

Sparse Learning Parity with Noise



$$(A, S \cdot A + E) \approx_c (A, U)$$

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Well-studied variant of LPN [Alekhnovich03] with prior cryptographic applications

(PKE [ABW10], correlated randomness [ADI+17, AK23, BCG+23], HSS [DIJK23], etc.)

Our Setting: $m \ll n^{k/2}$, k **is constant*** **or slightly super-constant** $(\approx \log \log n)$

* Requires non-uniformly random distribution of A [AK19]

LTDF Template Also Fails for Sparse LPN

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Compression: $\binom{m}{t} > \binom{n}{\leq kt}$ $\overset{\text{requires}}{\Longrightarrow}$ $t = \Omega\left(\left(\frac{n^k}{m}\right)^{1/(k-1)}\right)$

Example:

$k = 6$

$m = n^2$

$t \approx n^{0.8}$

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Noise rate: $\delta \approx \epsilon \cdot t = O(1)$ **requires** $\Longrightarrow$ $\epsilon = O\left(\frac{1}{t}\right) = O\left(\left(\frac{m}{n^k}\right)^{1/(k-1)}\right)$

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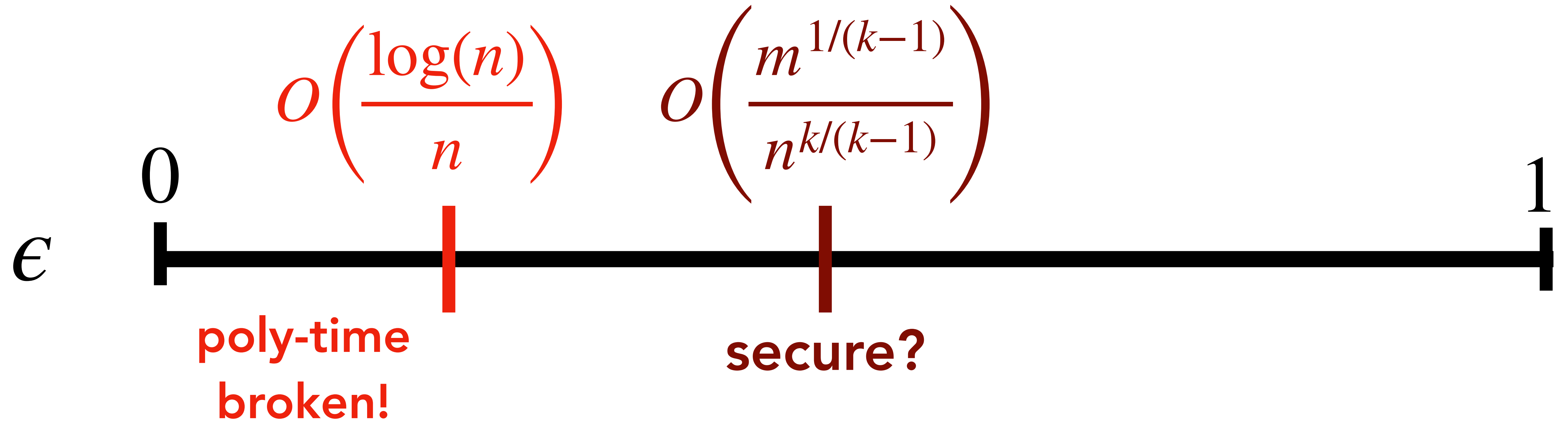
Sparse LPN Security:



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ϵ: 0 ... $O\left(\frac{\log(n)}{n}\right)$ ... $O\left(\frac{m^{1/(k-1)}}{n^{k/(k-1)}}\right)$ ... 1

poly-time broken!

secure?

no! (our new attack)

Attack on Sparse LPN in Compression Regime

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random k-sparse

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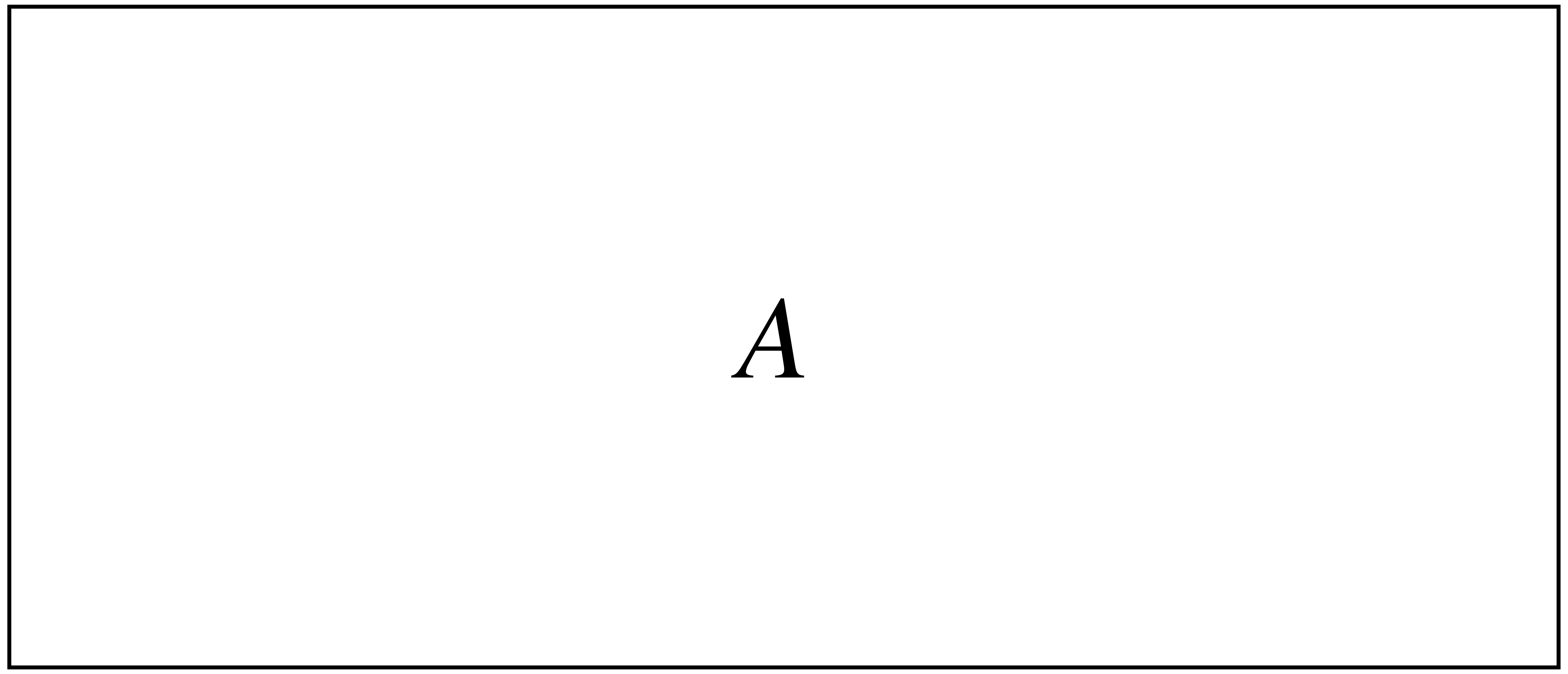

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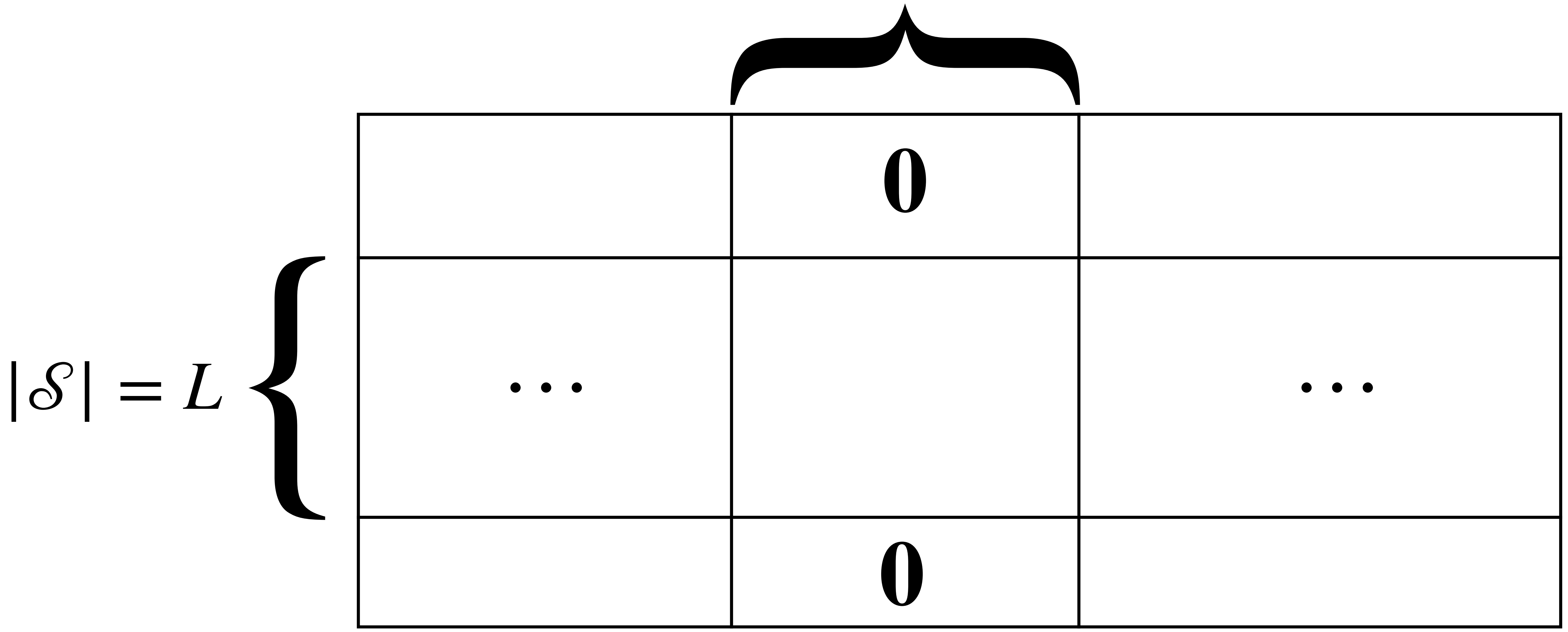

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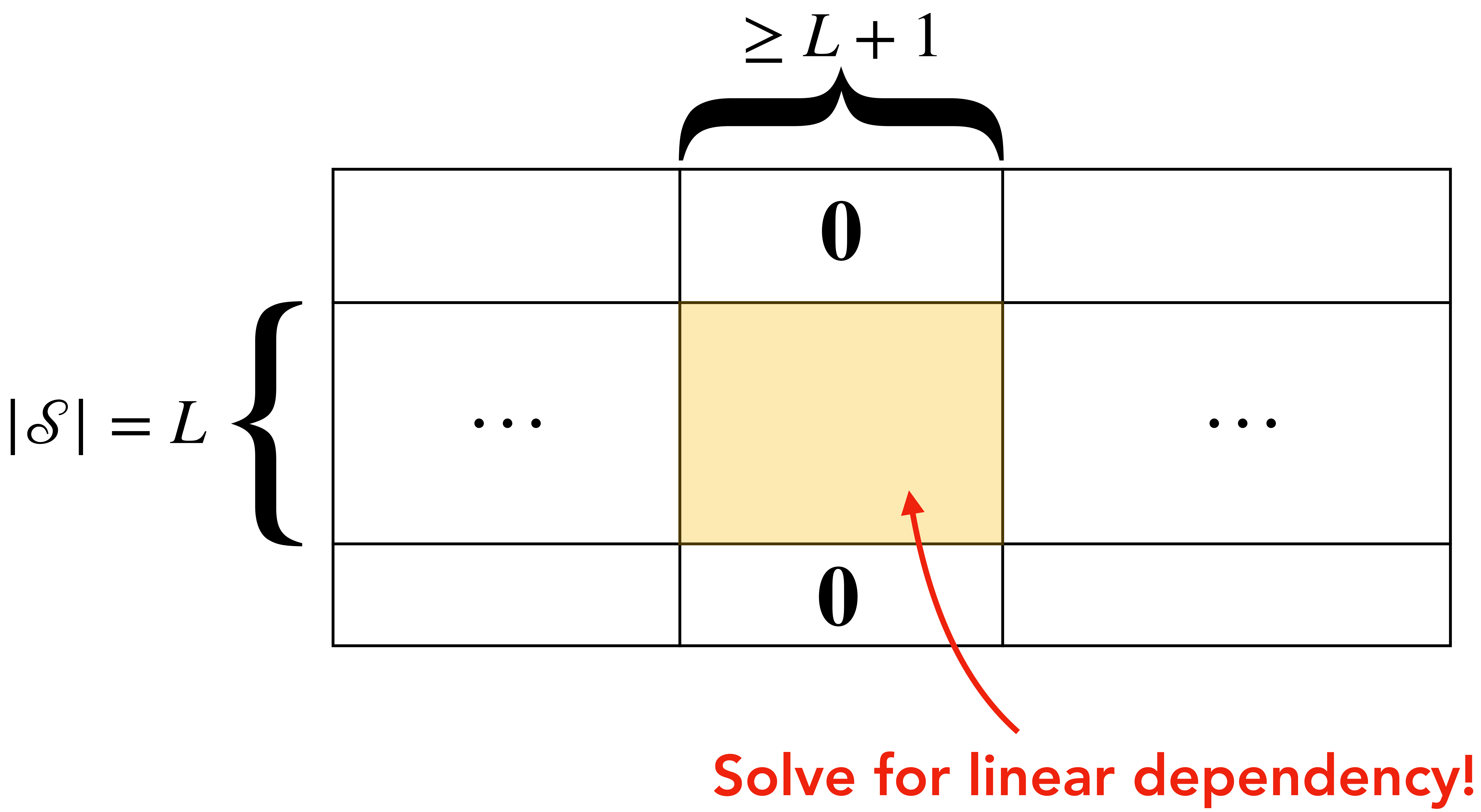

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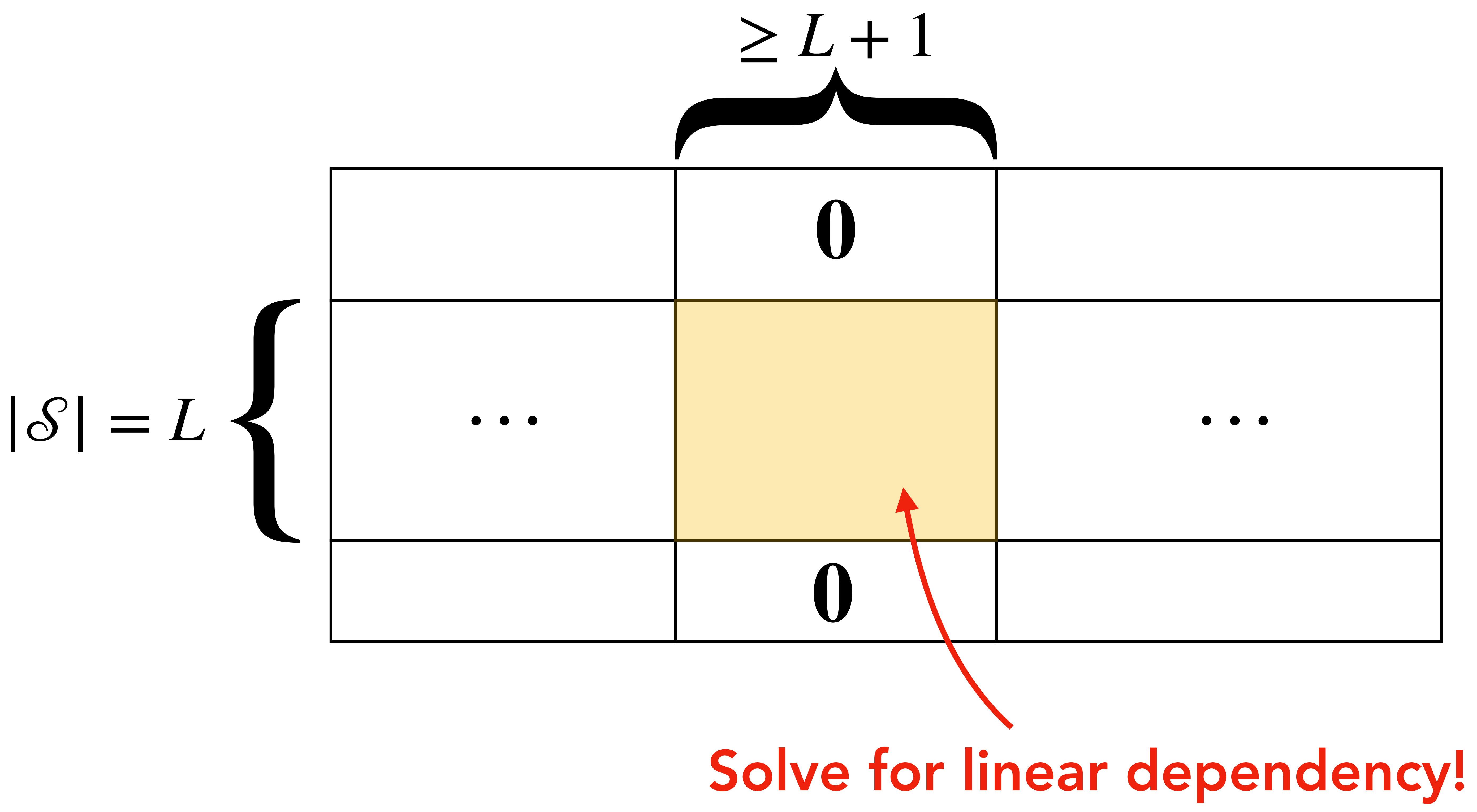
(more likely 0 if $u = sA + e$)



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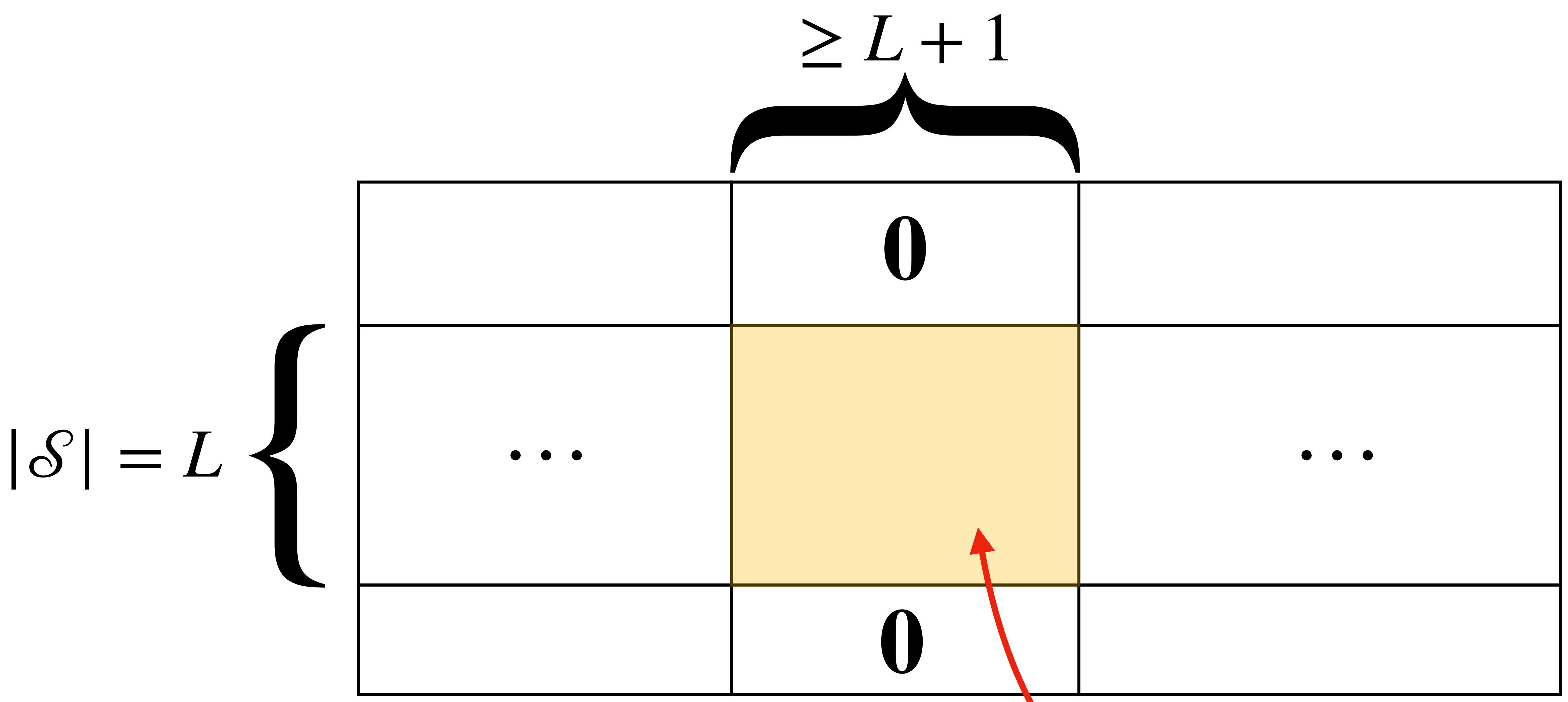

Solve for linear dependency!

Want: $\mathbb{E}[\#\,\text{cols}] = m \cdot \dfrac{\binom{L}{k}}{\binom{n}{k}} > L \iff L \approx \left(\dfrac{n^k}{m}\right)^{\frac{1}{k-1}}$

Same parameters for compression!

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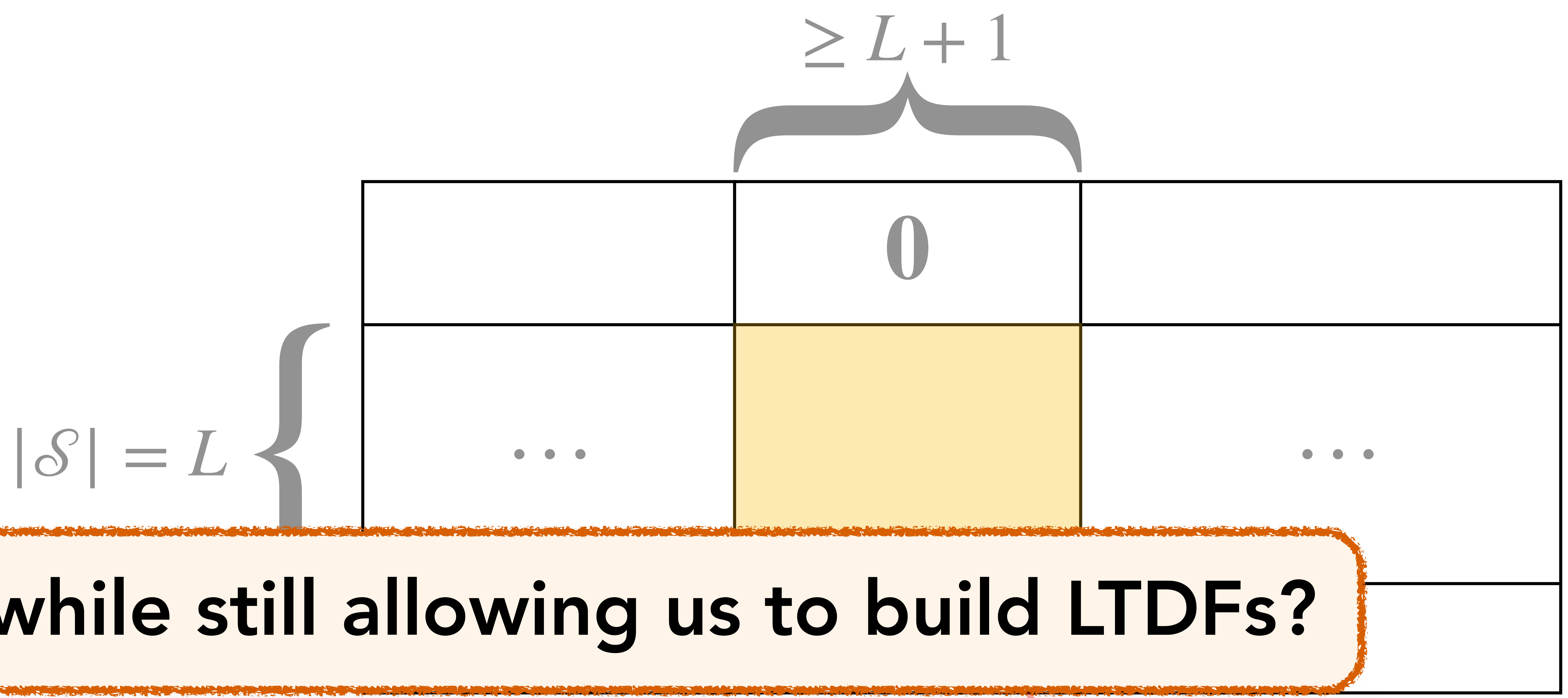

Can we avoid this attack while still allowing us to build LTDFs?

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2. Fin[...]

   ent[...]

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   [...]ear dependency, e.g. $\le$[...]

   $x \in \mathbb{F}_2^m$ such that $A \cdot x = 0$[...]

4. Compute $\langle u, x \rangle$ to detect bias.

   (more likely 0 if $u = sA + e$)

$$\ge L + 1$$

$$|\mathcal{S}| = L \left\{ \begin{array}{|c|c|c|} \hline & \mathbf{0} & \\ \hline \cdots & & \cdots \\ \hline & & \\ \hline \end{array} \right.$$

Can we avoid this attack while still allowing us to build LTDFs?

Perhaps, by masking sparsity pattern of A!

[...]pendency!

Want: $\mathbb{E}[\#\text{ cols}] = m \cdot \frac{[\ldots]}{\binom{n}{k}} > L \iff L \approx [\ldots] \left(\frac{n^k}{n[\ldots]}\right)^{\frac{1}{k-1}}$

Same parameters for compression!

Talk Outline

1. LTDF Template from Noisy Learning Problems

   (and why it fails from LPN)

2. Introducing Dense-Sparse LPN

3. Cryptanalysis & Open Questions

Dense-Sparse LPN

$$(A, sA + e) \approx_c (A, u)$$

$$\left(\boxed{A \leftarrow \mathcal{DS}(\mathbb{F}_2, n, m, k)} \,,\, \boxed{s \xleftarrow{R} \mathbb{F}_2^{1\times n}} \cdot \boxed{A} + \boxed{e \leftarrow \mathsf{Ber}(\epsilon)^{1\times m}} \right) \approx_c \left(\boxed{A \leftarrow \mathcal{DS}(\mathbb{F}_2, n, m, k)} \,,\, \boxed{u \xleftarrow{R} \mathcal{R}^{1\times m}} \right)$$

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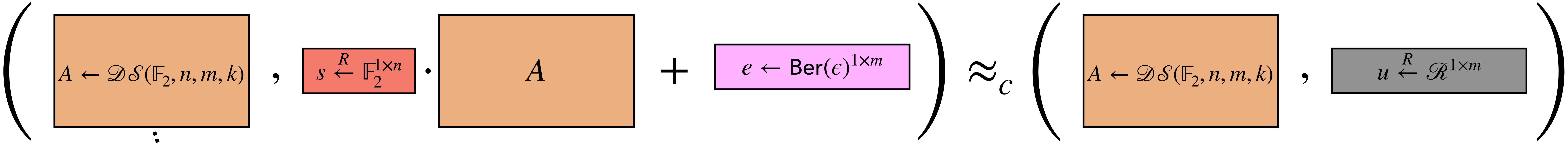

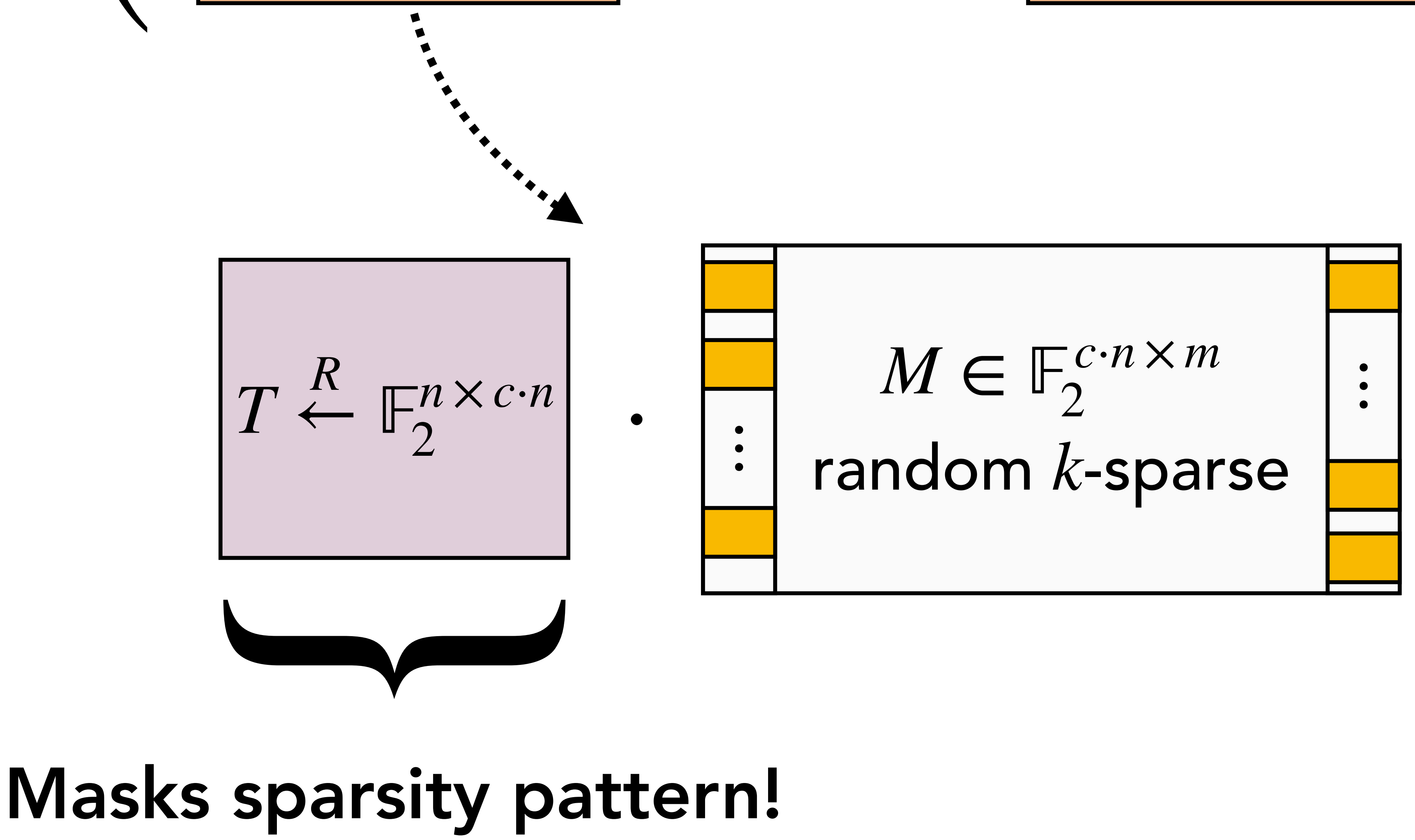

where $A = T \cdot M$ with $T \in \mathbb{F}_2^{n\times c\cdot n}$ random (dense),

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(say $c = 1.1$)

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$$\underbrace{T \xleftarrow{R} \mathbb{F}_2^{n\times c\cdot n}}_{\text{Masks sparsity pattern!}} \cdot \begin{matrix} M \in \mathbb{F}_2^{c\cdot n \times m} \\ \text{random } k\text{-sparse} \end{matrix}$$

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Inspiration from McEliece: hide code via linear transformation

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Now LTDF construction works! ($T \cdot M \cdot x$ has image size at most $M \cdot x$)

Putting Things Together: LTDF Parameters

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<u>Theorem:</u> LTDF from Dense-Sparse LPN that loses a factor $D > 1$ in lossy mode...

$\Longrightarrow$ requires Dense-Sparse LPN with $m \ll n^{k/2}$ and $\epsilon \ll \left(\frac{m}{n^{Dk}}\right)^{\frac{1}{Dk-1}}$

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Concrete settings: $k = 6$, $m = n^2$, any $\delta > 0$

- $D = 10$ (loses $90\,\%$ of input): $\epsilon = n^{-\frac{58}{59} - \delta} \approx n^{-0.984}$
- $D = 2$ (loses $50\,\%$ of input): $\epsilon = n^{-\frac{10}{11} - \delta} \approx n^{-0.91}$
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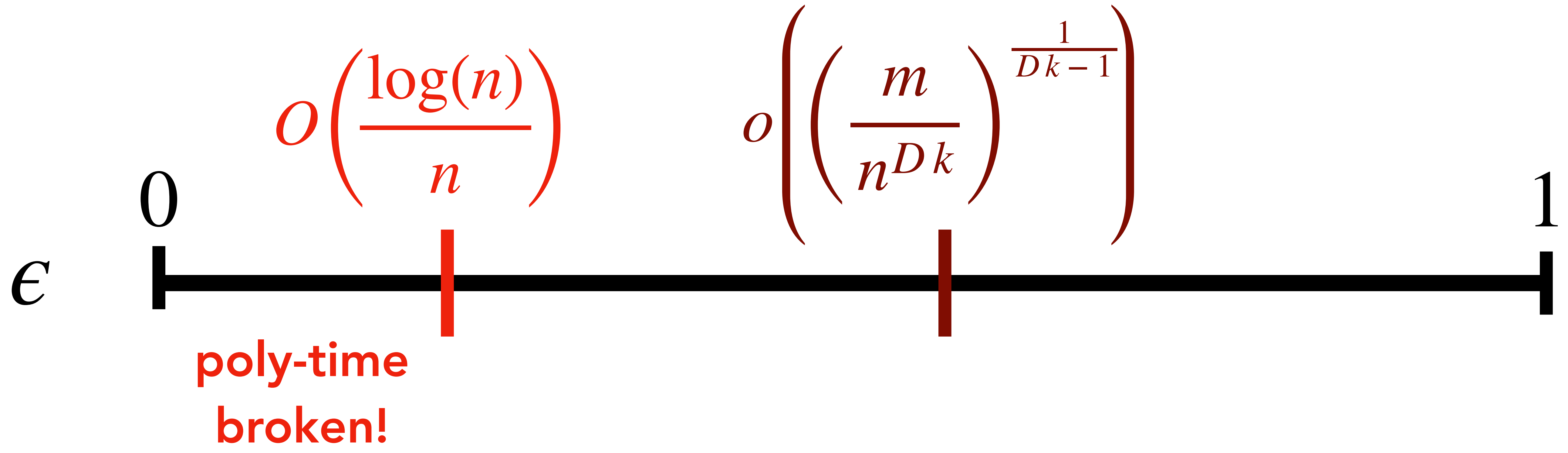
How secure is this?



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Summary of Cryptanalysis

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1. **Information Set Decoding: guess error coordinates of e $\implies$ time $2^{\tilde{\Omega}(\epsilon \cdot n)}$**

2. **Find a sparse vector x in the (right) kernel of $A = T \cdot M$ $\implies$ time $2^{\tilde{\Omega}(n^{\delta})}$**

   (inherited from M such that $M \cdot x = 0$)

   Parameter: $m = n^{1+\left(\frac{k}{2}-1\right)(1-\delta)}$

3. **Decompose Dense-Sparse matrix $\implies$ time $2^{\tilde{\Omega}(n)}$**

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3. Decompose Dense-Sparse matrix $\implies$ time $2^{\tilde{\Omega}(n)}$

Conjectured Security: secure against attackers w/ time $\ll 2^{\min\left(\tilde{O}(\epsilon\cdot n),\, \tilde{O}(n^\delta)\right)}$

Decomposing a Dense-Sparse Matrix

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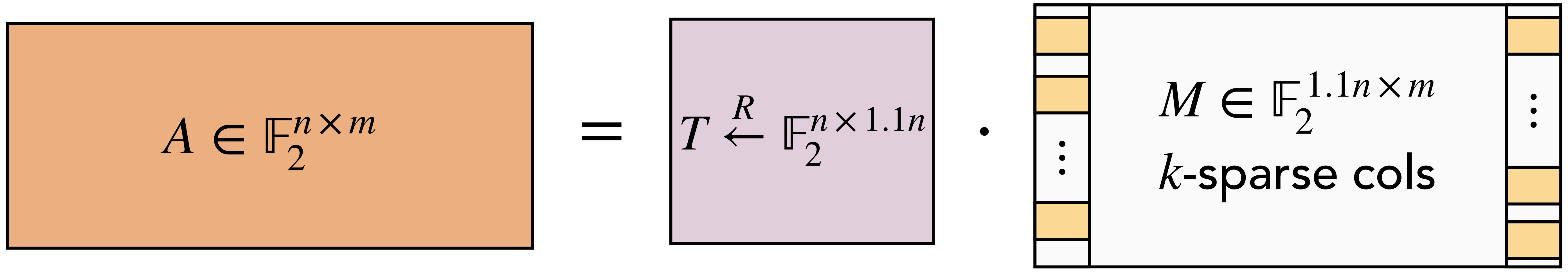
$A \in \mathbb{F}_2^{n \times m}$ $=$ $T \xleftarrow{R} \mathbb{F}_2^{n \times 1.1n}$ $\cdot$ $M \in \mathbb{F}_2^{1.1n \times m}$ k-sparse cols

Find T, M from $A = T \cdot M \implies$ break DS-LPN with compression parameters!

(using our earlier Sparse LPN attack on M)

Decomposing a Dense-Sparse Matrix



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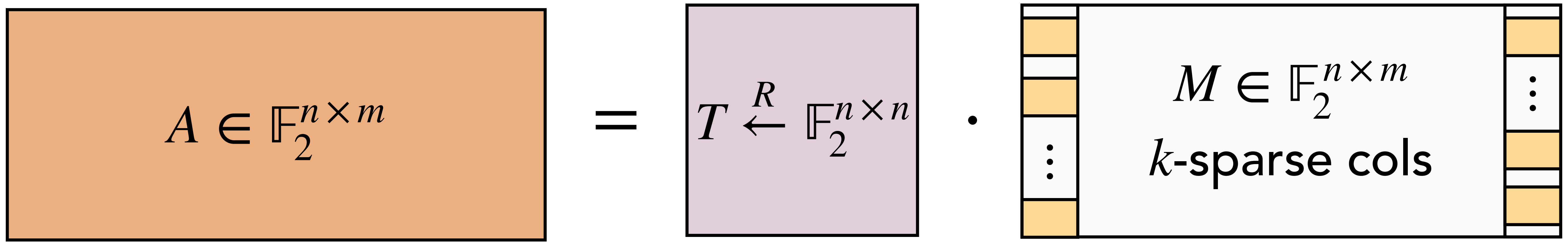

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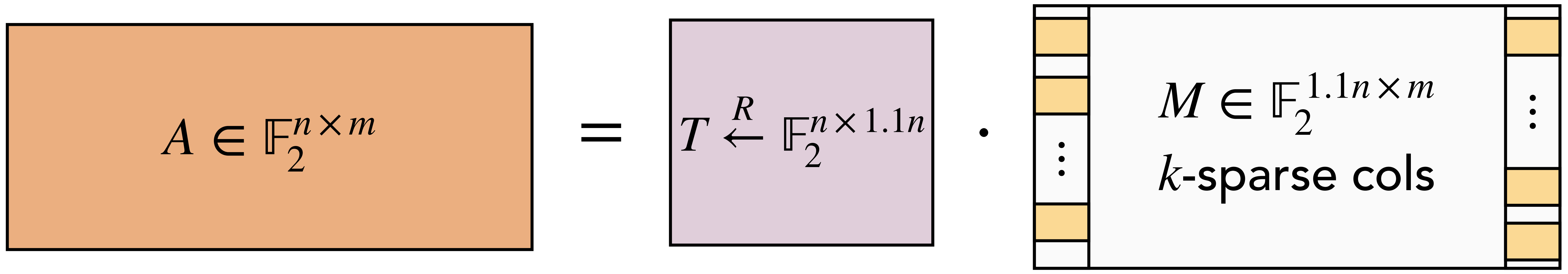

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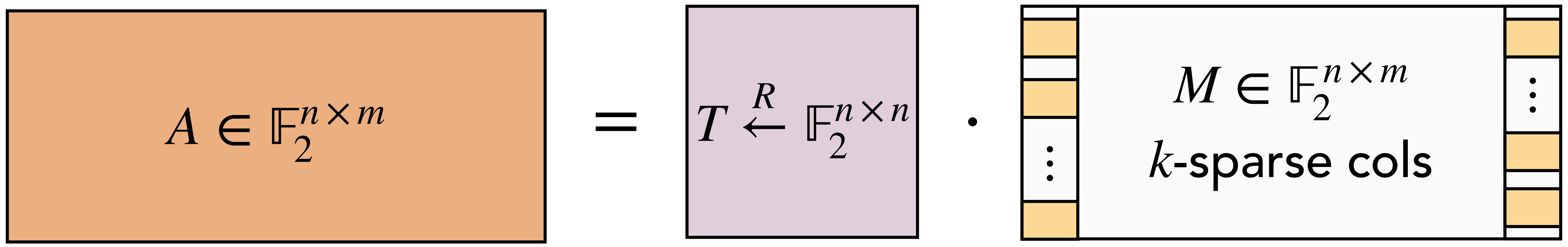
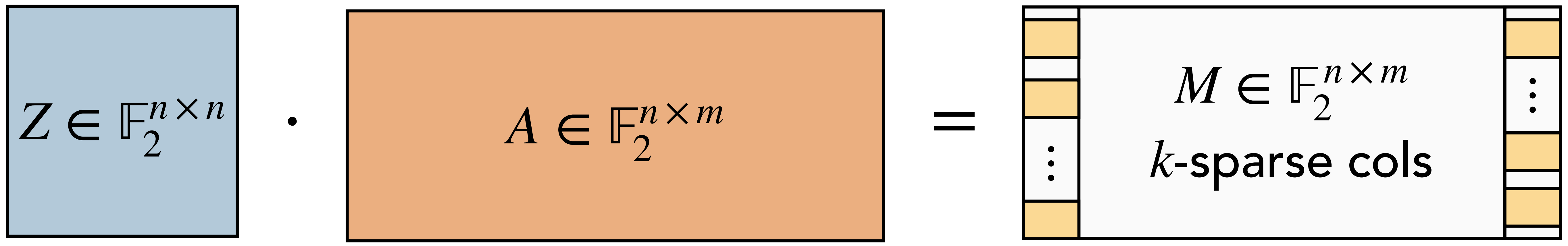


This is <u>insecure</u>! We can find Z such that $Z \cdot A$ is sparse



Decomposing a Dense-Sparse Matrix

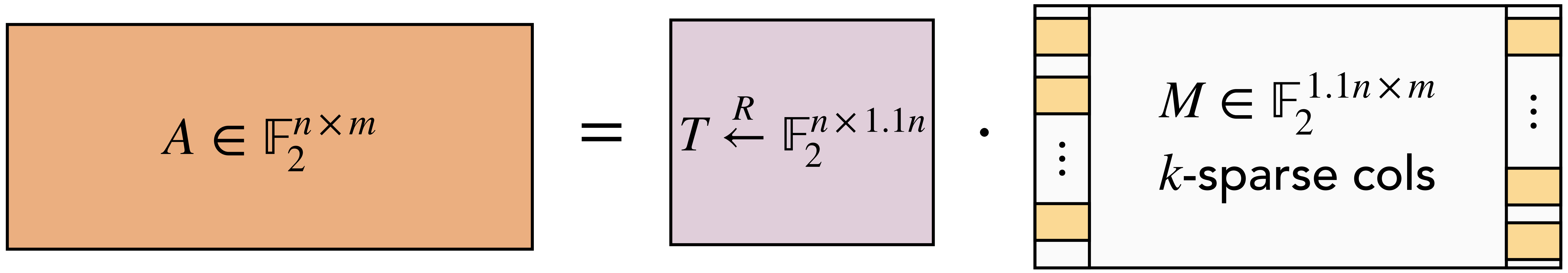
$$A \in \mathbb{F}_2^{n\times m} = T \xleftarrow{R} \mathbb{F}_2^{n\times 1.1n} \cdot \underbrace{M \in \mathbb{F}_2^{1.1n\times m}}_{k\text{-sparse cols}}$$

$$a_{1,j} \cdot z_1 + a_{2,j} \cdot z_2 + \cdots + a_{n,j} \cdot z_n = m_j \qquad \forall j \in [m]$$

This is <u>insecure</u>! We can find Z such that $Z \cdot A$ is sparse

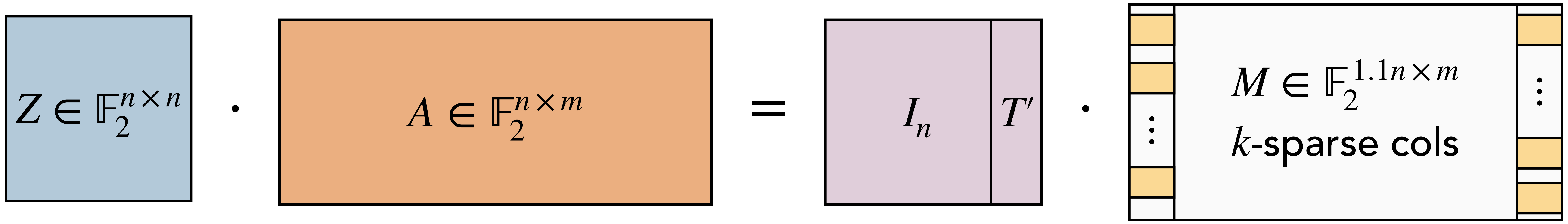
$$Z \in \mathbb{F}_2^{n\times n} \cdot A \in \mathbb{F}_2^{n\times m} = \underbrace{M \in \mathbb{F}_2^{n\times m}}_{k\text{-sparse cols}}$$

Decomposing a Dense-Sparse Matrix



The attack breaks down when $T \in \mathbb{F}_2^{n\times 1.1n}$ is rectangular

$\Longrightarrow$ time complexity is now $2^{\tilde{\Omega}(n)}$ due to guessing T'



Summary & Open Problems

Our Result: We introduce a **new** code-based assumption, Dense-Sparse LPN, and show how it gives rise to Lossy Trapdoor Functions.

Summary & Open Problems

<u>**Our Result:**</u> We introduce a **new** code-based assumption, **Dense-Sparse LPN**, and show how it gives rise to **Lossy Trapdoor Functions**.

<u>**Future Directions:**</u>

- <u>**Cryptanalysis:**</u>
  - <u>**Reductions:**</u> search-to-decision? worst-to-average-case?
  - <u>**Concrete parameters:**</u> we need help!
- <u>**Applications:**</u> PIR? Laconic OT? NIZK? IBE? ABE?
- <u>**Coding Theory:**</u> better constant-sparse matrix distributions?

Read our paper!
(ePrint 2024/175)

Thank you! Questions?