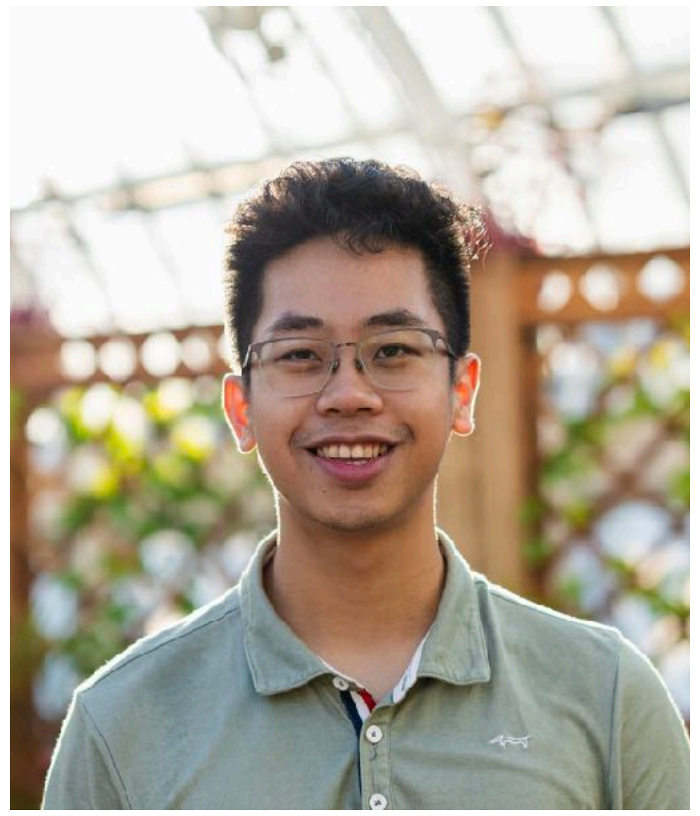
Non-Interactive Zero-Knowledge from LPN and MQ

Quang Dao

Aayush Jain

Zhengzhong Jin



Crypto 2024

Non-Interactive Zero-Knowledge (NIZK)

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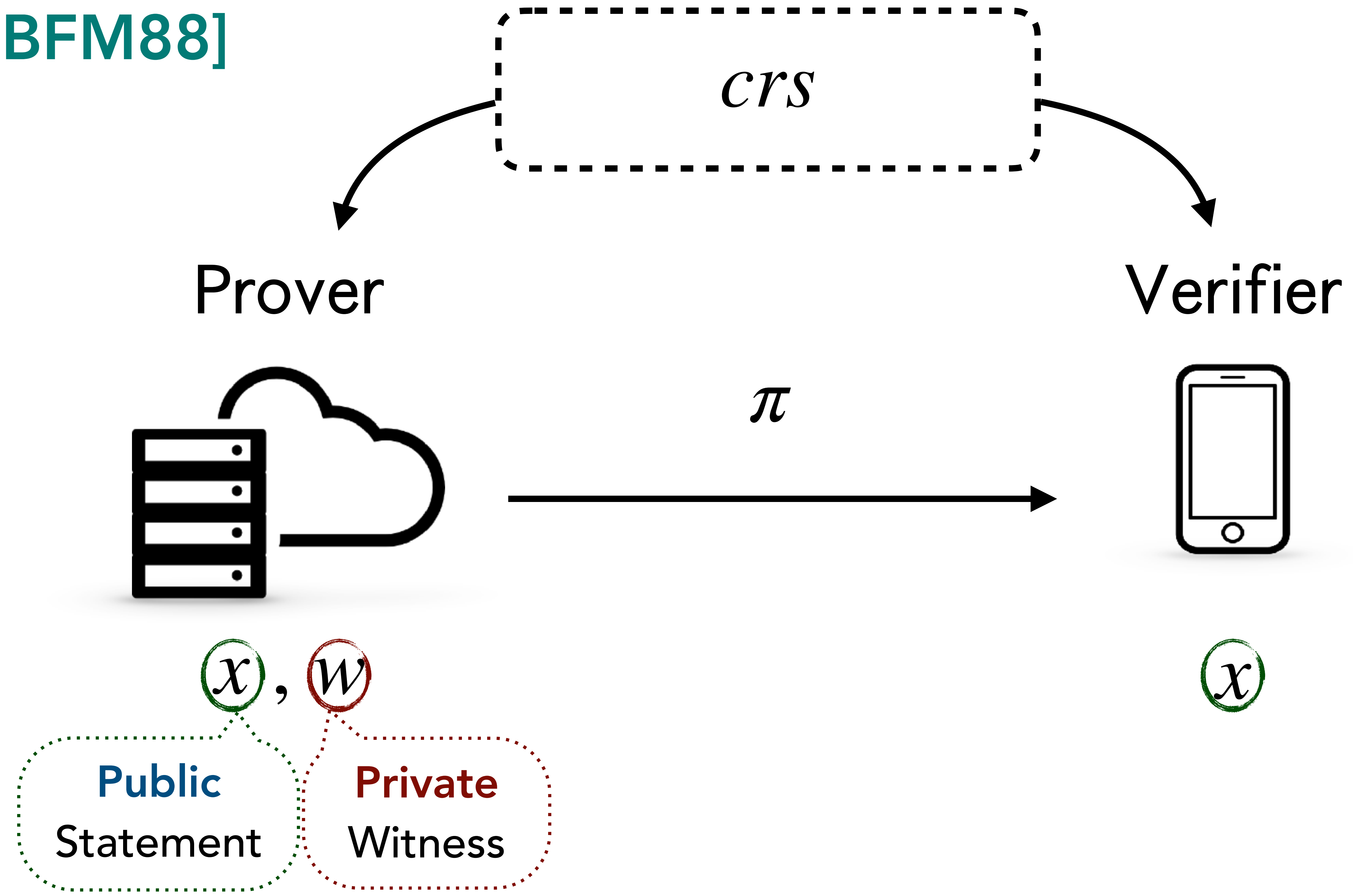
Fundamental notion in cryptography [GMR85, BFM88]



crs

Prover

Verifier

π

x, w

x

Public Statement

Private Witness

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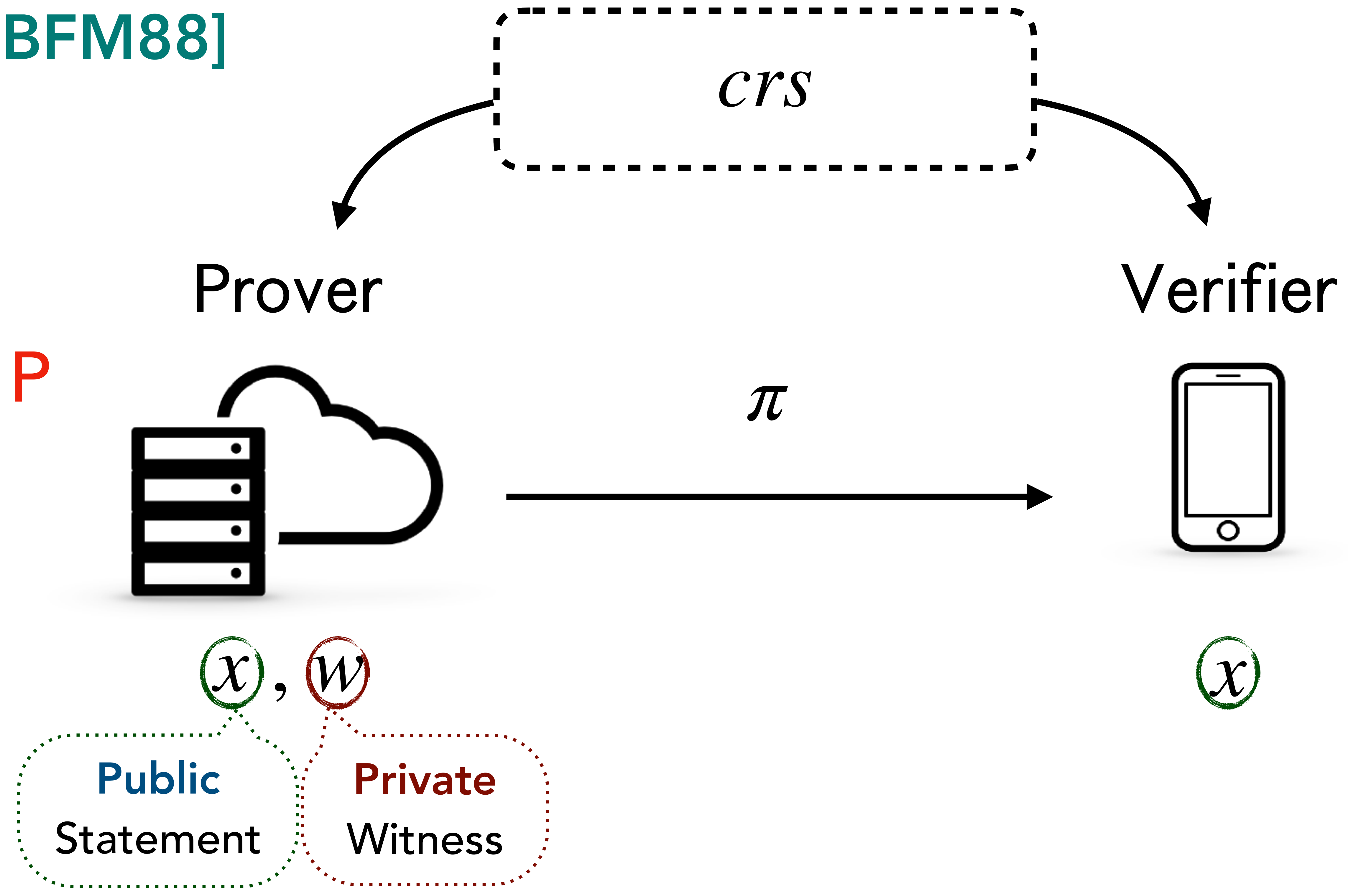
- **Completeness:** honest P convinces V
- **Soundness:** V rejects $x \notin L$ for any malicious P
- **Zero-knowledge:** there exists a simulator S that can simulate (crs, π)



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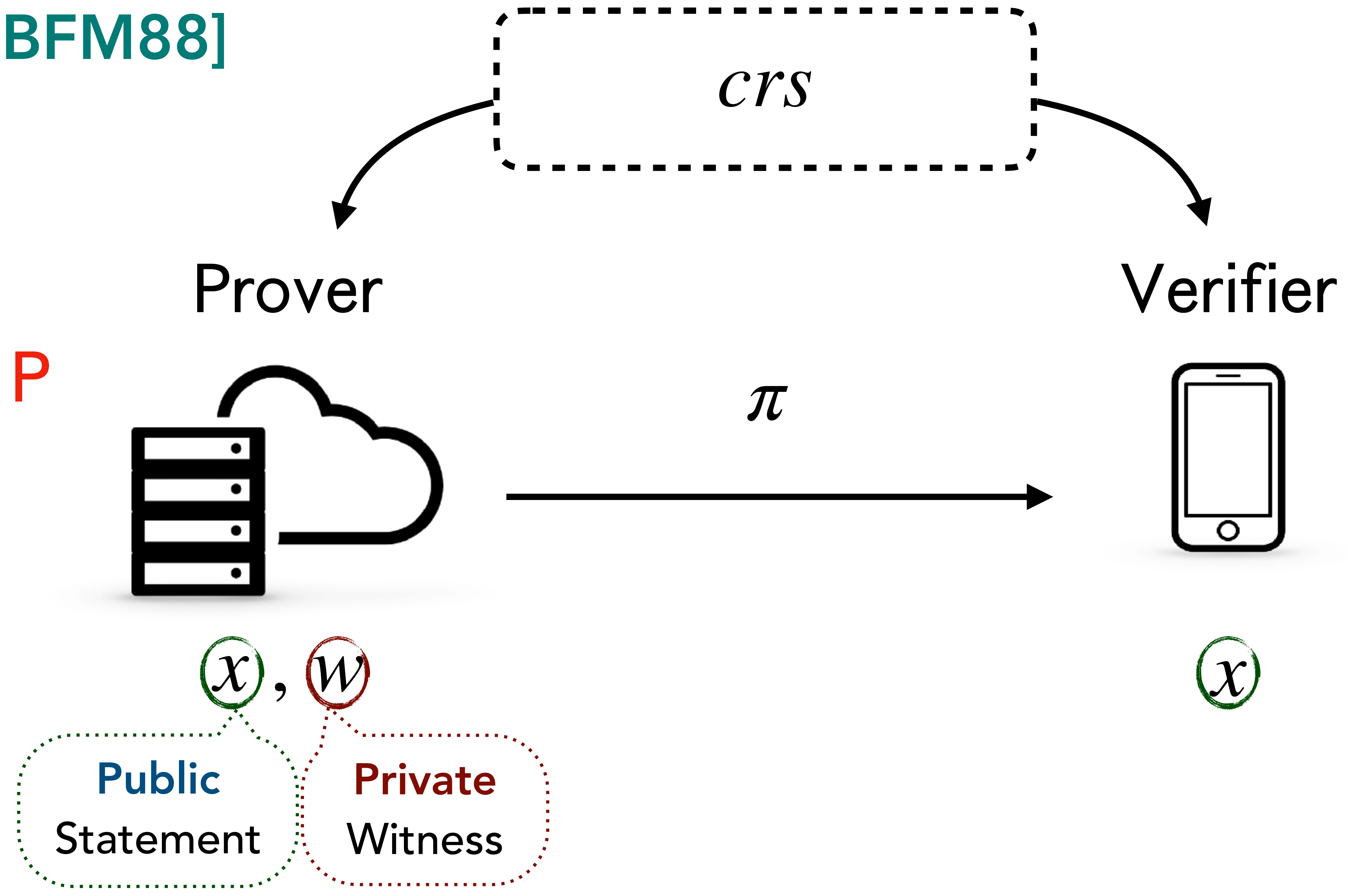

Many practical applications!

Private Cryptocurrencies

Voting Systems

Anonymous credentials

ZK-Rollups

Proving Image Transformations

...and more!

Achieving NIZK in Practice and Theory

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- Recent progress relies on ***correlation-intractable*** **(CI)** hash functions!

$\implies$ enables constructions from LWE [CCR+19, PS19], DDH/DCR + LPN [BKM20], or sub-exponential DDH [JJ21]

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  $\Longrightarrow$ enables constructions from LWE [CCR+19, PS19], DDH/DCR + LPN [BKM20], or sub-exponential DDH [JJ21]
- The only post-quantum secure construction is from LWE!

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Can we build NIZK from post-quantum assumptions other than lattices?

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   - Learning Parity with Noise (LPN), * with slightly-stronger-than-PKE noise rate
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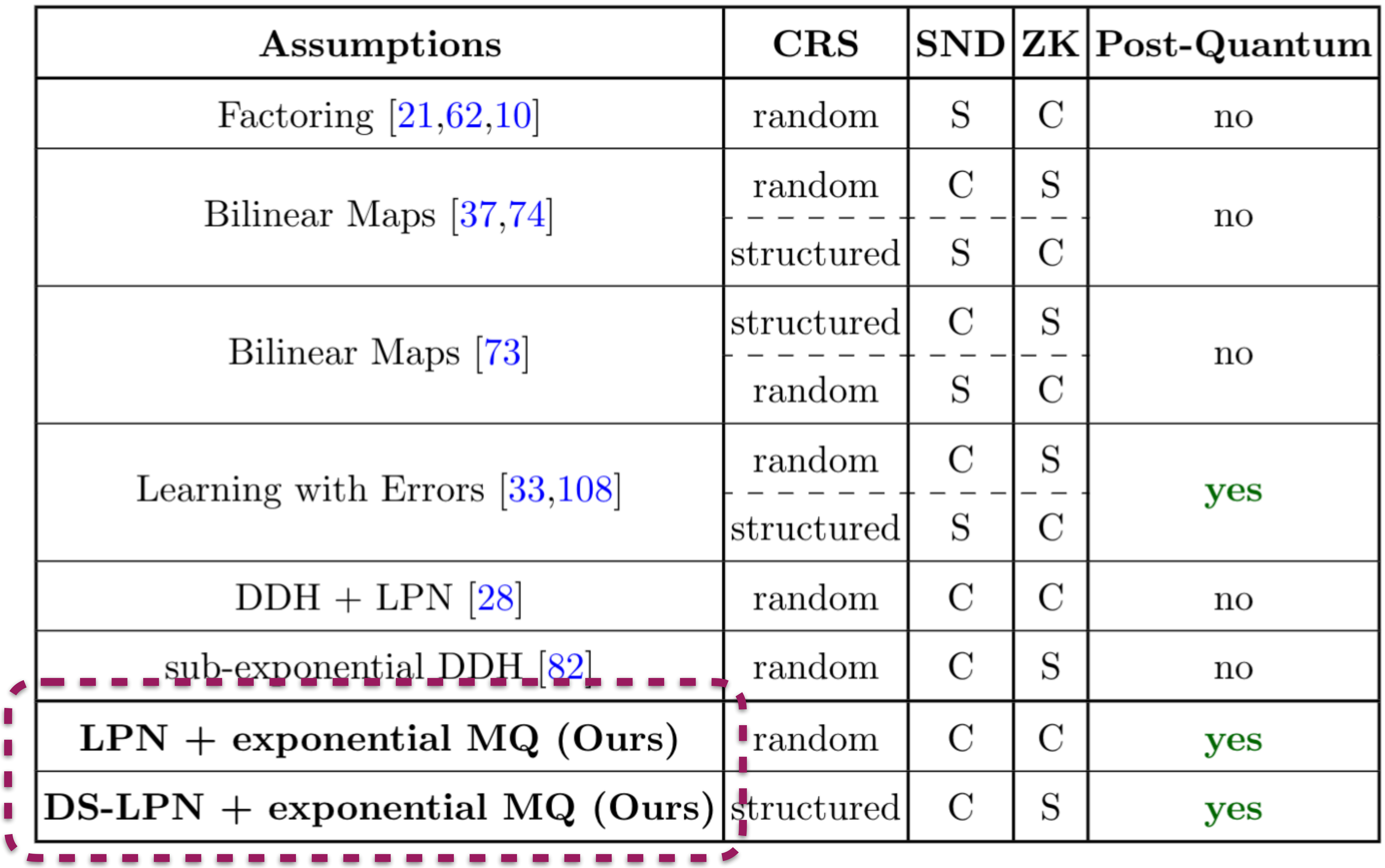
3. We can upgrade our **NIZK** to statistical zero-knowledge, assuming:
   - **Dense-Sparse LPN** [DJ24] * implies Lossy PKE with ≈ linear decryption & low correct. error

Our Result: NIZK from LPN and MQ



Assumptions	CRS	SND	ZK	Post-Quantum
Factoring [21,62,10]	random	S	C	no
Bilinear Maps [37,74]	random	C	S	no
	structured	S	C	
Bilinear Maps [73]	structured	C	S	no
	random	S	C	
Learning with Errors [33,108]	random	C	S	yes
	structured	S	C	
DDH + LPN [28]	random	C	C	no
sub-exponential DDH [82]	random	C	S	no
LPN + exponential MQ (Ours)	random	C	C	yes
DS-LPN + exponential MQ (Ours)	structured	C	S	yes

Talk Outline

1. Recap: NIZK from Correlation Intractability

2. CI Hashing from (Approximate) MQ

3. Putting Things Together

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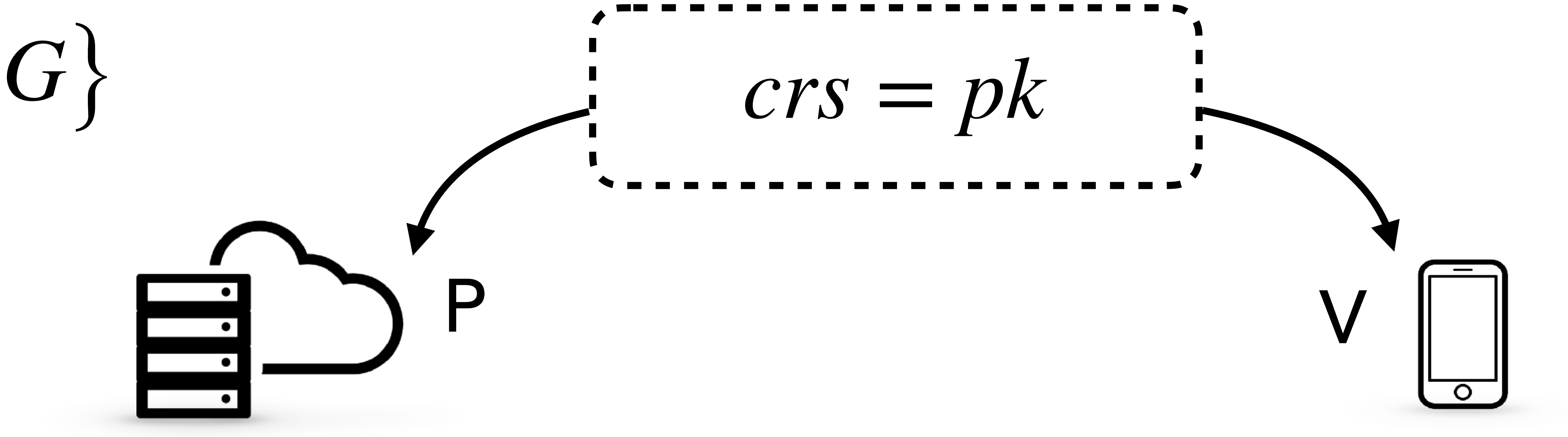
Relation: $\left\{(G, H) \,|\, H \text{ is a Hamiltonian cycle of } G\right\}$

contains all vertices of G



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V

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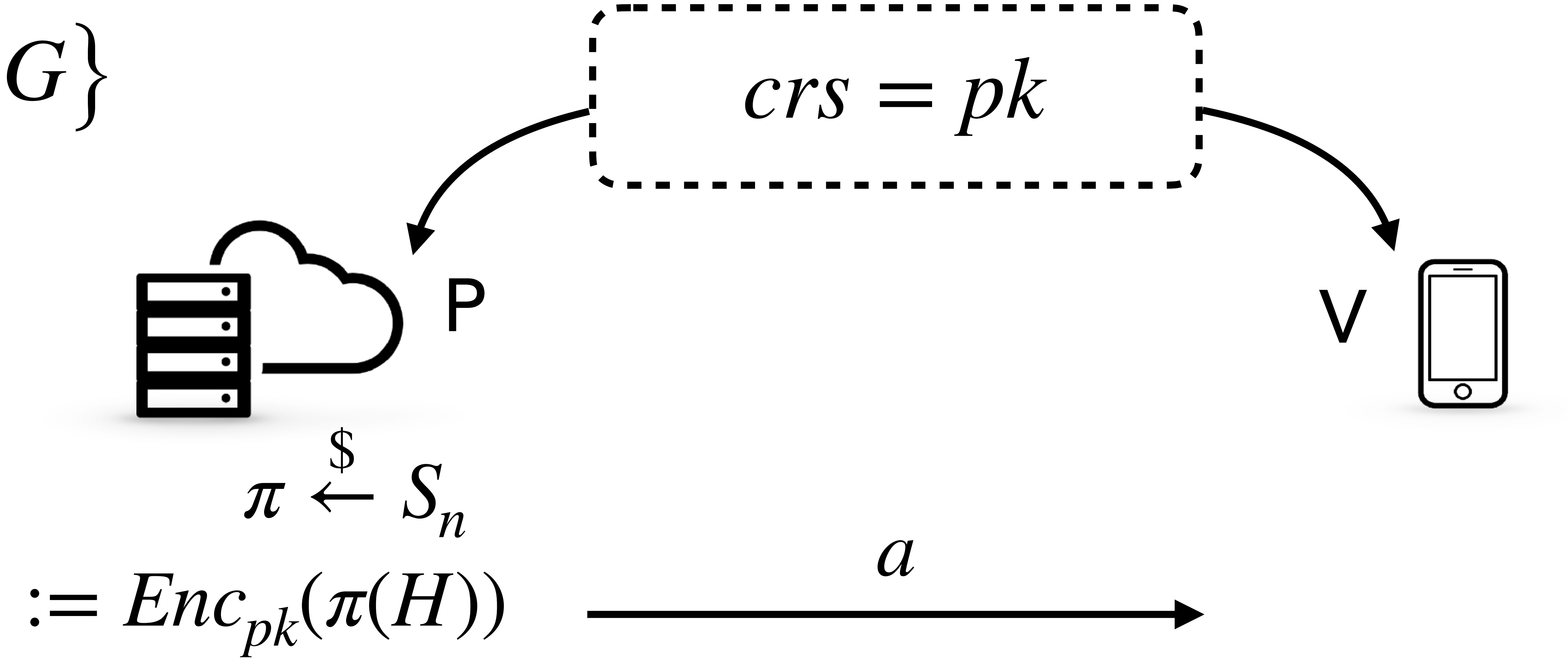

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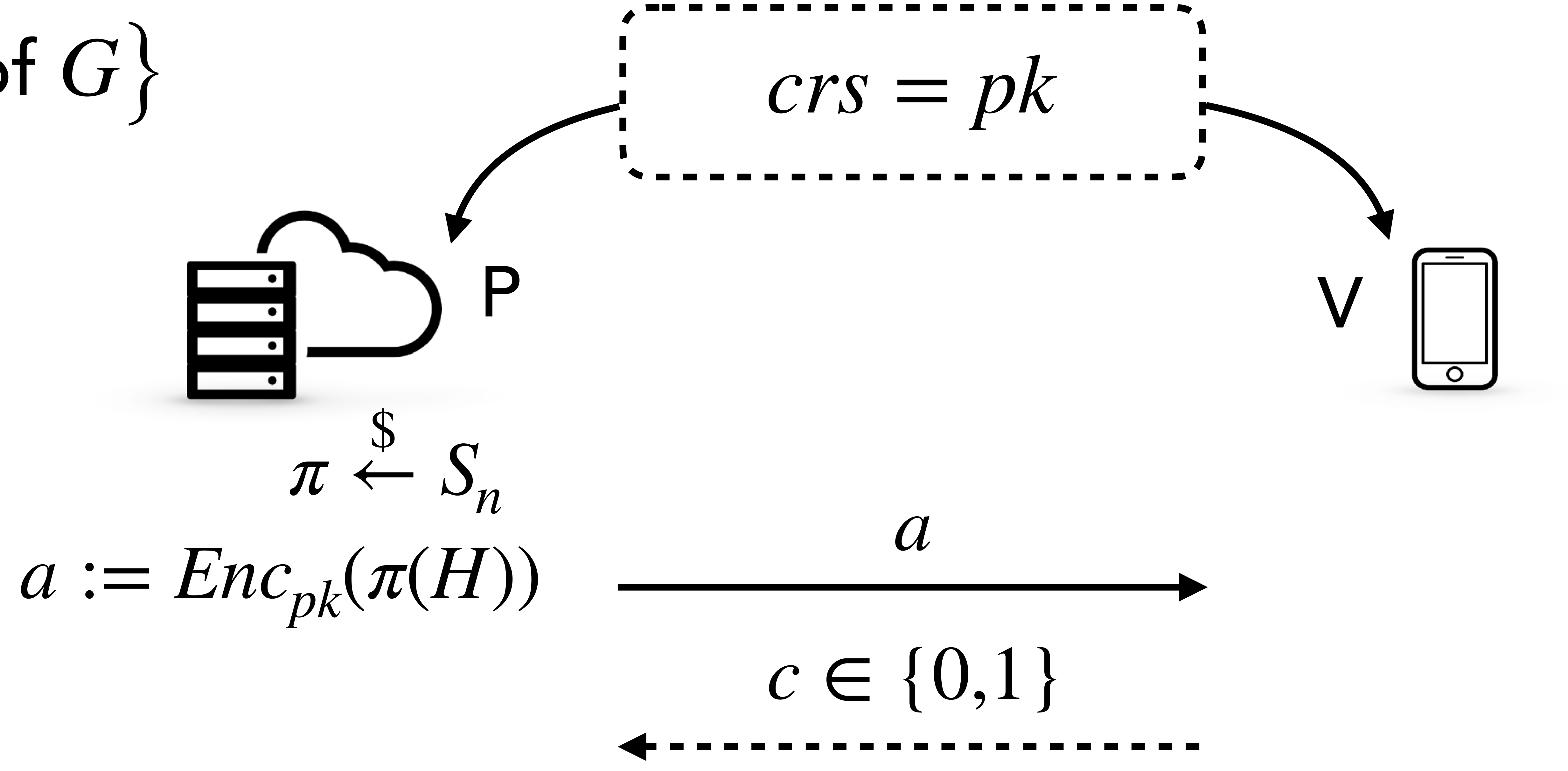

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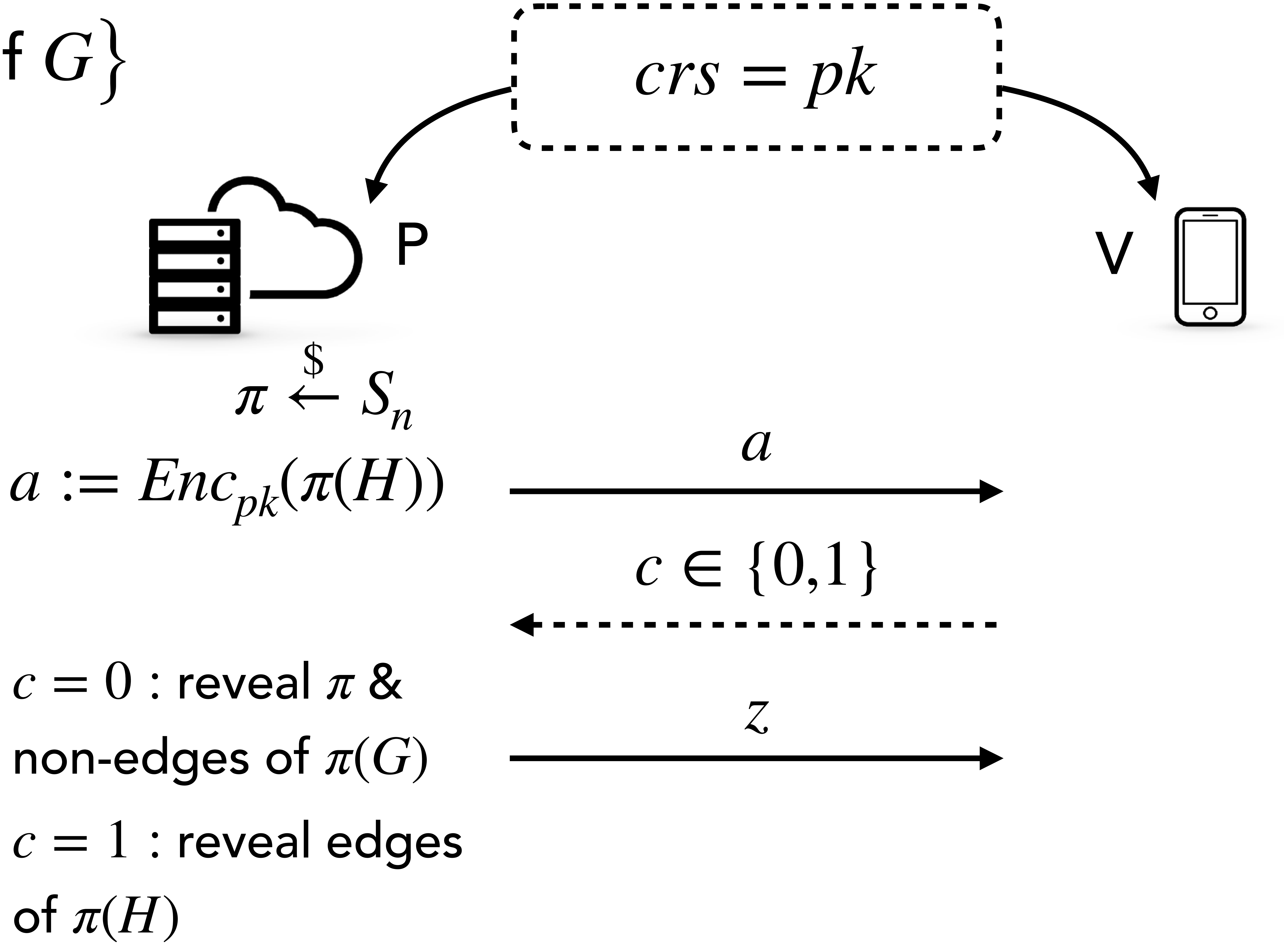

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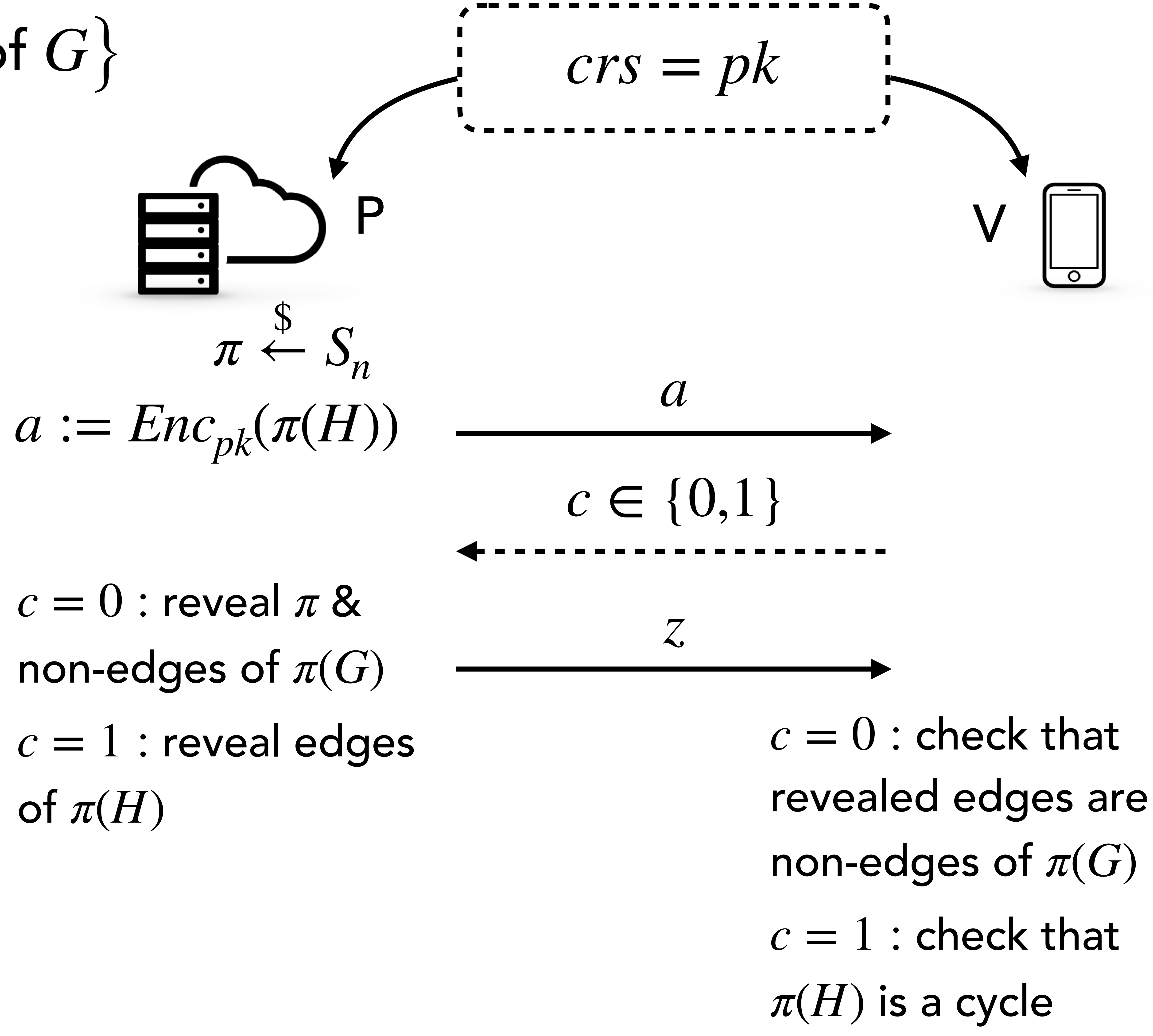

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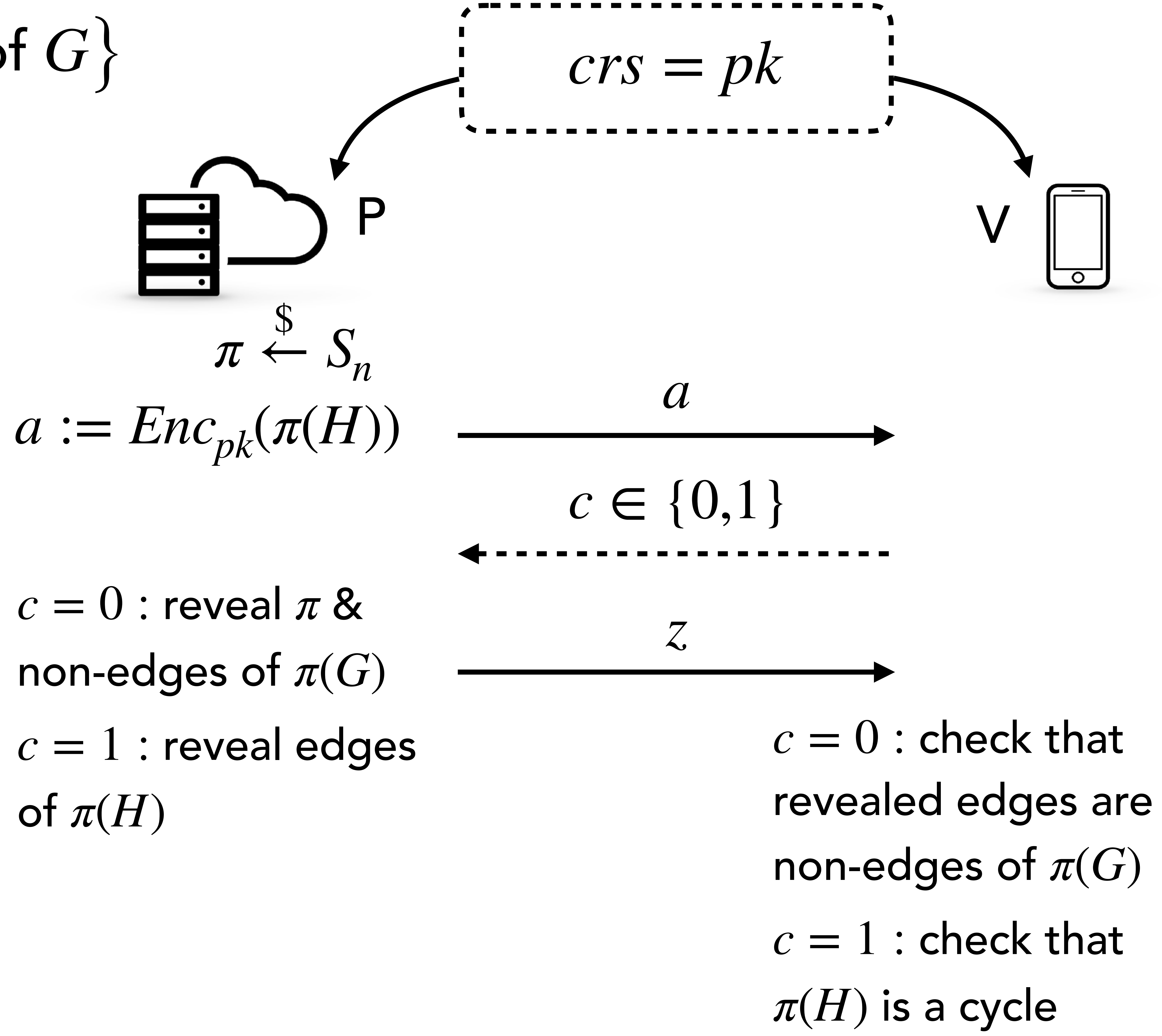
- **Perfect completeness**
- **Soundness error:** $1/2$
- **Honest-verifier zero-knowledge**



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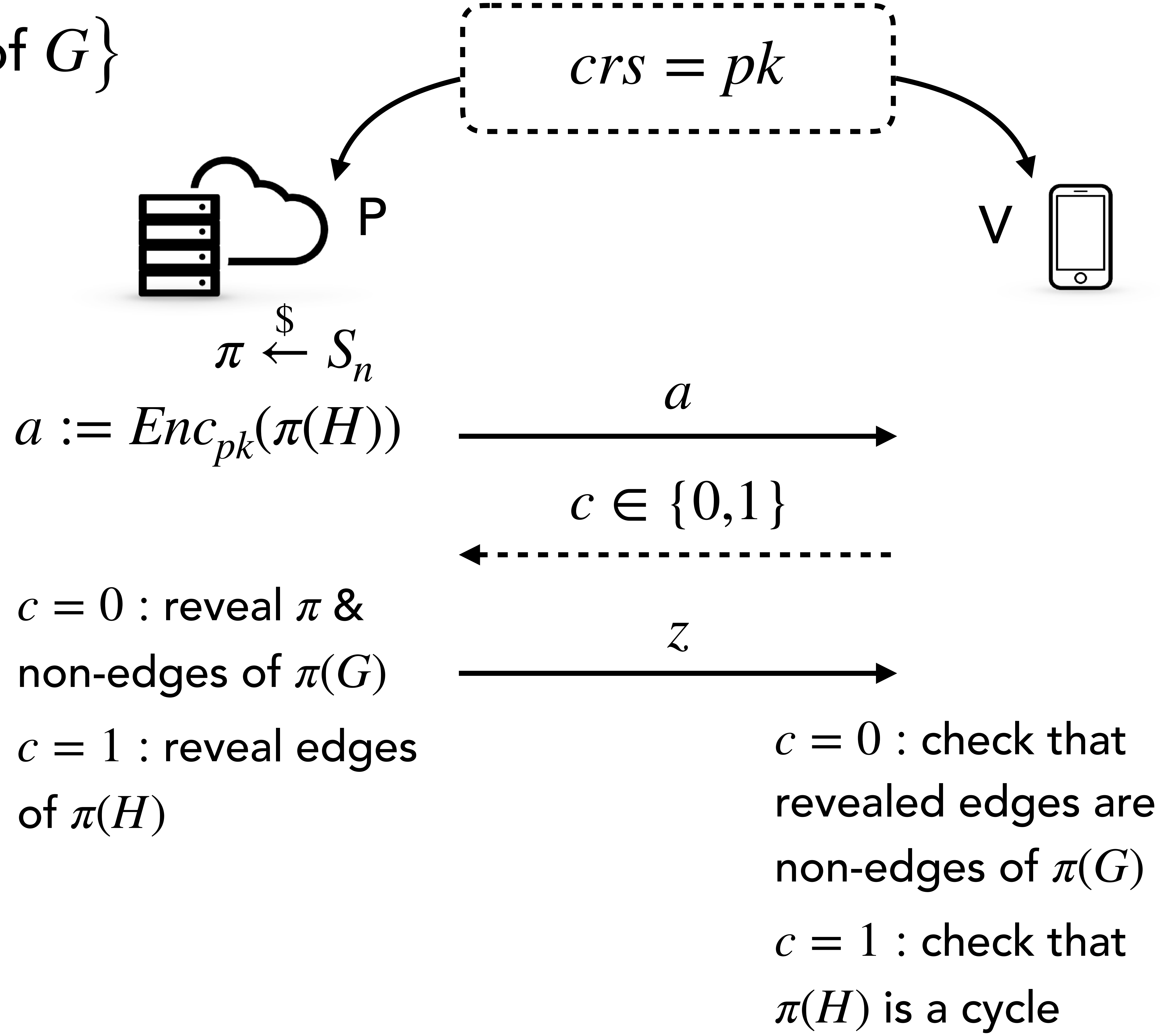

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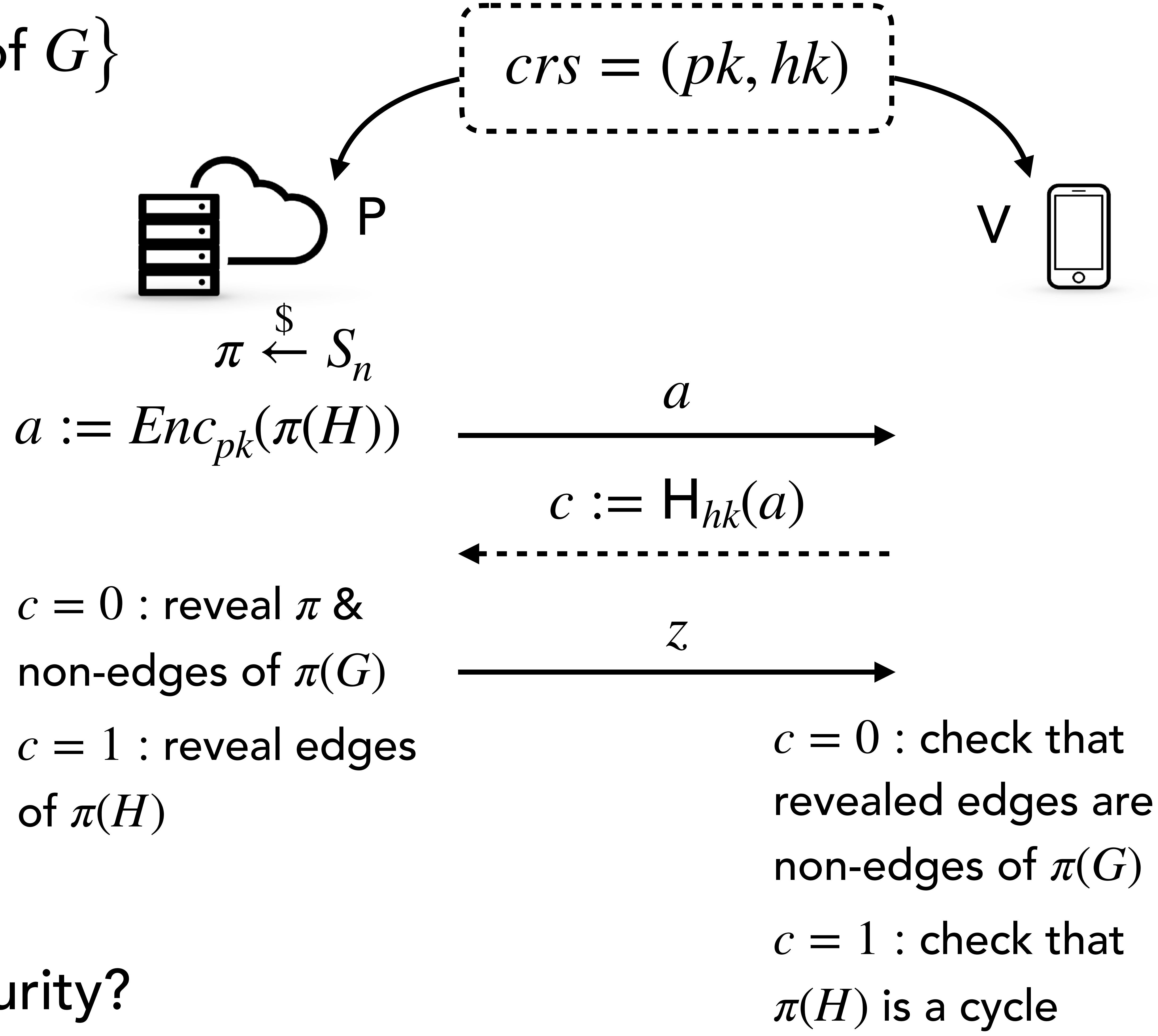
- Derive $c := \mathsf{Hash}_{hk}(a)$
- Which hash function would preserve security?



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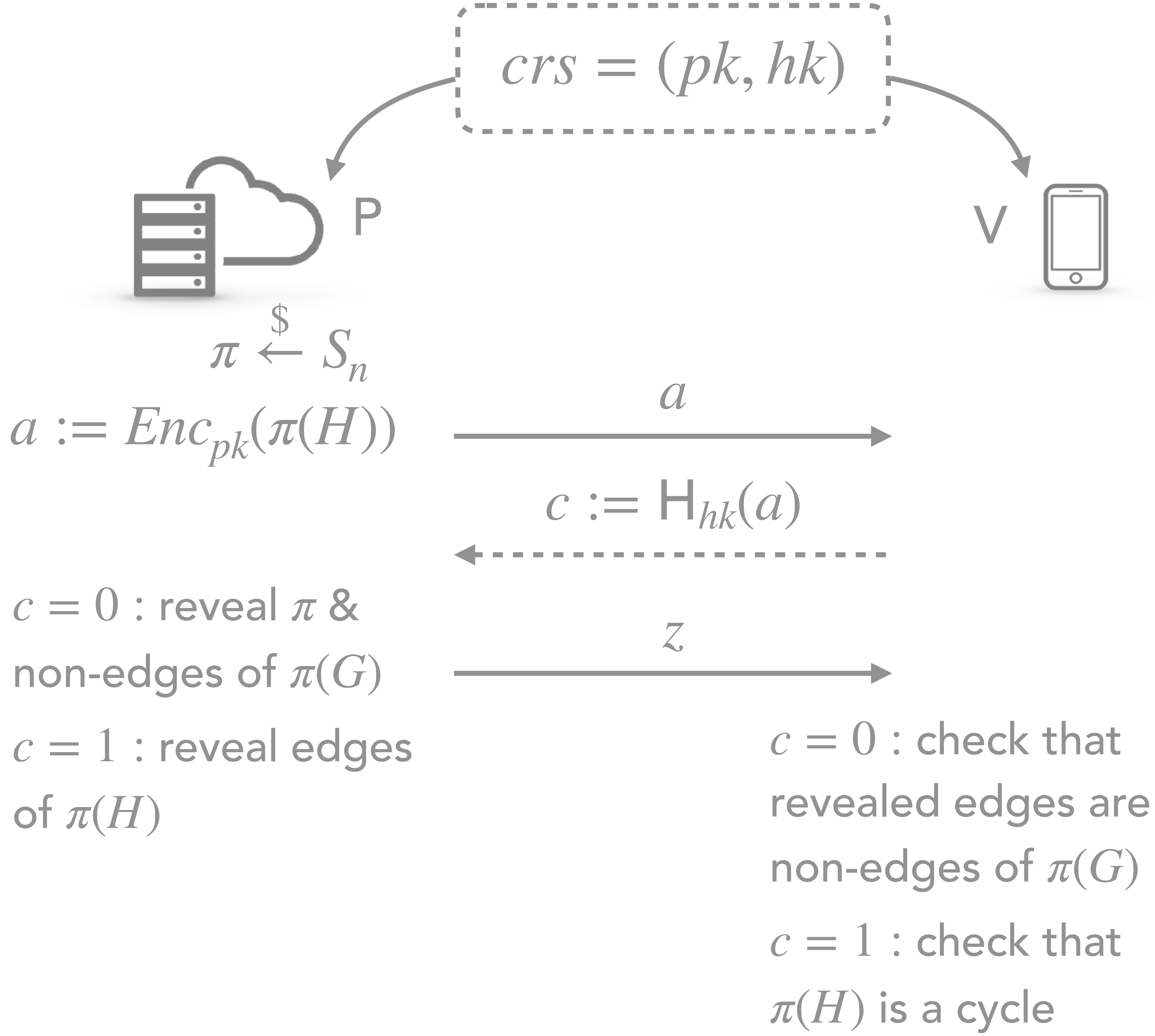

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Correlation Intractability: [CGH04]

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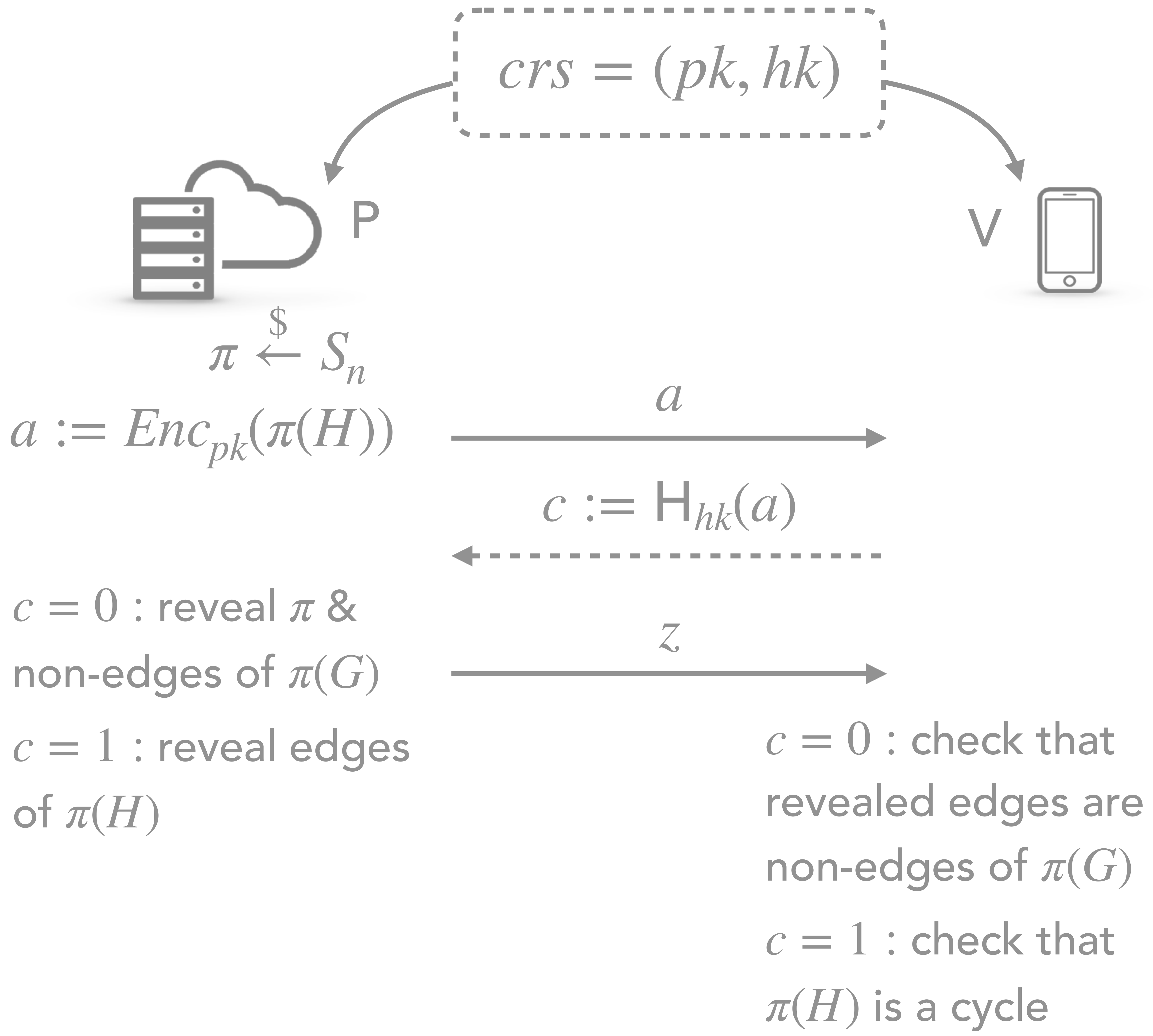

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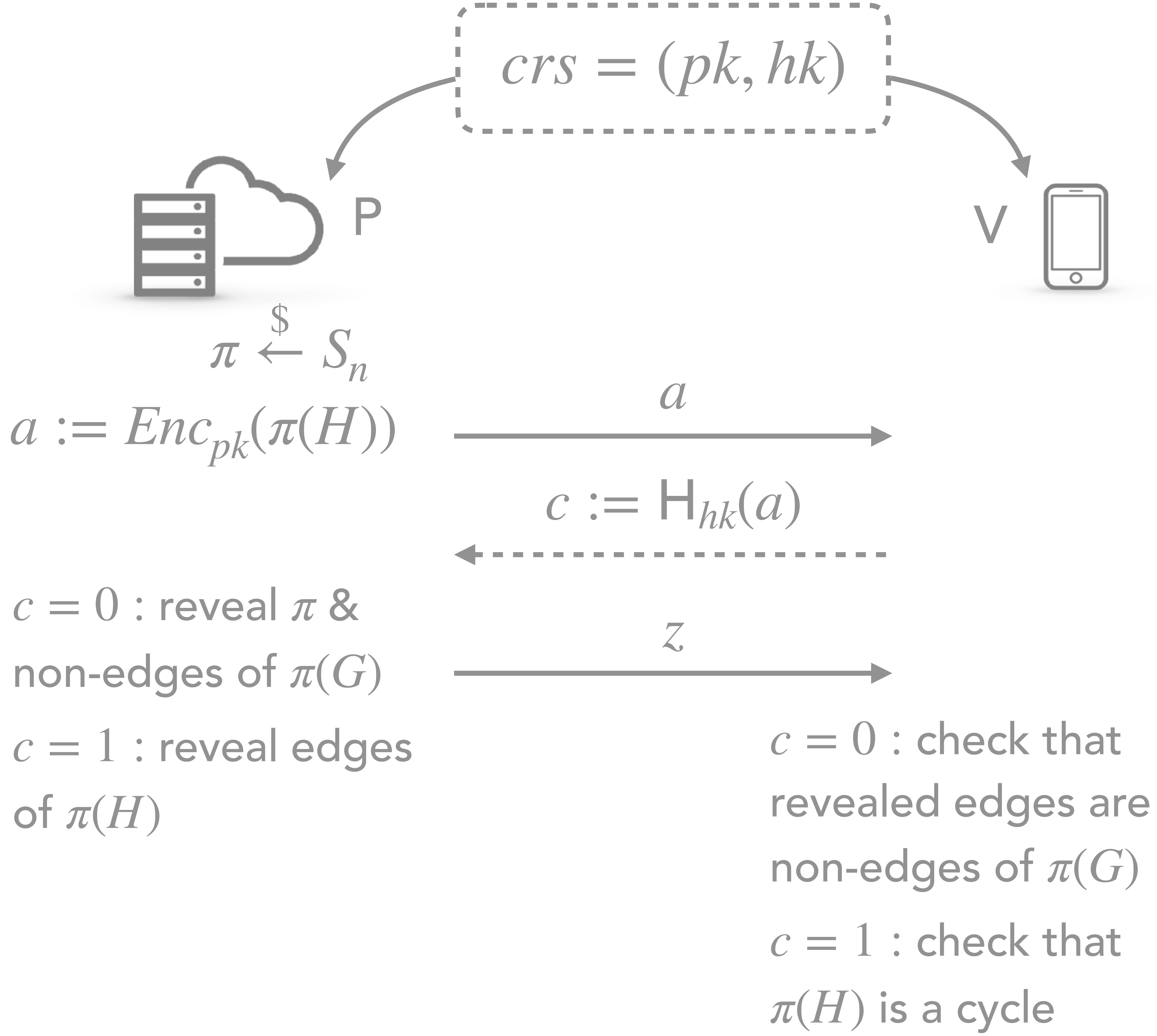

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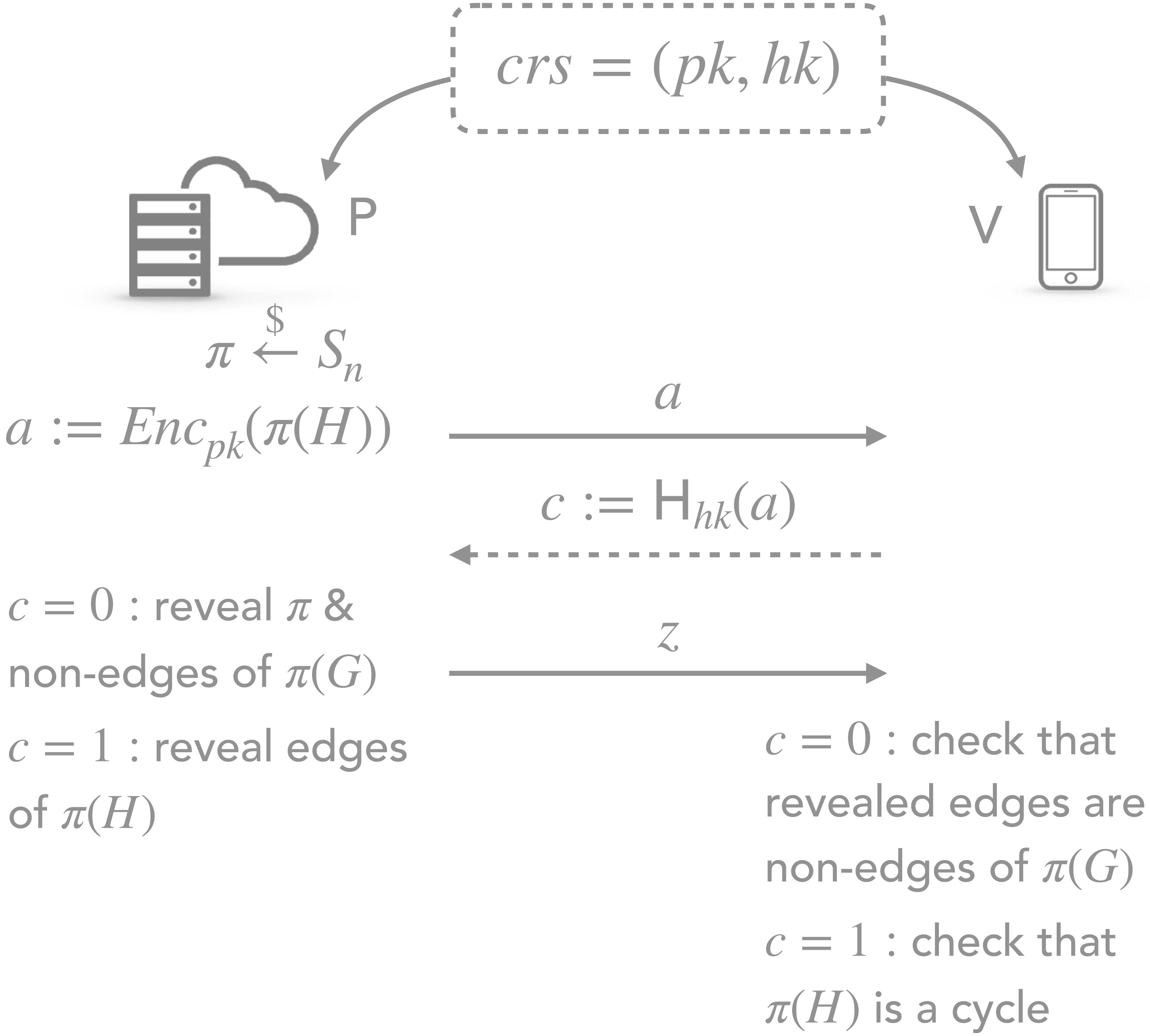

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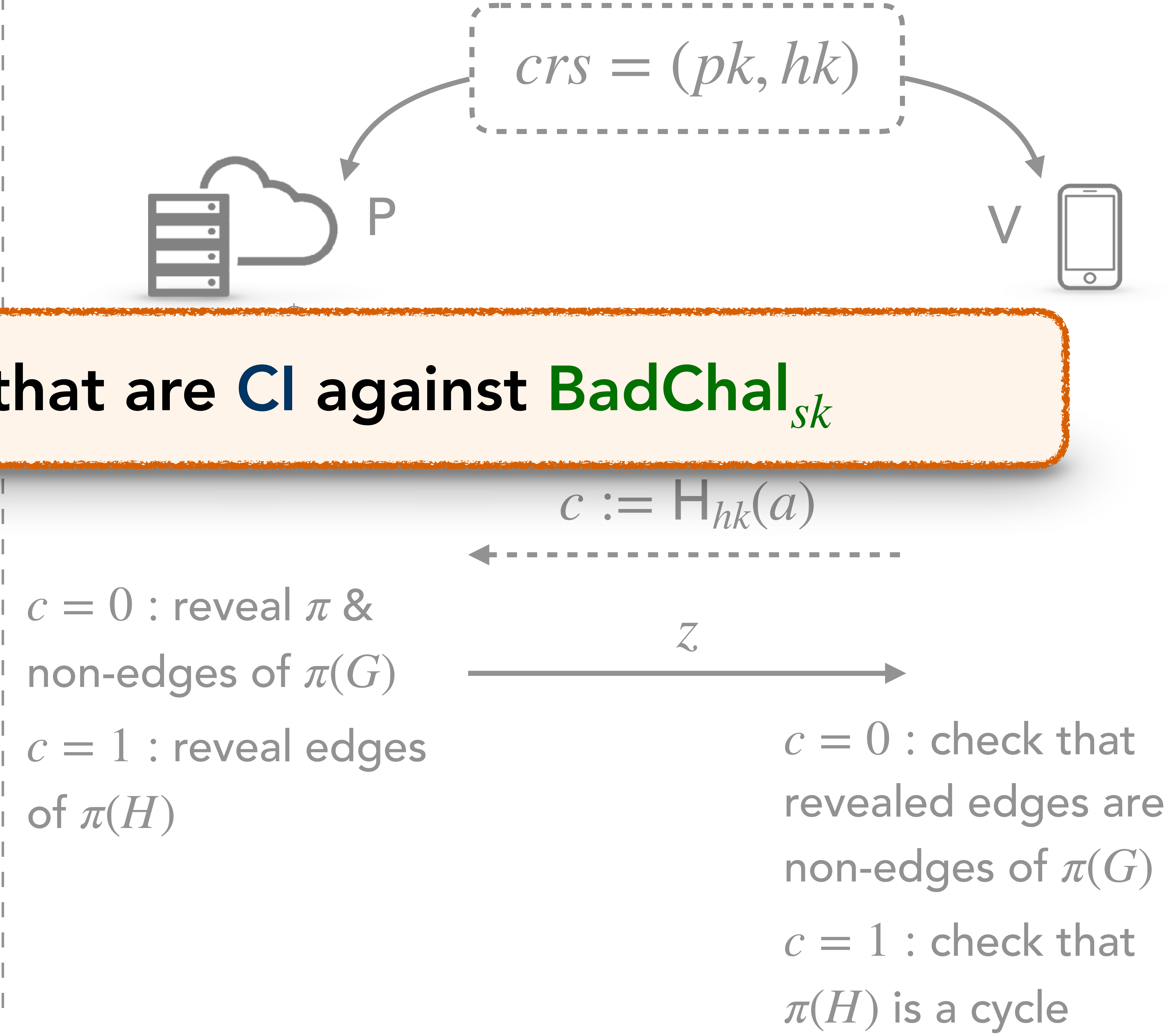

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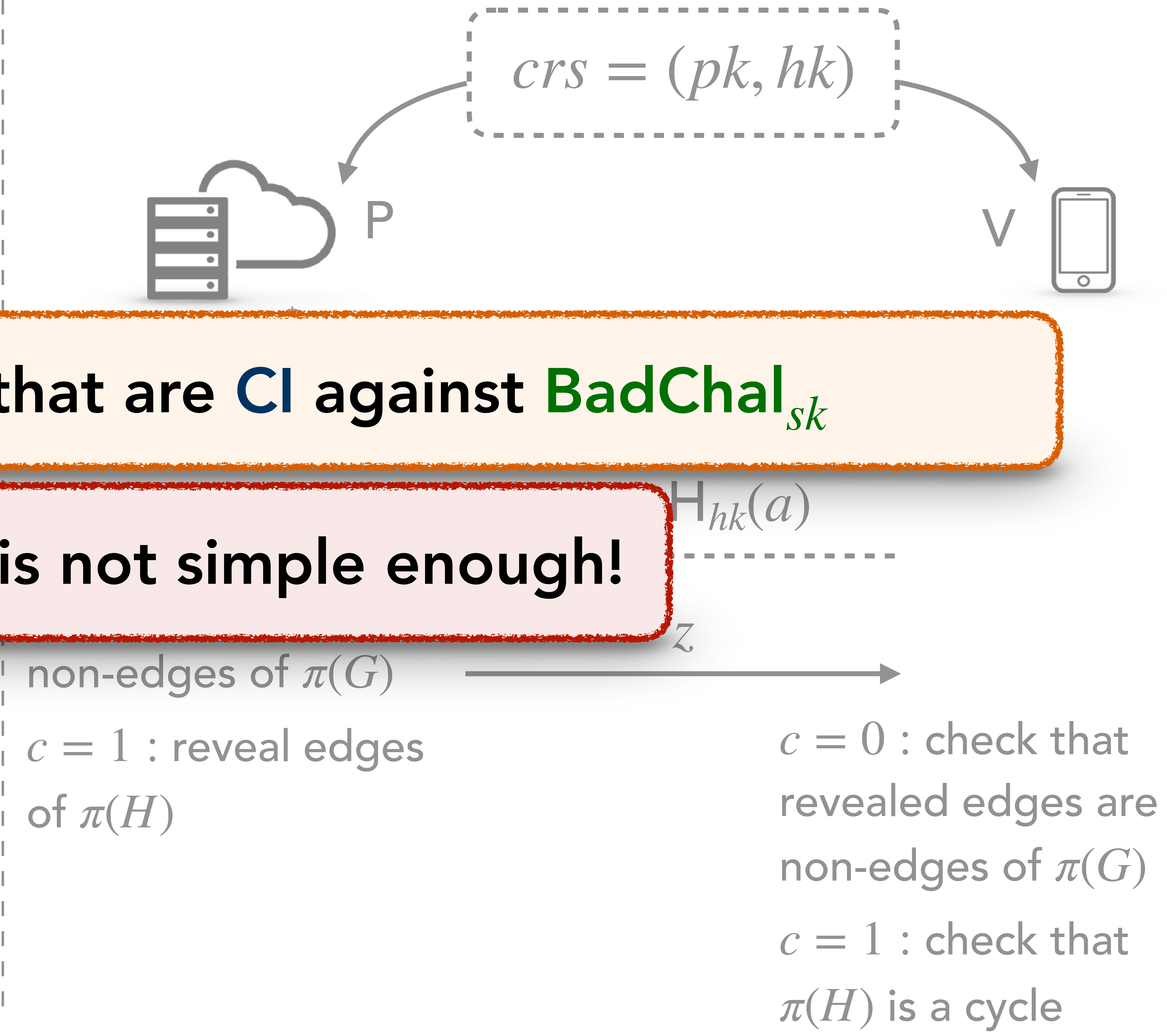
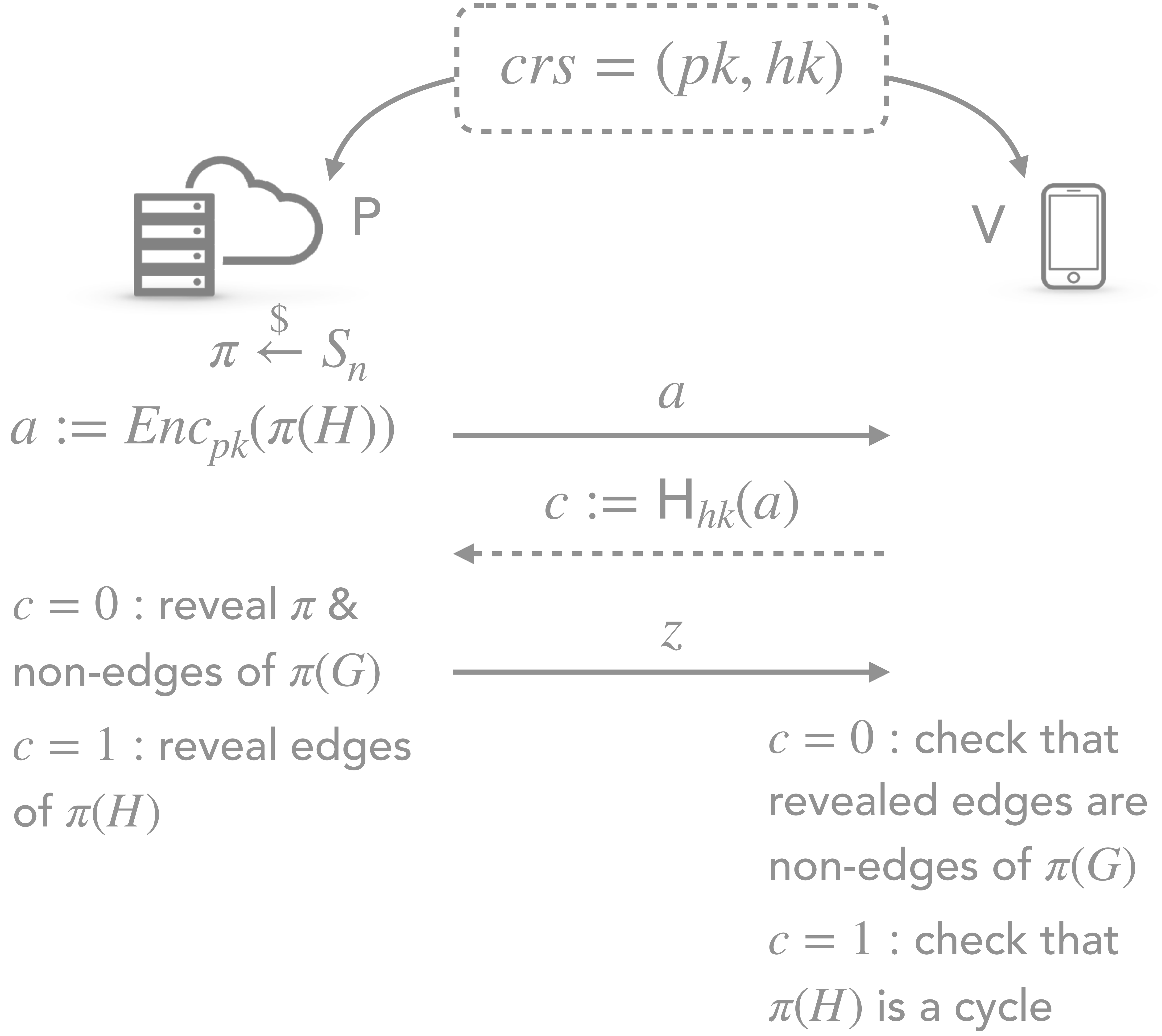
Problem: **BadChal**$_{sk}$ **is not simple enough!**



Simplifying Bad Challenge Function



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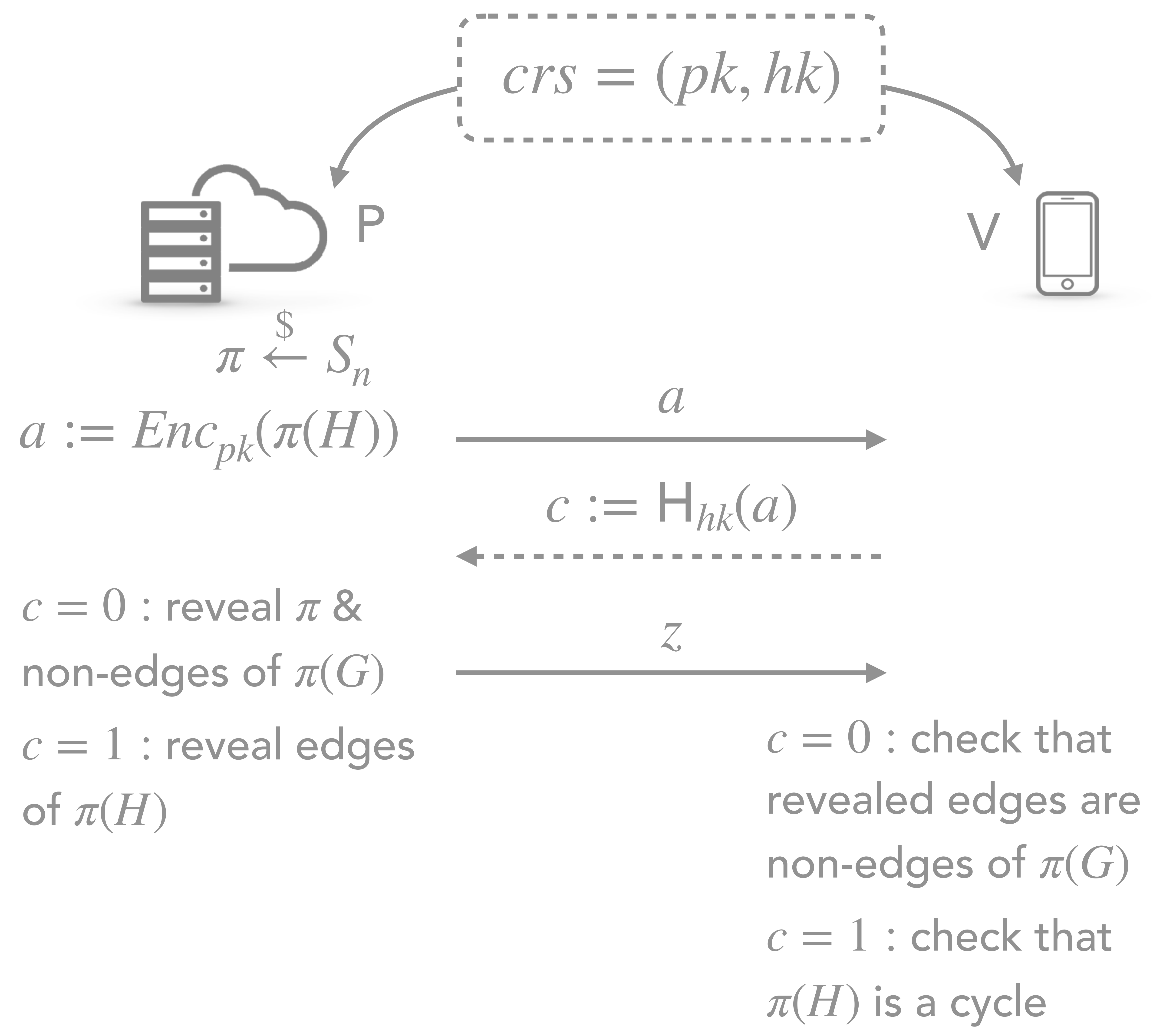

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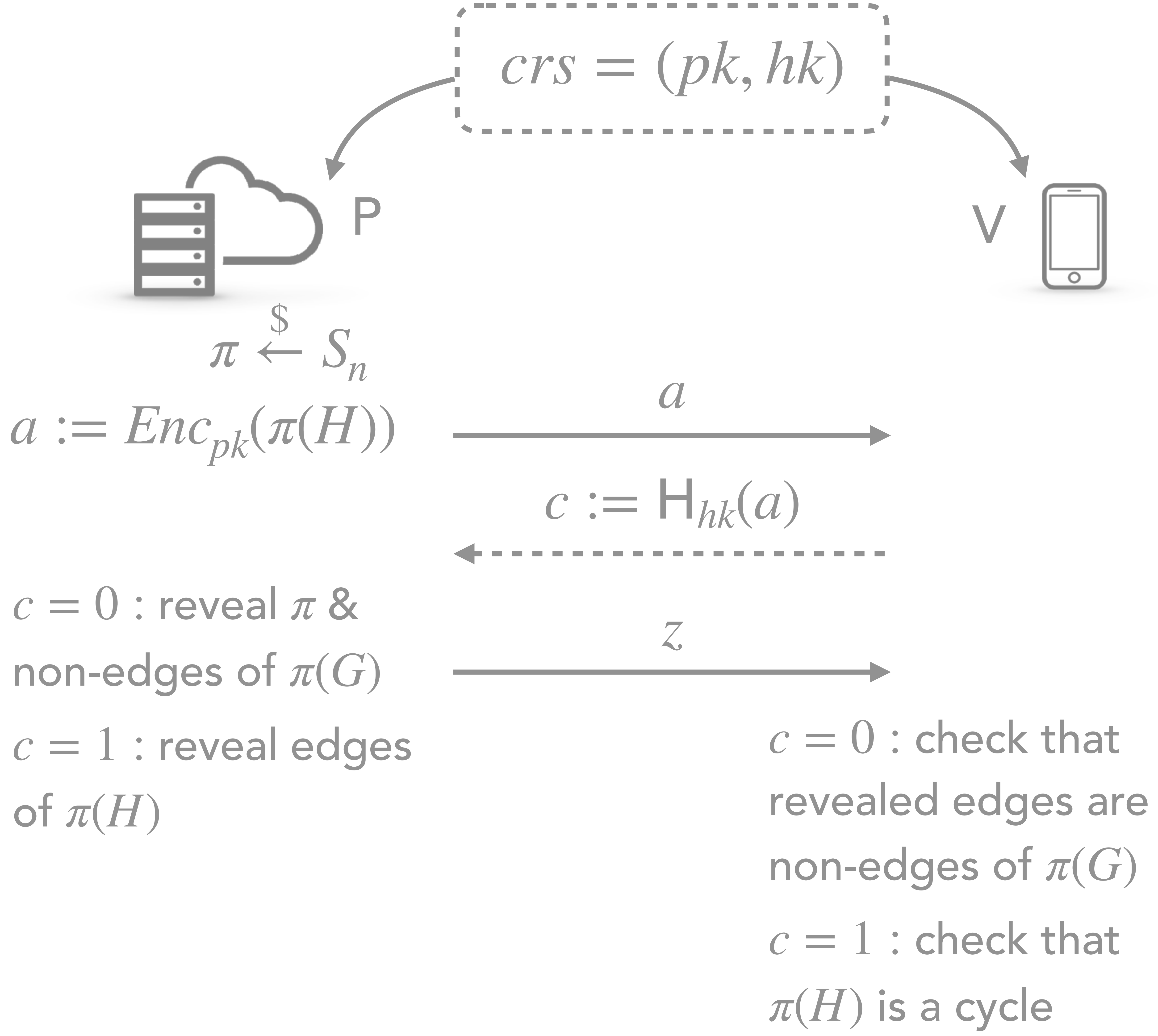
Can we modify BadChal$_{sk}$ to fall into this function class?



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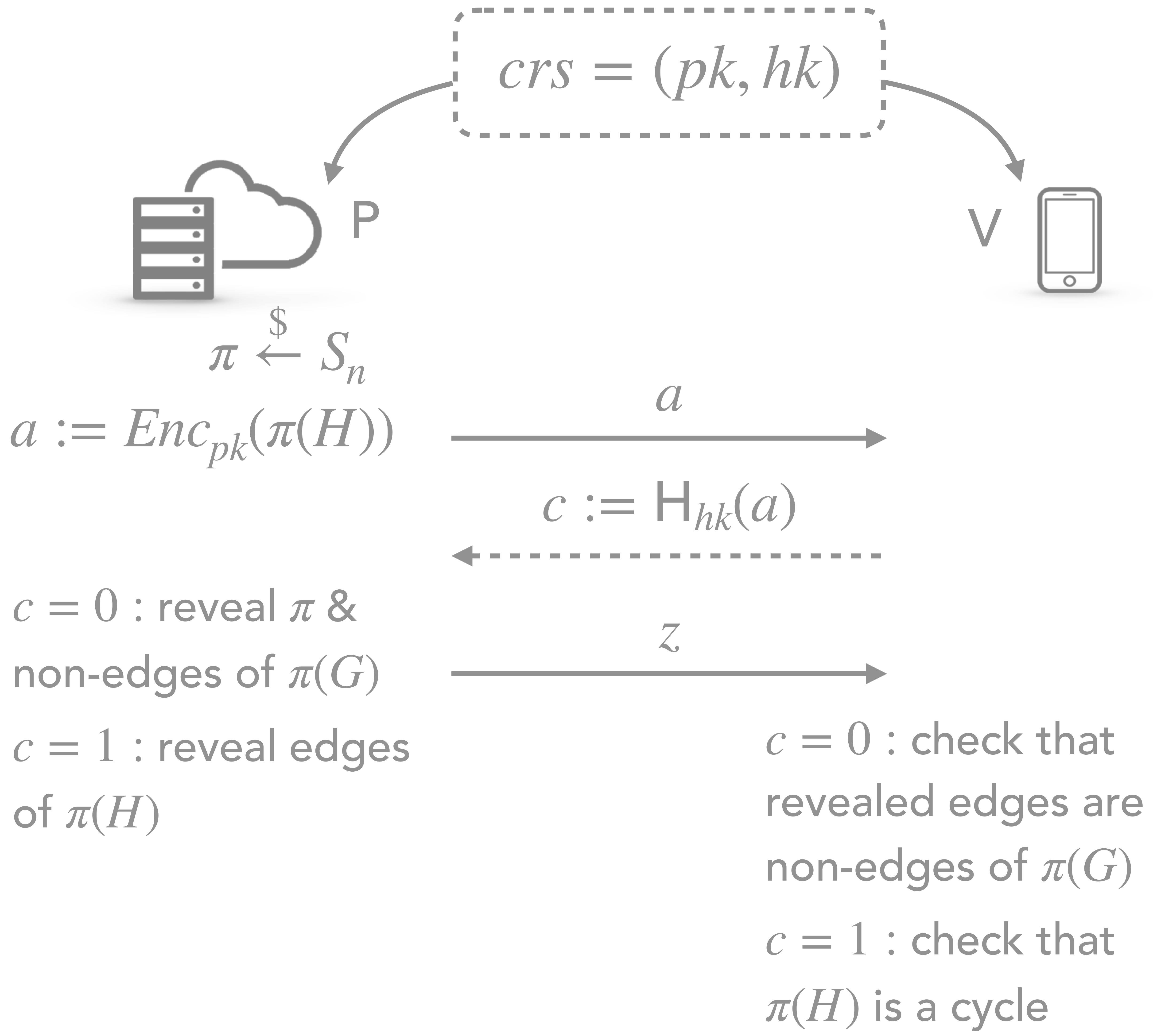

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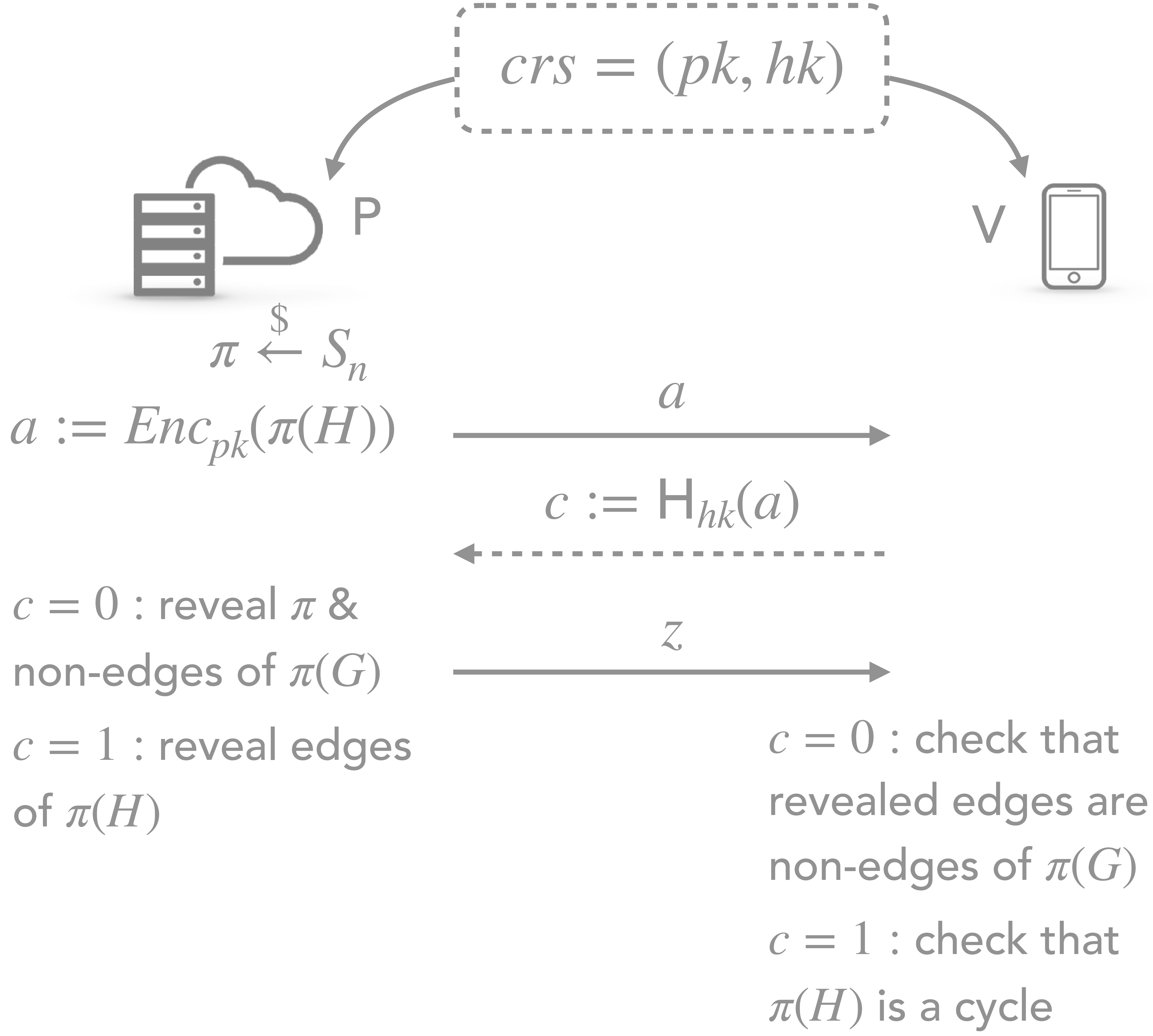
1. Have Dec_{sk} be approximately linear

$\implies$ achieved via **LPN**-based PKE



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P

V

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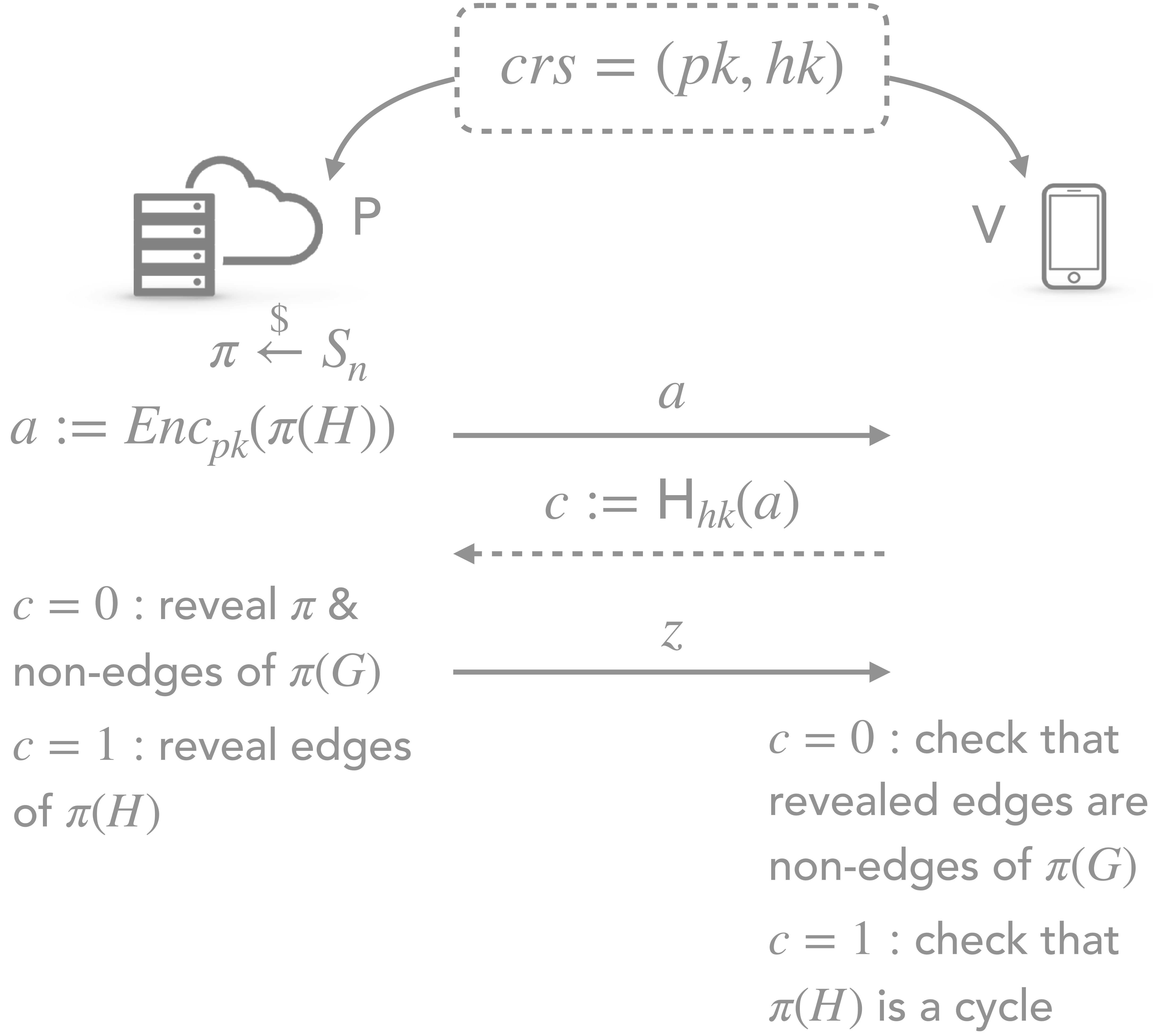

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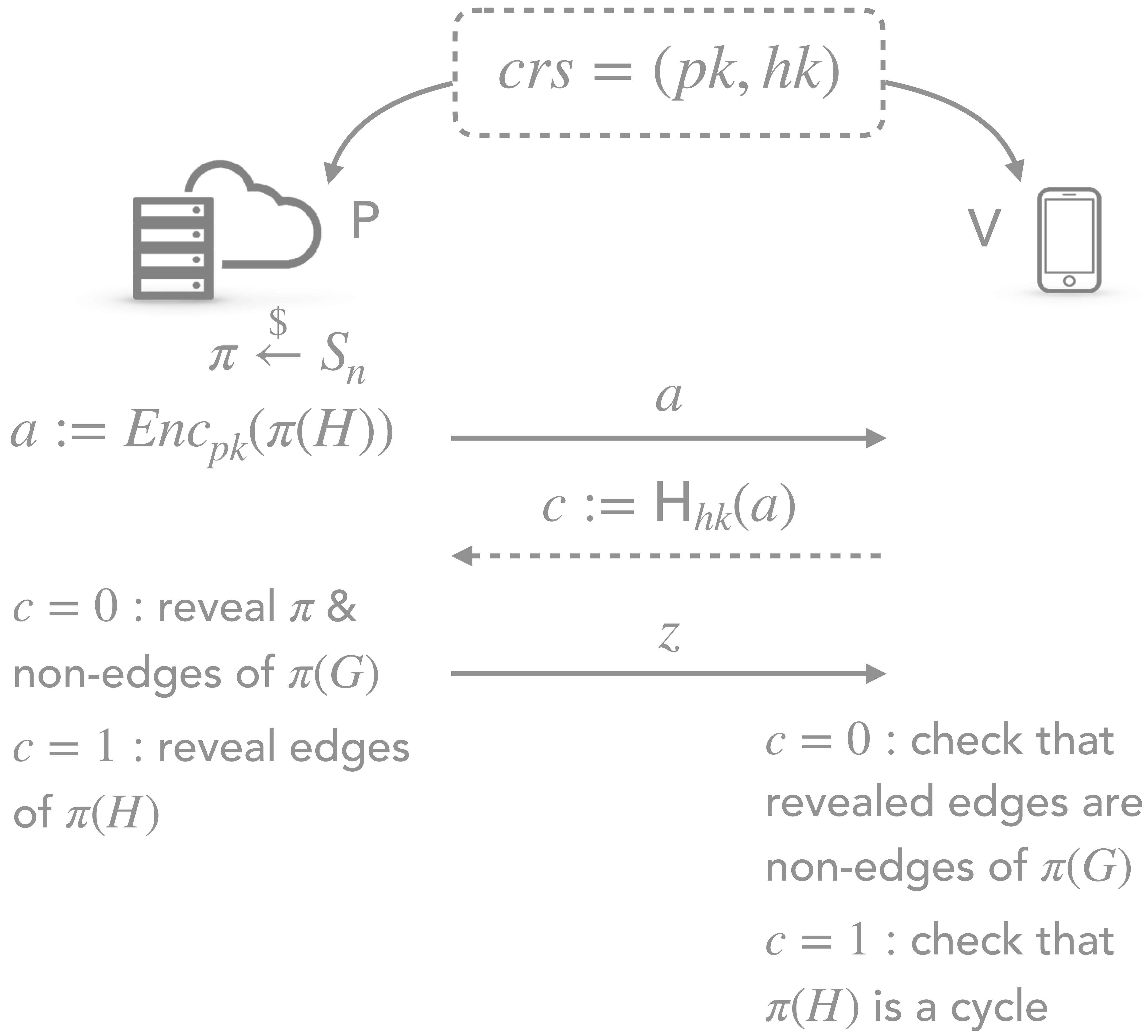

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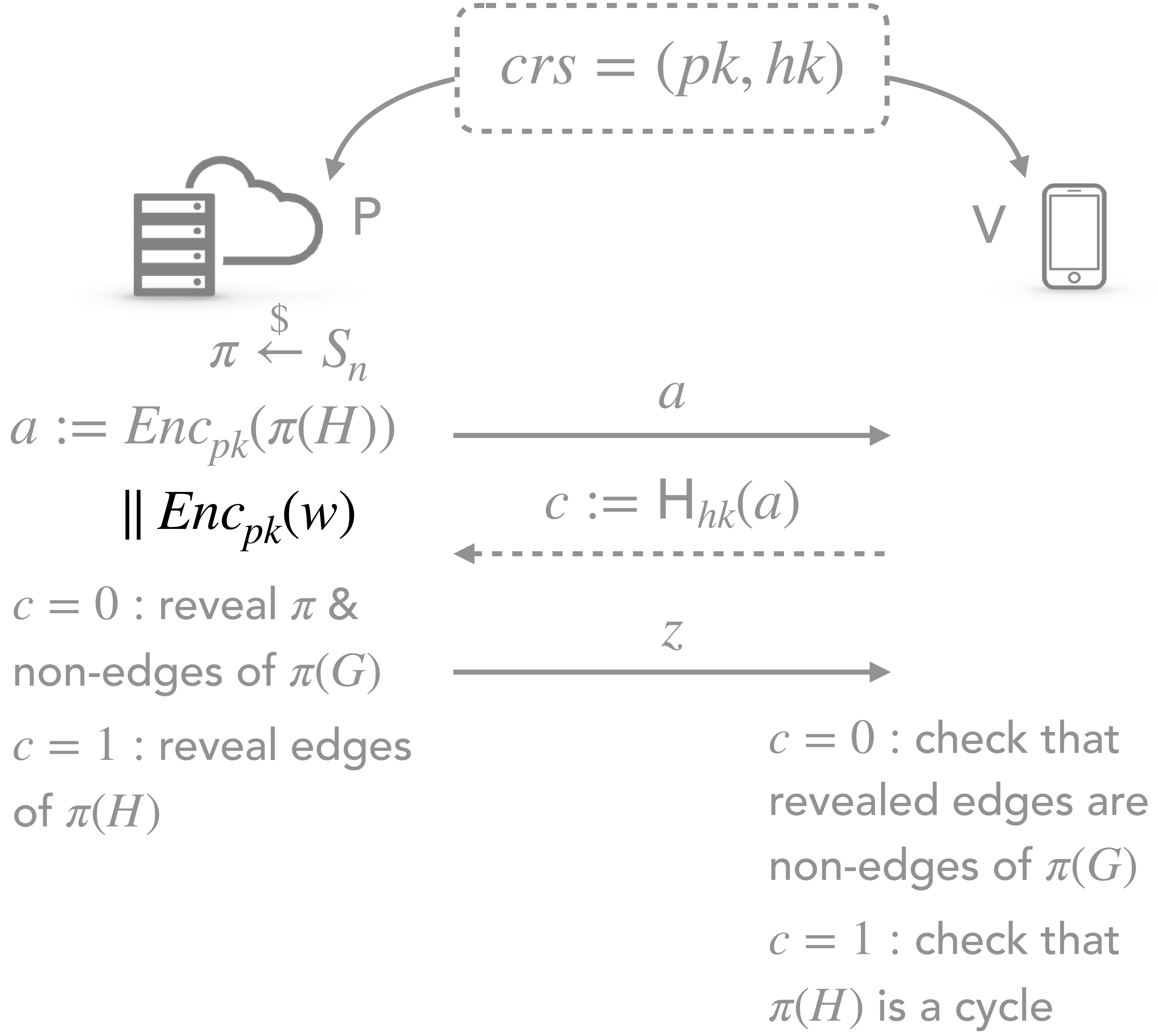

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<u>New Goal:</u> build hash functions that are <u>approximate CI</u> against constant-degree polynomials

1.

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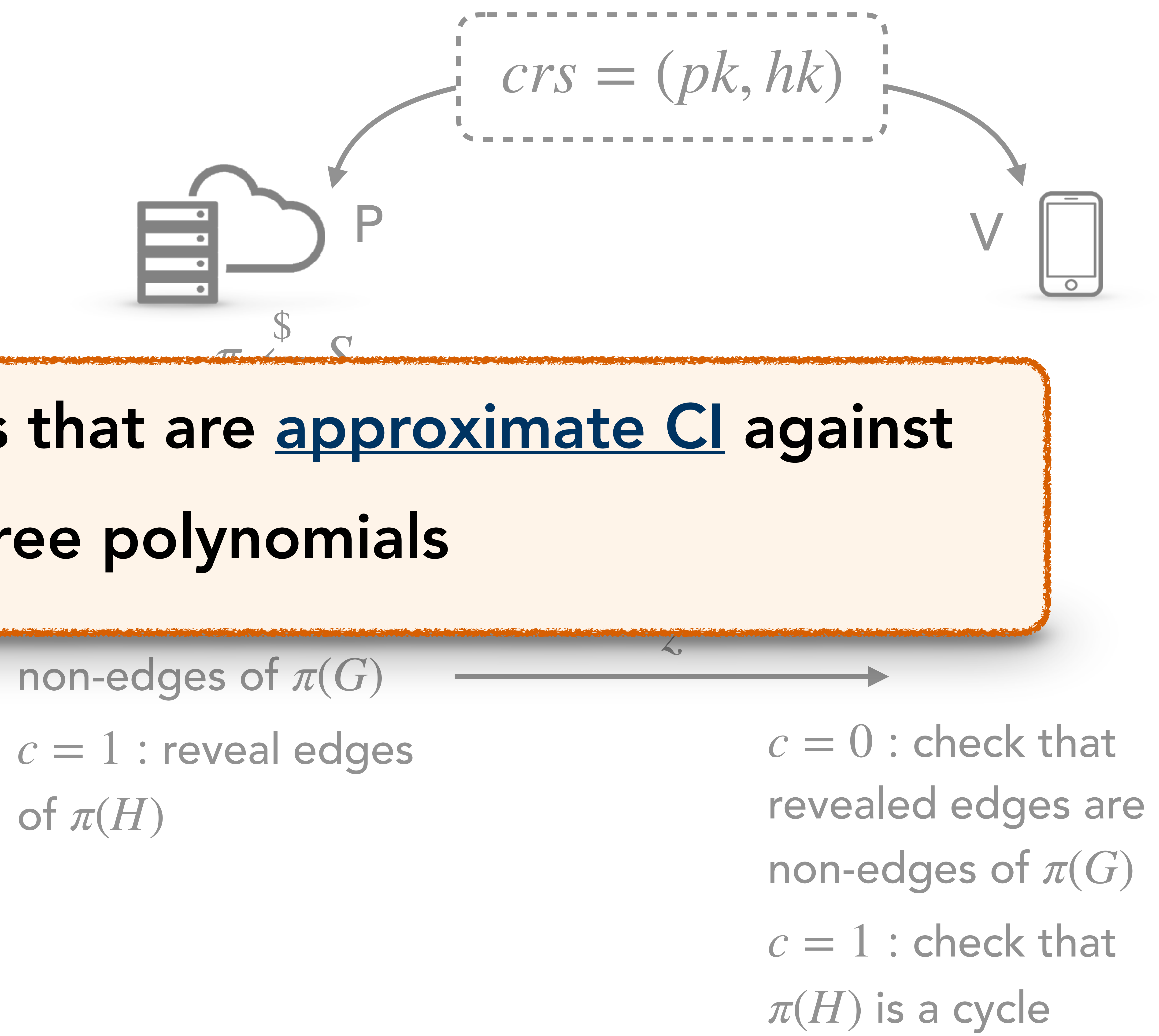

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Multivariate Quadratic

Multivariate Quadratic

Solving a random system of quadratic polynomial equations (over finite $\mathbb{F}$) is hard!

$$\begin{cases} \sum_{i,j=1}^{n} a_{i,j}^{(1)} \cdot x_i \cdot x_j + \sum_{i=1}^{n} b_i^{(1)} \cdot x_i + c^{(1)} = 0 \\ \vdots \\ \sum_{i,j=1}^{n} a_{i,j}^{(m)} \cdot x_i \cdot x_j + \sum_{i=1}^{n} b_i^{(m)} \cdot x_i + c^{(m)} = 0 \end{cases}, \quad \text{where} \quad \begin{cases} n = \#\ \text{variables} \\ m = \#\ \text{equations} \\ \text{eqns. over a finite field } \mathbb{F} \end{cases}$$

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- Hard for $\sqrt{n} \ll m \ll n^2$. Usual parameter regime: $m = \Theta(n)$

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- One of the main branches of assumptions in post-quantum cryptography
- Hard for $\sqrt{n} \ll m \ll n^2$. Usual parameter regime: $m = \Theta(n)$
- **<u>This work:</u>** under-determined setting, with $m = n^{1-\epsilon}$ for any $\epsilon > 0$

$\Longrightarrow$ best cryptanalysis [TW12, MHT13] suggests ***<u>exponential</u>*** security*

* poly-time attackers have ***<u>exponentially</u>*** small success probability

Hash Function from Multivariate Quadratic

Hash Function from Multivariate Quadratic

Hash key

$$Q := \left(a_{i,j}^{(k)},\, b_i^{(k)},\, c^{(k)}\right)_{i,j\in[n],\, k\in[m]}$$

Hash evaluation

$$\mathsf{H}_Q(x) := Q(x) = \left(\sum_{i,j} a_{i,j}^{(k)} x_i x_j + b_i^{(k)} x_i + c^{(k)}\right)_{k=1}^{m}$$

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However, it is **correlation-intractable** against quadratic polynomials!

- Assume $\mathsf{H}_Q(x) = f(x)$ for a quadratic function f.
- Switch to hybrid where $Q \mapsto Q + f \implies$ hash key is still random
- But we have: $\mathsf{H}_{Q+f}(x) = f(x) \iff Q(x) = 0$, which breaks **MQ**.

Approximate CI from Approximate MQ

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- **Approximate MQ** is implied by **MQ** with exponential security

* via guessing
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Question: How to achieve ApproxCI against degree$-d$ polynomials, for *any* constant d?

Approximate CI against Degree$-d$ Polys

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Two Solutions:

Approximate CI against Degree$-d$ Polys

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1. Evaluate random degree$-d$ polynomials on input
    - Hardness follows from degree$-d$ analogue of **(Approximate) MQ**
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Approximate CI against Degree$-d$ Polys

Two Solutions:

1. Evaluate random degree$-d$ polynomials on input
   - Hardness follows from degree$-d$ analogue of (Approximate) MQ
   - **Downsides:** not as well-studied, blows up key size & evaluation time
2. Achieve approximate CI against a ***sub-class*** of degree$-d$ polynomials
   - ***Concatenated*** degree$-d$ polys: $P\left(x_1 \| \ldots \| x_l\right) = P_1(x_1) \| \ldots \| P_l(x_l)$, $\deg(P_i) = d$
   - **Setting:** $|x_i| = s$ is fixed, $l = poly(\lambda)$ may grow
   - Hash evaluation: $\mathsf{H}_Q\left(x_1 \| \ldots \| x_l\right) := Q\left(x_1^{\otimes d/2}, \ldots, x_l^{\otimes d/2}\right)$

     $\Longrightarrow$ achieves compression for large enough $l = poly(\lambda)$

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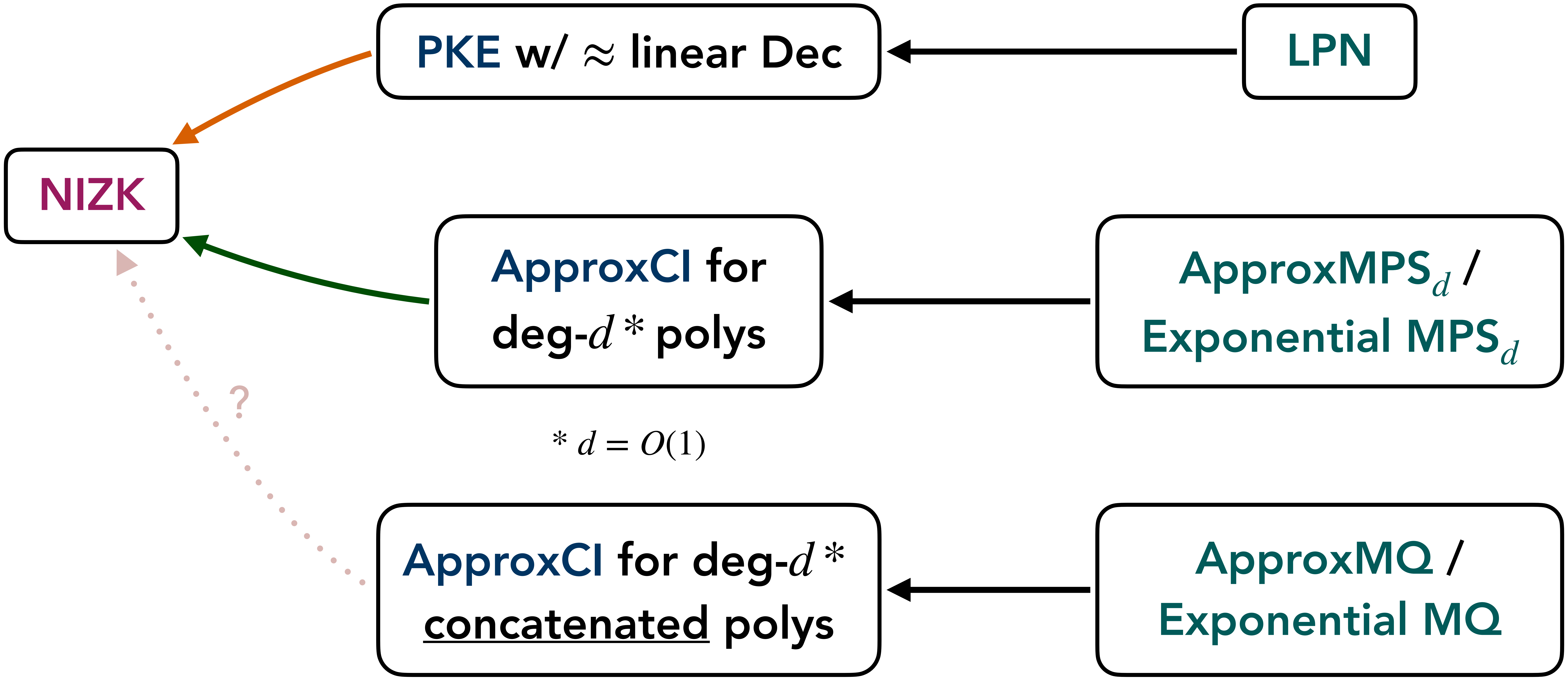
Road to Achieve NIZK

Road to Achieve **NIZK**



PKE w/ ≈ linear Dec

LPN

NIZK

ApproxCI for deg-d* polys

ApproxMPS$_d$ / Exponential MPS$_d$

?

* $d = O(1)$

ApproxCI for deg-d* concatenated polys

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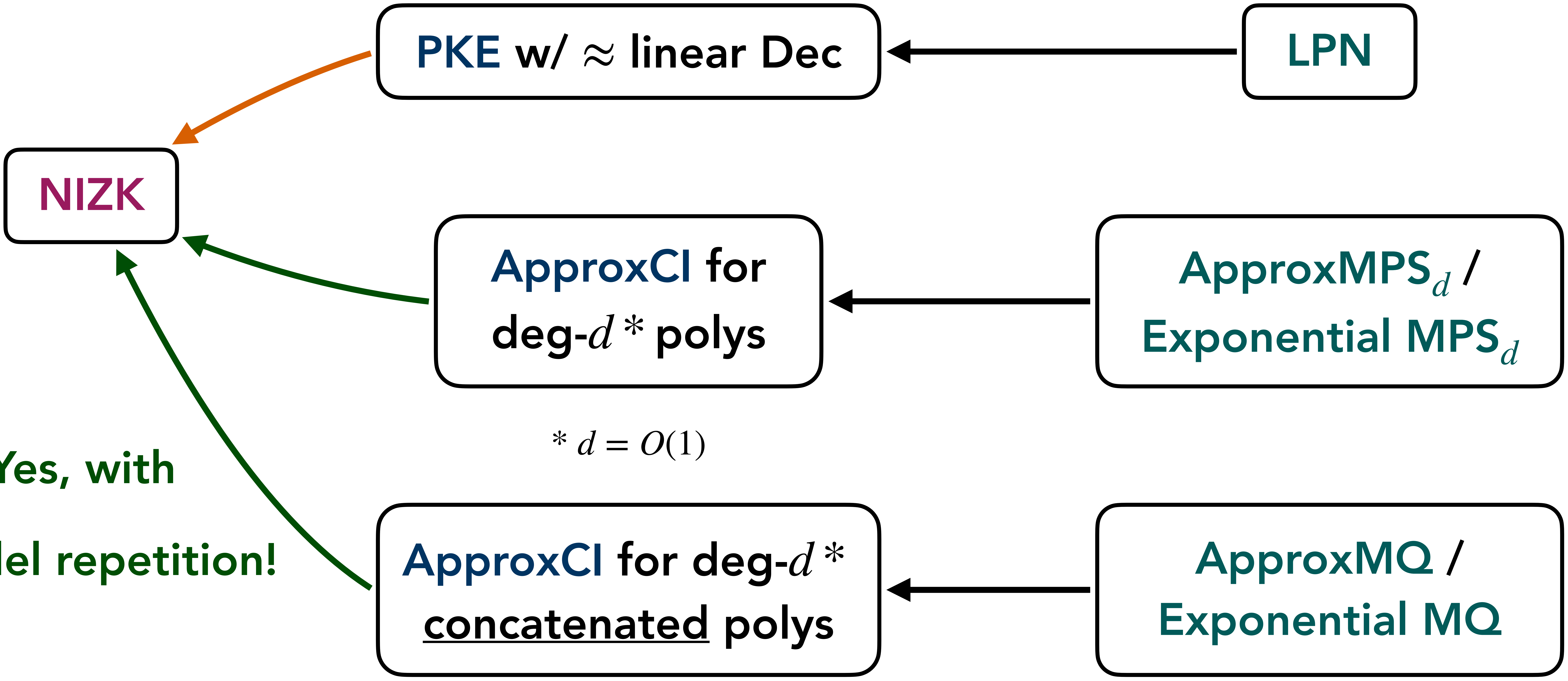

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Yes, with parallel repetition!

ApproxCI for deg-$d*$ concatenated polys

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ApproxCI for Concatenated Poly's Suffices

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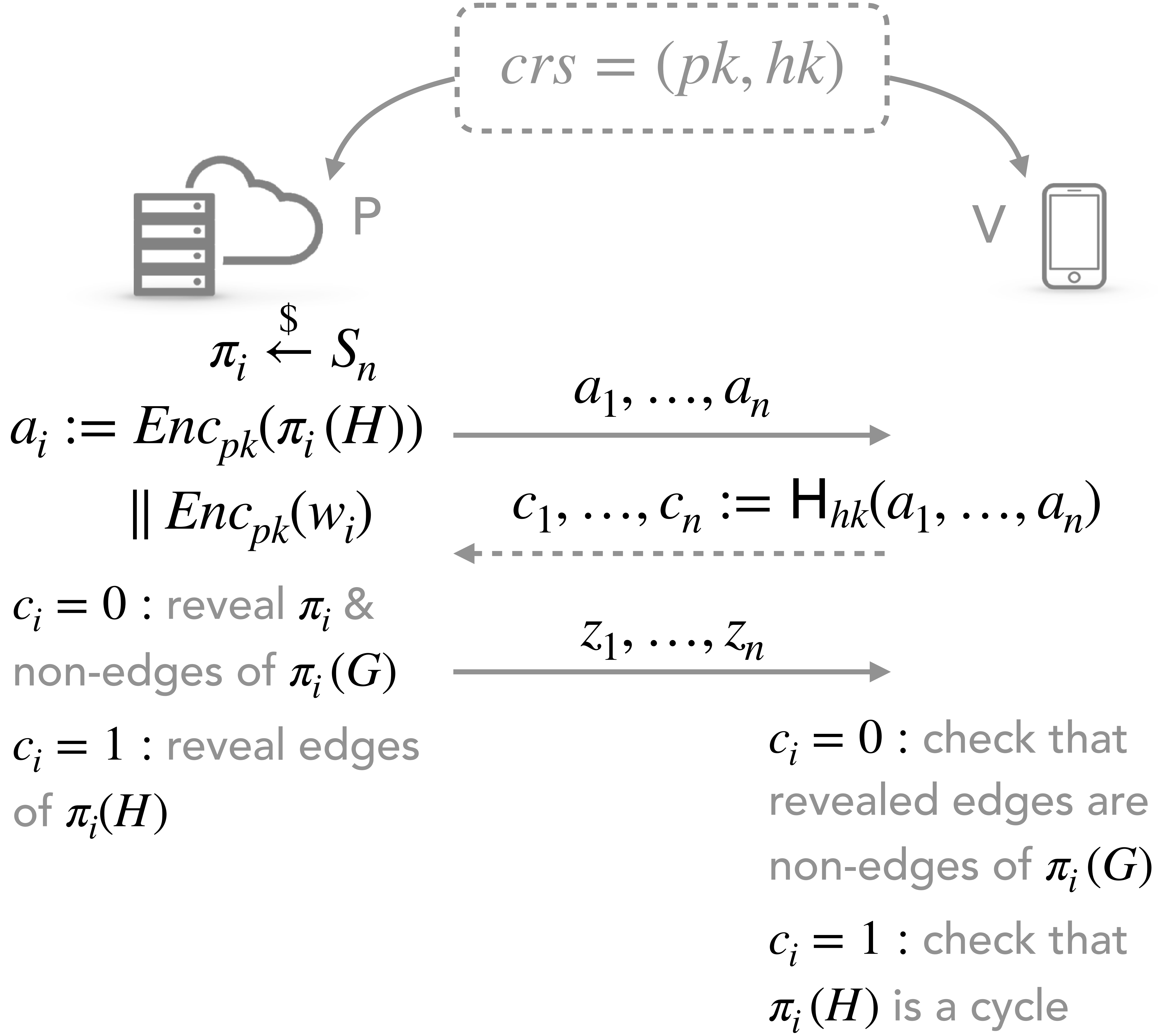
Bad challenge function of parallel-repeated protocol has ***concatenated*** format



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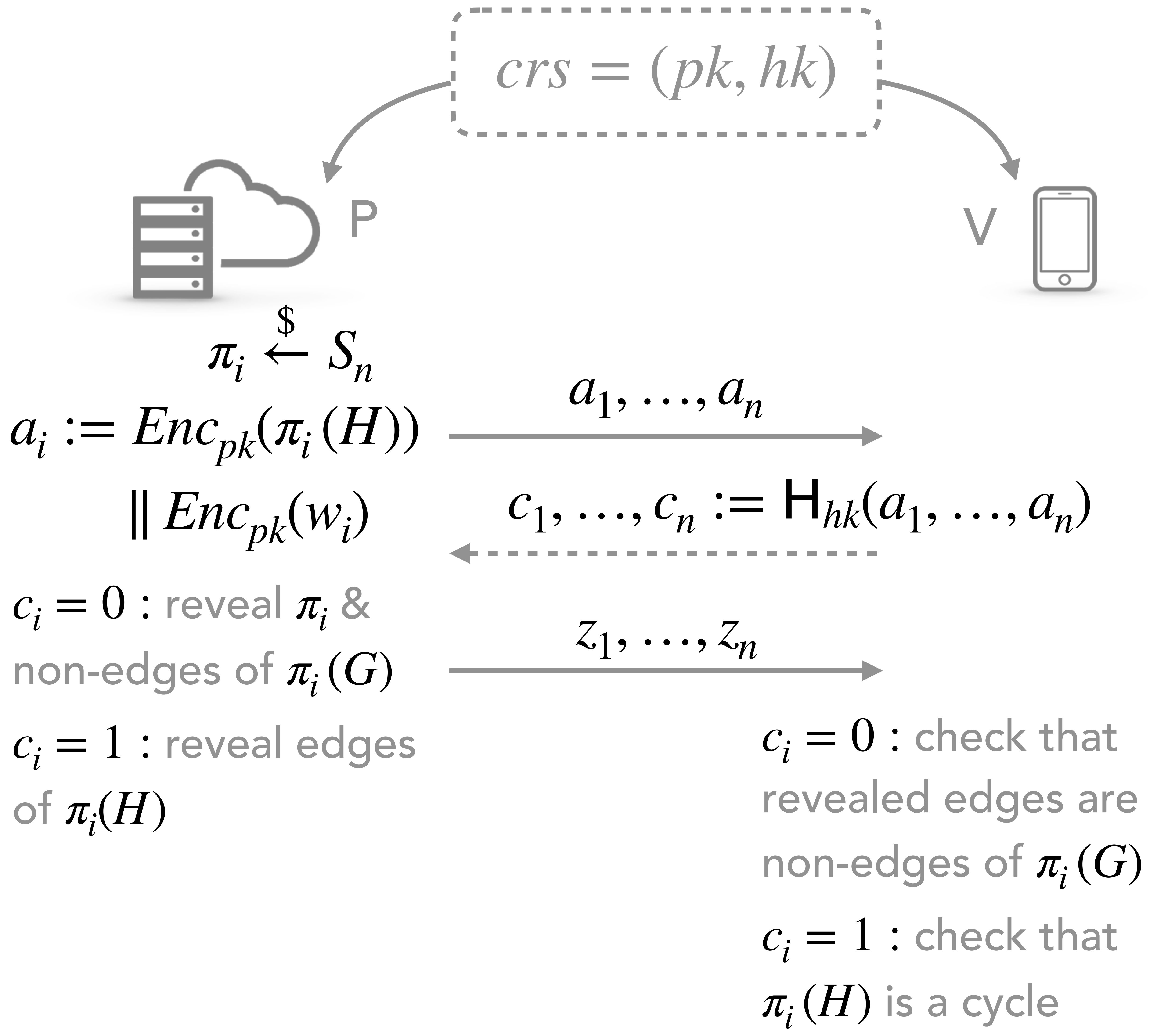
$= \textbf{BadChal}_{sk}(a_1)\|\ldots\|\textbf{BadChal}_{sk}(a_n)$

$\implies$ **BadChal**$_{sk}$ is approximable by ***concatenated*** constant-degree polynomials



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Statistical ZK via Dense-Sparse LPN

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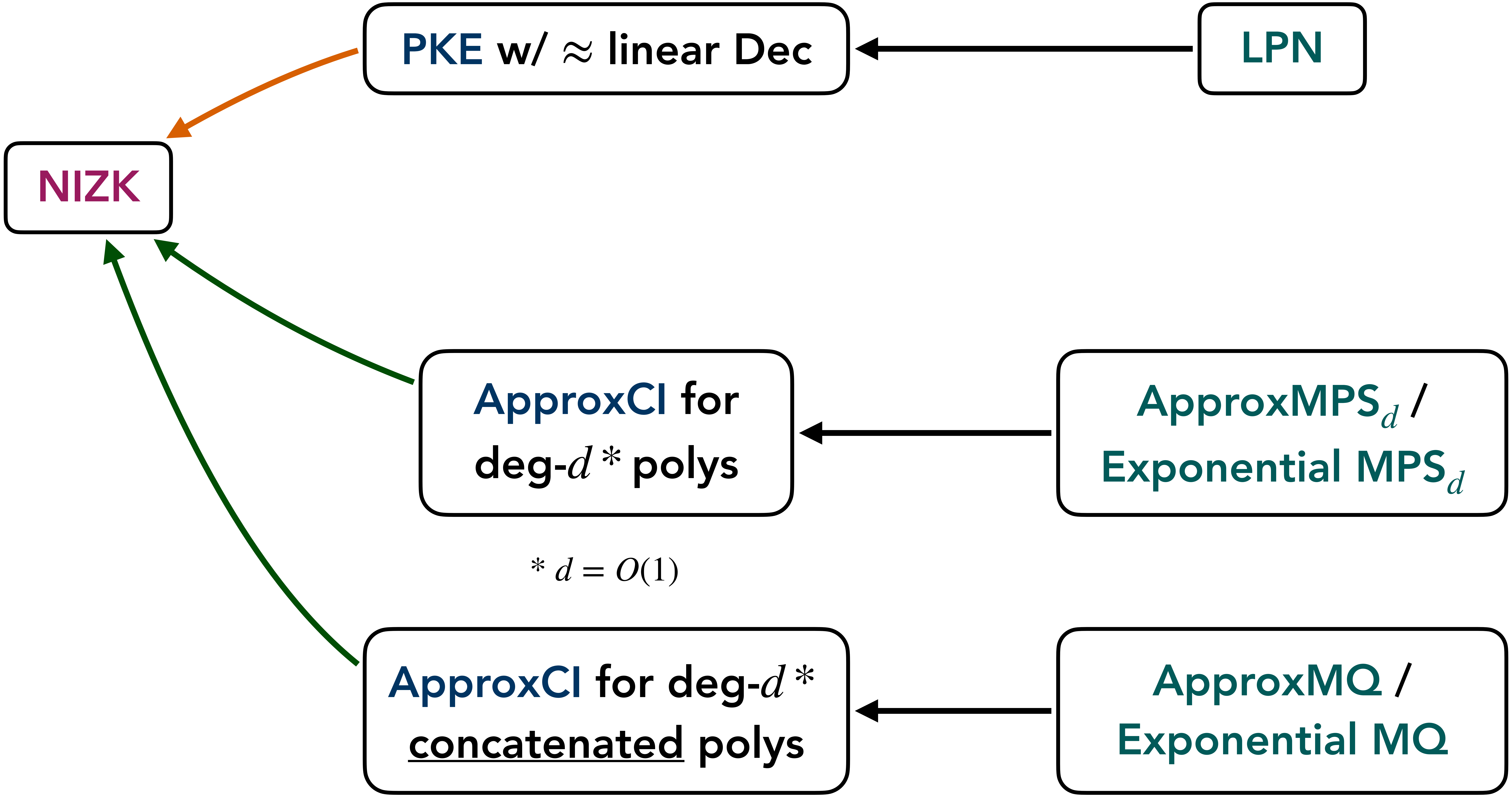

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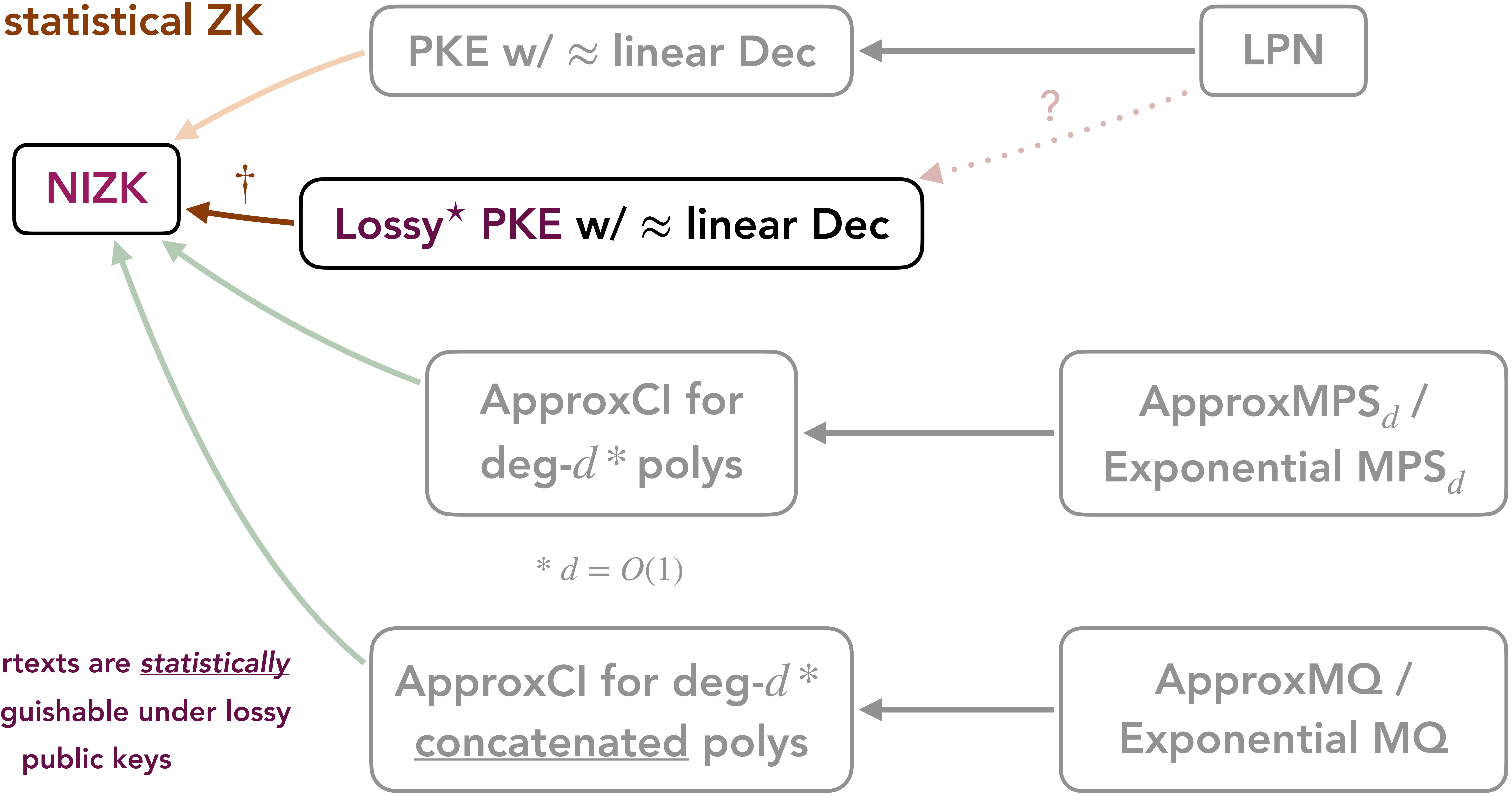
† For statistical ZK



PKE w/ $\approx$ linear Dec

LPN

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Lossy* PKE w/ $\approx$ linear Dec

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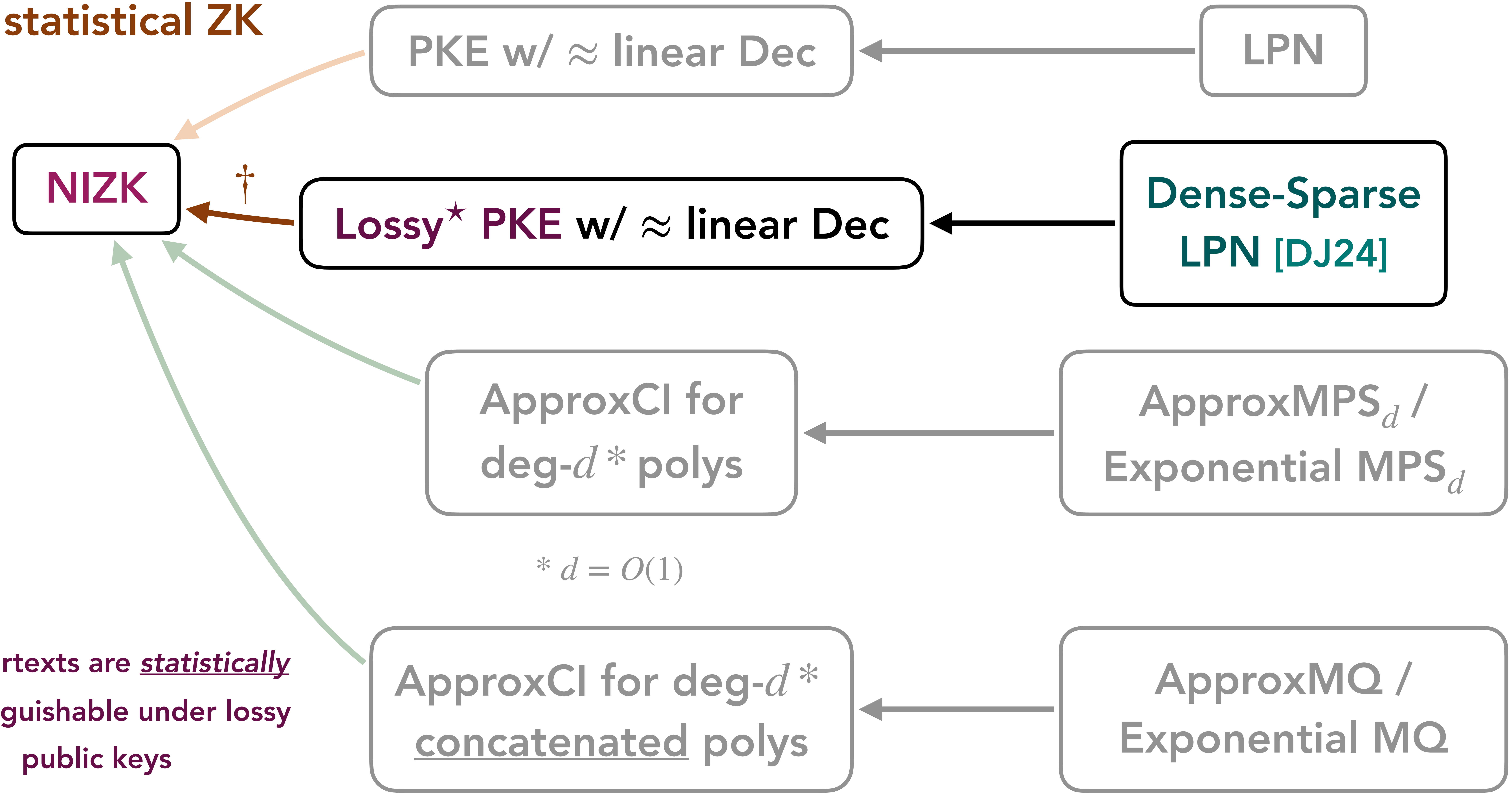

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Dense-Sparse LPN [DJ24]

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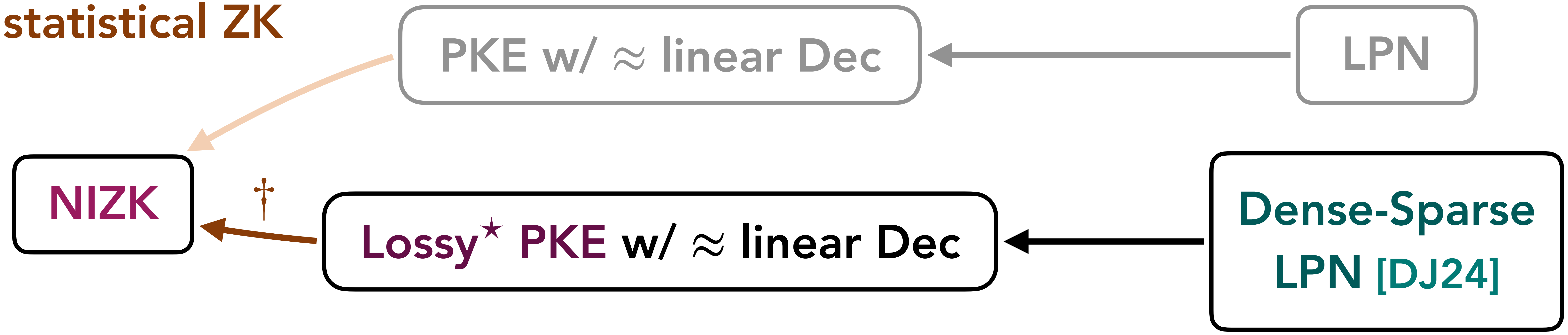

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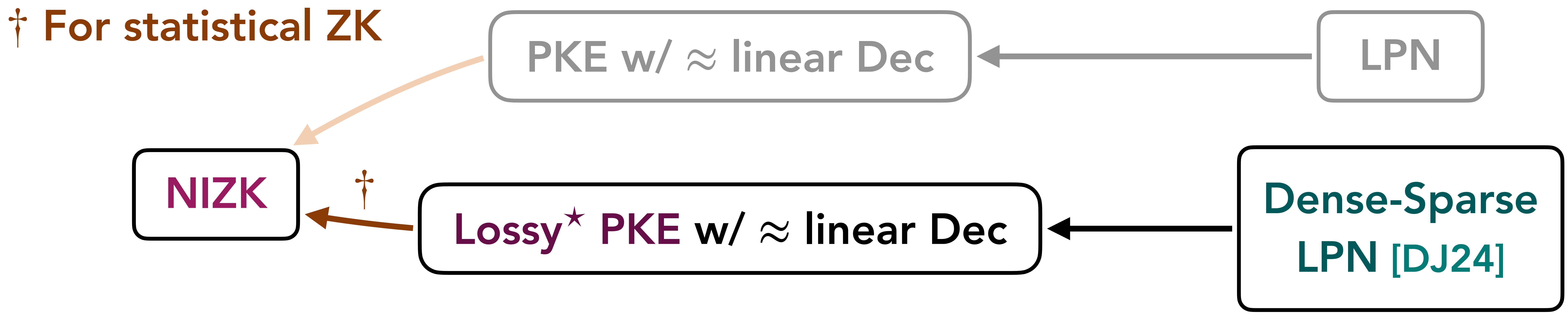

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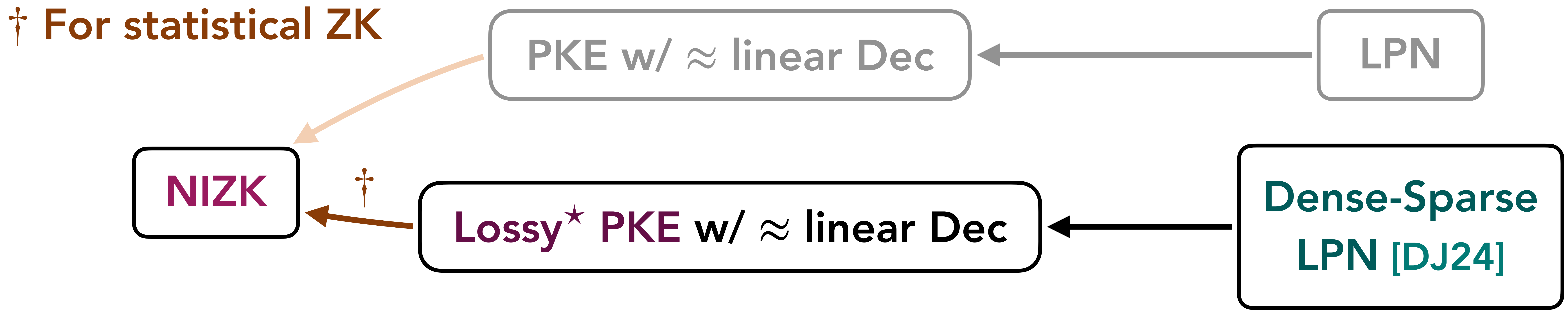

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$\implies$ not strong enough to achieve NIZK

Instead, use Dense-Sparse LPN (variant of LPN with structured public matrix)

$\implies$ lossy PKE from DS-LPN with correctness error $\frac{1}{poly(\lambda)}$

Summary & Open Problems

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Read our paper!
(ePrint 2024/1254)

Thank you! Questions?