The Complexity of Algebraic Algorithms for LWE

Matthias Johann Steiner

Alpen-Adria-Universität Klagenfurt, Klagenfurt am Wörthersee, Austria

May 27, 2024

M. Steiner has been supported in part by the European Research Council

(ERC Grant No. 725042)

Outline

Motivation

Learning With Errors (LWE)

- $q, n \in \mathbb{Z}_{\geq 1}$, q a prime.
- χ probability distribution on $\mathbb{Z}$.
- $\boldsymbol{s} \in \mathbb{F}_q^n$ secret vector.
- LWE sample $(\boldsymbol{a}, b) \in \mathbb{F}_q^n \times \mathbb{F}_q$

$$b = \langle \boldsymbol{s}, \boldsymbol{a} \rangle + e = \sum_{i=1}^{n} s_i \cdot a_i + e \in \mathbb{F}_q^n,$$

where $e \leftarrow \chi$.

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- Arora-Ge polynomial model [AG11].
  - Choose interval $[-N, N] \subset \mathbb{Z}$ and set up the polynomial

$$f = x \cdot \prod_{i=1}^{N} (x - i) \cdot (x + i).$$

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$$f = x \cdot \prod_{i=1}^{N} (x - i) \cdot (x + i).$$

  - For LWE sample $(\boldsymbol{a}, b = \langle \boldsymbol{s}, \boldsymbol{a} \rangle + e)$, the polynomial

$$f\left(b - \langle \boldsymbol{x}, \boldsymbol{a} \rangle\right) = 0 \in \mathbb{F}_q[x_1, \ldots, x_n] = \mathbb{F}_q[\boldsymbol{x}]$$

    has root $\boldsymbol{s}$ with probability, $N = t \cdot \sigma$,

$$1 - \mathbb{P}\left[|e| > t \cdot \sigma\right] \geq 1 - \frac{2}{t \cdot \sqrt{2 \cdot \pi}} \cdot \exp\left(-\frac{t^2}{2}\right).$$

Motivation

- LWE polynomial system, $m \geq n$,

$$\mathcal{F}_{\mathrm{LWE}} \colon f\left(b_1 - \langle \boldsymbol{x}, \boldsymbol{a}_1 \rangle\right) = \ldots = f\left(b_m - \langle \boldsymbol{x}, \boldsymbol{a}_m \rangle\right) = 0.$$

- Previous secret recovery via
  - Linearization [AG11].
  - Gröbner bases under semi-regularity assumption [ACF+15].
  - Linearization under semi-regularity assumption [STA20].

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  - Linearization under semi-regularity assumption [STA20].
- **This work:**
  - Proven Gröbner basis complexity estimation.
  - Framework for worst-case (designer's perspective) Gröbner basis complexity estimation.
  - Incorporating hints into LWE polynomial systems.

DRL Gröbner Basis

$P = K[x_1, \dots, x_n]$, $m = \prod_{i=1}^{n} x_i^{d_i}$, $\boldsymbol{d} = (d_1, \dots, d_n)$

Degree Reverse Lexicographic (DRL) Term Order

$\boldsymbol{a} >_{DRL} \boldsymbol{b}$ if

- $\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i$, or
- $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the last non-zero entry of $\boldsymbol{a} - \boldsymbol{b}$ is negative.

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$I = (f_1, \ldots, f_m) \subset P$ ideal, $I = \{ f \mid f = \sum_{i=1}^{m} h_i \cdot f_i,\ h_i \in P \}$

DRL Gröbner Basis [Buc65]

- $I \subset P$ ideal.
- $\mathcal{G} \subset I$ finite basis.
- $(\mathrm{LM}_{DRL}(f) \mid f \in I) = (\mathrm{LM}_{DRL}(g) \mid g \in \mathcal{G})$.

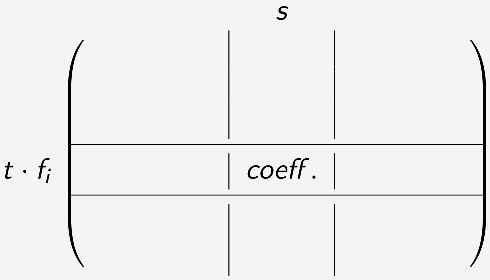
$\mathcal{F} = \{f_1, \ldots, f_m\} \subset P = K[x_1, \ldots, x_n]$

Macaulay Matrix $M_{\leq d}$

- $d \in \mathbb{Z}_{\geq 0}$:

Monomials: $s \in P, \ \deg(s) \leq d$

Polynomials:
$t \in P, \ f_i \in \mathcal{F}$,
$\deg(t \cdot f_i) \leq d$



- Columns sorted via DRL.

Solving Degree

$\mathcal{F} = \{f_1, \ldots, f_m\} \subset P = K[x_1, \ldots, x_n]$

Solving Degree [CG21, Definition 6]

- $\text{sd}_{DRL}(\mathcal{F})$ least $d \in \mathbb{Z}_{\geq 0}$ such that Gaussian elimination on $M_{\leq d}$ produces Gröbner basis.
- Complexity estimate [Sto00]

$$\mathcal{O}\left(m \cdot \text{sd}_{DRL}(\mathcal{F}) \cdot \binom{n + \text{sd}_{DRL}(\mathcal{F}) - 1}{\text{sd}_{DRL}(\mathcal{F})}^{\omega} \right),$$

where $2 \leq \omega \leq 3$.

Degree of Regularity

$\mathcal{F} = \{f_1, \ldots, f_m\} \subset P = K[x_1, \ldots, x_n]$

Degree of Regularity [BFS04, Definition 4]

- $f^{\text{top}} = f^{\text{hom}} \mod (x_0)$.
- $d_{\text{reg}}(\mathcal{F}) = \min\{d \geq 0 \mid (\mathcal{F}^{\text{top}})_d = P_d\}$.
- Heuristic: The larger $|\mathcal{F}|$ the lower $d_{\text{reg}}(\mathcal{F})$.

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$m \geq n + 1$, $d_1 = \deg(f_1) \geq \ldots \geq d_m = \deg(f_m)$

Solving Degree Bound [CG21, Theorem 10, Corollary 2], [Ste24, Theorem 3.2]

- If $d_{\text{reg}}(\mathcal{F}) < \infty$, then
- $\text{sd}_{DRL}(\mathcal{F}) \leq \text{reg}\left(\mathcal{F}^{\text{hom}}\right) \leq \sum_{i=1}^{n+1}(d_i - 1) + 1$.

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Proven Complexity for LWE Polynomial Systems

- LWE sample vector matrix: $\operatorname{rank}\begin{pmatrix}\boldsymbol{a}_1 & \dots & \boldsymbol{a}_m\end{pmatrix} = n \Rightarrow d_{\text{reg}}\left(\mathcal{F}_{\text{LWE}}\right) < \infty$.

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- General LWE $q, n, \sigma = \sqrt{\frac{n}{2\cdot\pi}}$

$$\mathcal{O}\left(n \cdot 2^{\omega\cdot\mathcal{O}\left(n^{1.2787}\right)}\right), \quad p_{success} \geq 1 - \frac{2}{\pi \cdot \sqrt{n}}.$$

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- Small error LWE $|\mathcal{E}| = D$

$$\mathcal{O}\left(m \cdot (D - 1) \cdot n \cdot 2^{\omega \cdot \mathcal{O}(n)}\right).$$

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- Small secret LWE $|\mathcal{S}| = D$

$$\mathcal{O}\left(m \cdot (D-1) \cdot n^2 \cdot 2^{\omega \cdot \mathcal{O}\left(n^{1.2787}\right)}\right).$$

- Previous analysis [ACF+15] assumed that $\mathcal{F}_{\text{LWE}}$ is semi-regular.
  - $d_{\text{reg}}\left(\mathcal{F}_{\text{LWE}}\right)$ computable.
  - $d_{\text{reg}}\left(\mathcal{F}_{\text{LWE}}\right)$ DRL Gröbner basis complexity measure.

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- $\overline{\text{sd}}_{DRL}\left(\mathcal{F}\right)$ least $d \in \mathbb{Z}_{\geq 0}$ such that Gaussian elimination on iterated Macaulay matrices produces DRL-Gröbner basis.

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Theorem

- $\mathcal{F} \subset K[x_1, \ldots, x_n]$, $d_{\text{reg}}(\mathcal{F}) < \infty$.
- $\overline{\text{sd}}_{DRL}(\mathcal{F}) \leq 2 \cdot d_{\text{reg}}(\mathcal{F}) - 1$.
- *Complexity*

$$\mathcal{O}\left(m \cdot \overline{\text{sd}}_{DRL}(\mathcal{F})^3 \cdot \binom{n + \overline{\text{sd}}_{DRL}(\mathcal{F}) - 1}{\overline{\text{sd}}_{DRL}(\mathcal{F})}^{\omega + 2} \right).$$

Two Approaches To LWE Complexity Estimation

Gaussian elimination on

- Iterated Macaulay matrices.
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- Single Macaulay matrix.
- $\text{sd}_{DRL}\left(\mathcal{F}\right) \leq \text{reg}\left(\mathcal{F}^{\text{hom}}\right)$.
- $d_{\text{reg}}\left(\mathcal{F}\right) \leq \text{reg}\left(\mathcal{F}^{\text{hom}}\right)$ [CG23, Theorem 5.3].
- Complexity:

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How To Compute Degree of Regularity?

$\mathcal{F}_{\text{LWE}} = \{f_1, \dots, f_m\} \subset P = \mathbb{F}_q[x_1, \dots, x_n]$, $\forall i : \deg(f_i) = D$

- Compute DRL Gröbner basis of homogeneous system

$$\left(\mathcal{F}_{\text{LWE}}^{\text{top}}\right) = \left(\mathcal{F}_{\text{LWE}}^{\text{hom}}, x_0\right).$$

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  - Infeasible in practice.

- But we can estimate **"Lowest Achievable Degree of Regularity"** via necessary condition:

$$|\mathcal{F}_{\text{LWE}}| \cdot \underbrace{\binom{n+d-1}{d}}_{=\dim_{\mathbb{F}_q}(P_d)} \overset{!}{\geq} \underbrace{\binom{n+D+d-1}{D+d-1}}_{=\dim_{\mathbb{F}_q}(P_{D+d})}.$$

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LWE Worst Case Complexity estimation

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Binary Secret LWE

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- Necessary condition simplifies to: $m \cdot \binom{n}{d} \overset{!}{\geq} \binom{n}{D+d}$.
- If $m \in \mathcal{O}(n^D)$, then complexity $\mathcal{O}\left(n^D \cdot D \cdot 2^{\omega \cdot \mathcal{O}\left(n^{0.6393}\right)}\right)$.

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- Complexity estimation for Kyber768 [SAB+22] with $\omega = 2$.
  - Small secret small error LWE instance with $|\mathcal{S}| = |\mathcal{E}| = 5$.
  - $n = m = 3 \cdot 256$.
  - Complexities of lattice-based attacks are computed via the lattice estimator [APS15].

Method	BKW	USVP	BDD	BDD Hybrid	BDD MiTM Hybrid	Dual	Dual Hybrid	Proven complexity estimate		Iterated Macaulay Matrices		Single Macaulay Matrix	
Samples	2^{226}	768	768	768	768	768	768	768	768^4	768	768^4	768	768^4
Complexity (bits)	239	205	201	201	357	214	206	5554	5581	4717	419	1588	203

LWE With Hints

- Hardness of LWE in presence of side-information.
  - Improvement of lattice attacks [DDGR20, DGHK23].

Types of Hints

- Perfect hints: $\langle \boldsymbol{s}, \boldsymbol{v} \rangle = l \in \mathbb{F}_q$.
- Modular hints: $\langle \boldsymbol{s}, \boldsymbol{v} \rangle \equiv l \mod k$.
- Approximate hints: $\langle \boldsymbol{s}, \boldsymbol{v} \rangle + e_\sigma = l \in \mathbb{F}_q$.
- Short vector hints: $\boldsymbol{v} \in \Lambda_{\text{LWE}}$.

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- In principle any hint which can be modeled in $\mathbb{F}_q[\boldsymbol{x}]$ can be added to $\mathcal{F}_{\mathsf{LWE}}$.
- The closer one can get to binary secret/error LWE the better:

$$m \cdot \binom{n+d-1}{d} \overset{!}{\geq} \binom{n+D+d-1}{D+d} \text{ vs. } m \cdot \binom{n}{d} \overset{!}{\geq} \binom{n}{D+d}.$$

Finite Degree of Regularity for LWE

$$f(x) = x \cdot \prod_{i=1}^{N} (x - i) \cdot (x + i),$$
$$\mathcal{F}_{\text{LWE}} = \left\{ f\left(b_i - \langle \boldsymbol{x}, \boldsymbol{a}_i \rangle \right) \right\}_{1 \leq i \leq m} \subset \mathbb{F}_q[\boldsymbol{x}]$$

- Then
$$\mathcal{F}_{\text{LWE}}^{\text{top}} = \left\{ \left(\langle \boldsymbol{x}, \boldsymbol{a}_i \rangle \right)^{2 \cdot N + 1} \right\}_{1 \leq i \leq m}.$$

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$$\mathcal{F}_{\text{LWE}} = \left\{ f\left(b_i - \langle \boldsymbol{x}, \boldsymbol{a}_i \rangle\right) \right\}_{1 \leq i \leq m} \subset \mathbb{F}_q[\boldsymbol{x}]$$

- Then
$$\mathcal{F}_{\text{LWE}}^{\text{top}} = \left\{ \left(\langle \boldsymbol{x}, \boldsymbol{a}_i \rangle \right)^{2 \cdot N + 1} \right\}_{1 \leq i \leq m}.$$
- If $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \dots & \boldsymbol{a}_m \end{pmatrix}^\intercal$ has rank n, then perform change of coordinates
$$\boldsymbol{y} = \hat{\boldsymbol{A}} \boldsymbol{x}.$$

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Finite Degree of Regularity for LWE

$f(x) = x \cdot \prod_{i=1}^{N} (x - i) \cdot (x + i)$,
$\mathcal{F}_{\text{LWE}} = \left\{ f\left(b_i - \langle \boldsymbol{x}, \boldsymbol{a}_i \rangle\right) \right\}_{1 \leq i \leq m} \subset \mathbb{F}_q[\boldsymbol{x}]$

- Then

$$\mathcal{F}_{\text{LWE}}^{\text{top}} = \left\{ \left(\langle \boldsymbol{x}, \boldsymbol{a}_i \rangle \right)^{2 \cdot N + 1} \right\}_{1 \leq i \leq m}.$$

- If $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \dots & \boldsymbol{a}_m \end{pmatrix}^\intercal$ has rank n, then perform change of coordinates

$$\boldsymbol{y} = \hat{\boldsymbol{A}} \boldsymbol{x}.$$

- Then

$$\left\{ y_1^{2 \cdot N + 1}, \dots, y_n^{2 \cdot N + 1} \right\} \subset \mathcal{F}_{\text{LWE}}^{\text{top}} \quad \Rightarrow \quad d_{\text{reg}}\left(\mathcal{F}_{\text{LWE}} \right) < \infty.$$

Hints Complexity Example

- Impact of hints on small secret small error LWE:
  $|\mathcal{S}| = |\mathcal{E}| = D = 5,\ n = 256,\ m = 256^{\frac{3}{2}}$.

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- Impact of hints on small secret small error LWE: $|\mathcal{S}| = |\mathcal{E}| = D = 5,\ n = 256,\ m = 256^{\frac{3}{2}}$.
- The closer one can move a LWE to binary secret/error LWE the better.
  - Necessary degree of regularity condition: $m \cdot \binom{n+d-1}{d} \overset{!}{\geq} \binom{n+D+d-1}{D+d}$ vs. $m \cdot \binom{n}{d} \overset{!}{\geq} \binom{n}{D+d}$.

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- Impact of hints on small secret small error LWE: $|\mathcal{S}| = |\mathcal{E}| = D = 5,\ n = 256,\ m = 256^{\frac{3}{2}}$.
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- Column d lists degree for the lowest achievable degree of regularity $\hat{d}_{\text{reg}}\left(\mathcal{F}_{\text{LWE}}\right) = D + d$.

	Small Secret Small Error LWE $D=5$		Binary Secret LWE $D=5$		Binary Secret Binary Error LWE $D=2$	
Perfect hints	d	Single Macaulay Matrix (bits)	d	Single Macaulay Matrix (bits)	d	Single Macaulay Matrix (bits)
$\omega = 2$						
0	57	481	38	370	3	92
50	45	393	30	303	2	78
150	22	221	15	174	1	59

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