SQIsignHD: New Dimensions in Cryptography

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Joint work with Antonin Leroux, Damien Robert and Benjamin Wesolowski

Acknowledgements to Luca De Feo

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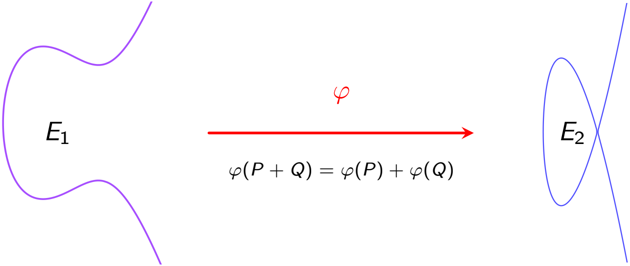
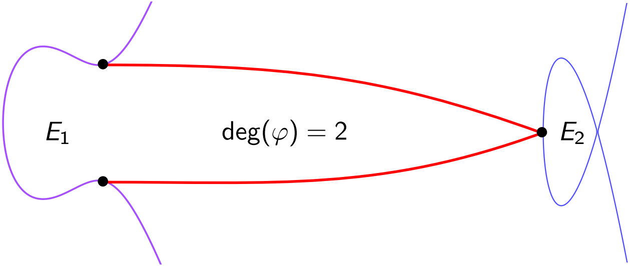
The Deuring correspondence

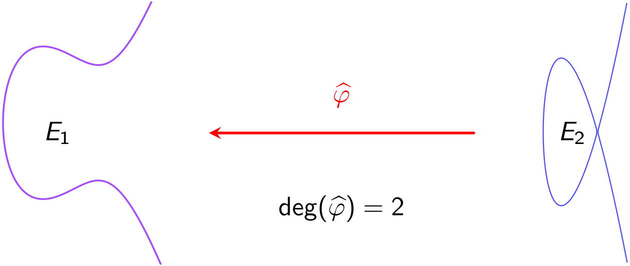
Isogenies



Isogenies - degree



Isogenies - the dual isogeny



Isogeny chains

$$\deg(\varphi_n \circ \cdots \circ \varphi_1) = \prod_{i=1}^{n} \deg(\varphi_i)$$

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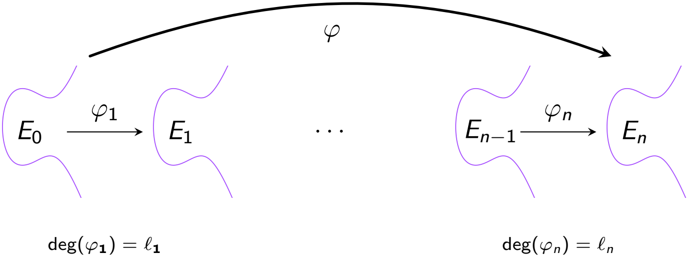

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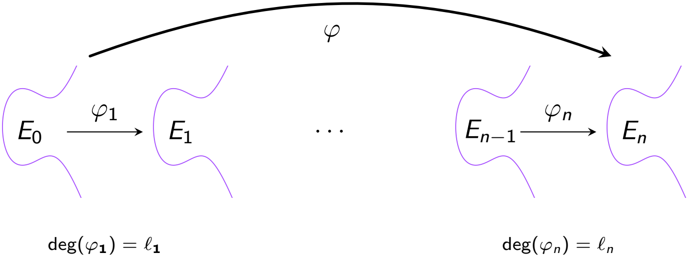

- Knowing $\ker(\varphi)$, φ can be computed in polynomial time.

Quaternions and supersingular elliptic curves

Definition (Quaternion algebra)

- Let p be a prime $\equiv 3 \mod 4$. The **quaternion algebra** ramifying at p and ∞ is:
$$\mathcal{B}_{p,\infty} := \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij,$$
with $i^2 = -1$, $j^2 = -p$, $ij = -ji$.
- An **order** $\mathcal{O} \subseteq \mathcal{B}_{p,\infty}$ is a rank 4 lattice which is also a subring.

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Let E be an elliptic curve, the **endomorphism ring** of E is:

$$\text{End}(E) = \{\text{Isogenies } \varphi : E \longrightarrow E\} \cup \{0\}$$

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Definition (Supersingular elliptic curve)

An elliptic curve E defined over $\overline{\mathbb{F}}_p$ is **supersingular** if $\mathrm{End}(E)$ is isomorphic to a maximal order of $\mathcal{B}_{p,\infty}$ (maximal for the inclusion).

The Deuring correspondence

Supersingular elliptic curves	Quaternions
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$\widehat{\varphi}$	$\overline{I_\varphi}$
$\varphi \circ \psi$	$I_\psi \cdot I_\varphi$
$\deg(\varphi)$	$\mathrm{nrd}(I_\varphi) = \sqrt{[\mathcal{O} : I_\varphi]}$

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- Compute a connecting ideal I between $\mathcal{O}_1$ and $\mathcal{O}_2$ (left $\mathcal{O}_1$-ideal and right $\mathcal{O}_2$-ideal).
- Compute $J \sim I$ of smooth norm via [KLPT14].
- Translate J into an isogeny $\varphi_J : E_1 \longrightarrow E_2$.

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✗ Slow in practice because of the red steps.

Effective Deuring correspondence and higher dimensional isogenies

Kani's embedding lemma

Theorem (Robert, 2022)

Let $\sigma : E_1 \longrightarrow E_2$ *such that* $\deg(\sigma) + a_1^2 + a_2^2 = 2^e$*. Then:*

- $\sigma : E_1 \longrightarrow E_2$ *can be represented by the dimension 4 isogeny:*

$$F := \begin{pmatrix} a_1 & a_2 & \widehat{\sigma} & 0 \\ -a_2 & a_1 & 0 & \widehat{\sigma} \\ -\sigma & 0 & a_1 & -a_2 \\ 0 & -\sigma & a_2 & a_1 \end{pmatrix} \in \mathrm{End}(E_1^2 \times E_2^2).$$

- F *can be computed by evaluating* σ *on* $E_1[2^e]$*.*

Context: This idea comes from the attacks against SIDH [CD23; MM22; Rob23].

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More on Robert's theorem:

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$$E_1^2 \times E_2^2 \xrightarrow{F_1} \mathcal{A}_1 \xrightarrow{F_2} \mathcal{A}_2 \quad \cdots \quad \mathcal{A}_{e-1} \xrightarrow{F_e} E_1^2 \times E_2^2$$

- Using theta coordinates, this chain can be computed with $O(e \log(e))$ finite field operations.

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- Using theta coordinates, this chain can be computed with $O(e\log(e))$ finite field operations.
- We have:

$$F(P,0,0,0) = ([a_1]P, -[a_2]P, -\sigma(P), 0)$$

so we can evaluate σ by evaluating F.

Kani's embedding lemma

Corollary (Robert, 2022)

Let $\sigma : E_1 \longrightarrow E_2$ *of degree* $q < 2^e$ *such that* $2^e - q$ *is a prime* $\equiv 1 \mod 4$. *There exists a polynomial time algorithm with:*

- ***Input:*** $(\sigma(P_1), \sigma(P_2))$, *where* (P_1, P_2) *is a basis of* $E_1[2^e]$ *and* $Q \in E_1(\mathbb{F}_{p^2})$.
- ***Output:*** $\sigma(Q)$.

A new algorithm for effective Deuring correspondence

Problem: Given $\phi : E_1 \longrightarrow E_2$, I_ϕ, $\mathcal{O}_1 \cong \mathrm{End}(E_1)$ and $\mathcal{O}_2 \cong \mathrm{End}(E_2)$ (secret), find another isogeny $\sigma : E_1 \longrightarrow E_2$.

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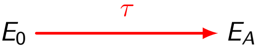
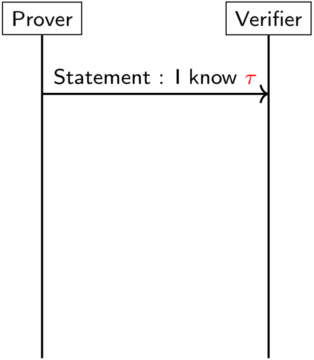
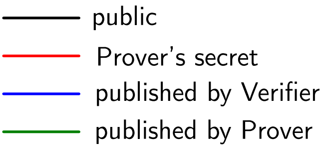
3. $(q, \sigma(E_1[2^e]))$, is sufficient to represent σ.
4. Compute $F \in \mathrm{End}(E_1^2 \times E_2^2)$ embedding σ.

SQIsignHD

The SQIsignHD identification scheme







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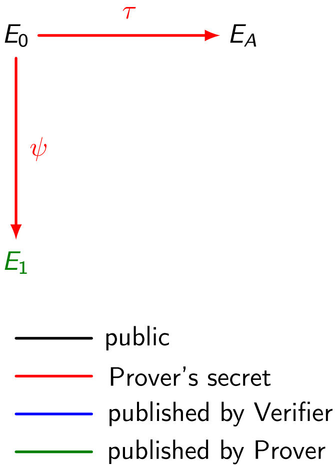

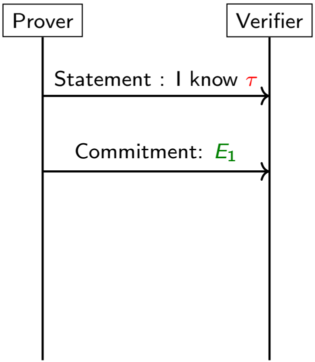

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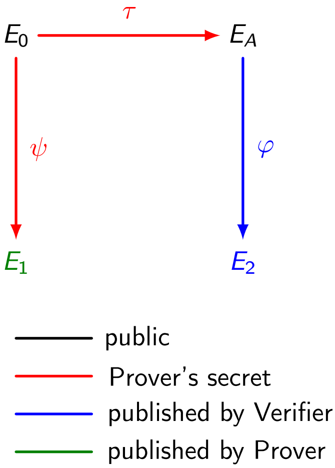

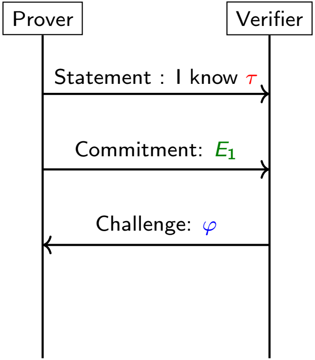

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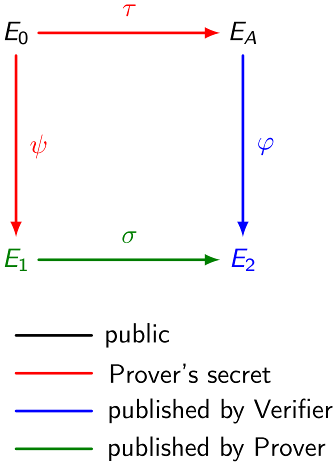

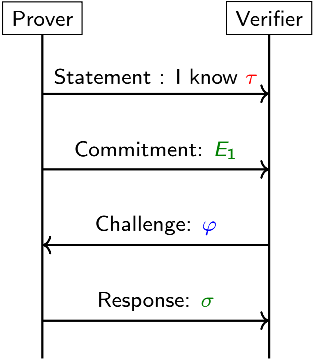

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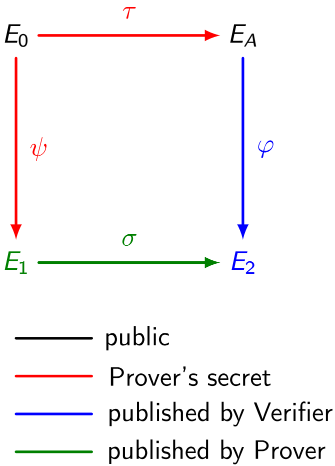

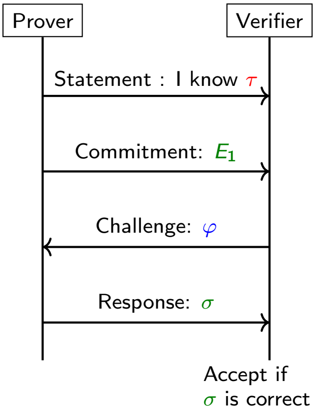

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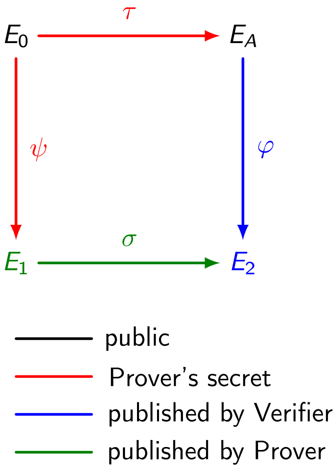

Response: $(q, \sigma(P_1), \sigma(P_2))$, where:

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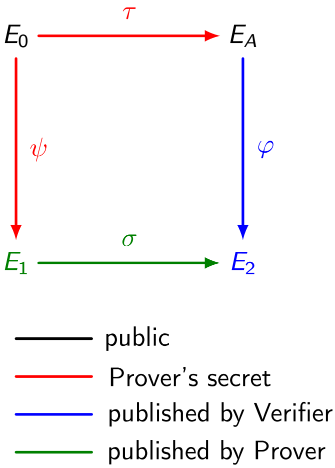

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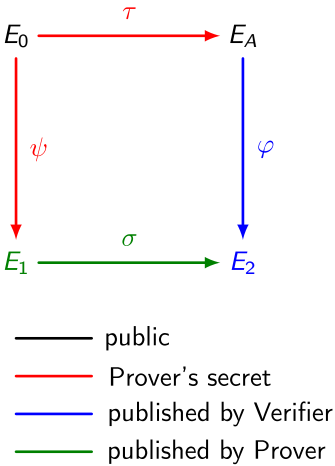

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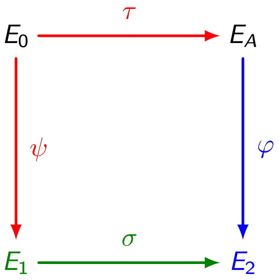

- public
- Prover's secret
- published by Verifier
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Proof of concept.
600 ms in `sagemath`.

Comparison of SQIsignHD with SQIsign

	SQIsign	SQIsignHD
Security	✗ Ad-hoc heuristic: • Distribution of σ.	✓ Simpler heuristics: • Oracle (RUGDIO); • Distribution of E_1.
Scalability	✗ $\prod_{i=1}^{n} \ell_i \| p^2 - 1$	✓ $p = c \cdot 2^f \cdot 3^{f'} - 1$
Signing time	✗ 400 ms for NIST-1	✓ 28 ms for NIST-1
Signature size	✓ 204 bytes for NIST-1	✓ 109 bytes for NIST-1
Verification	✓ Fast (6 ms for NIST-1)	✗ 600 ms for NIST-1 in sagemath