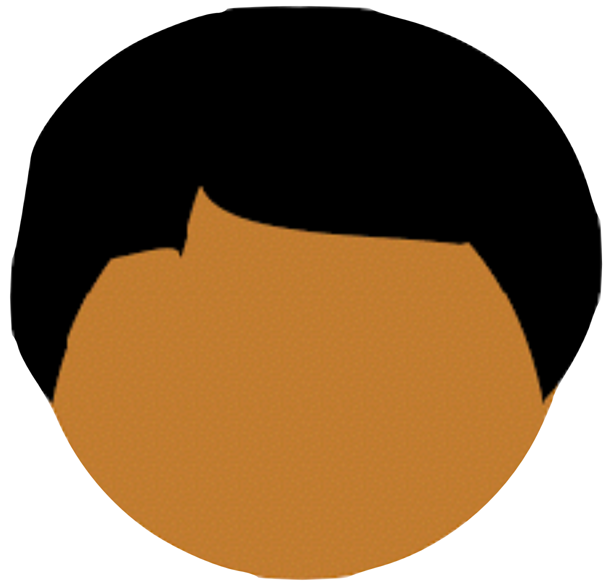
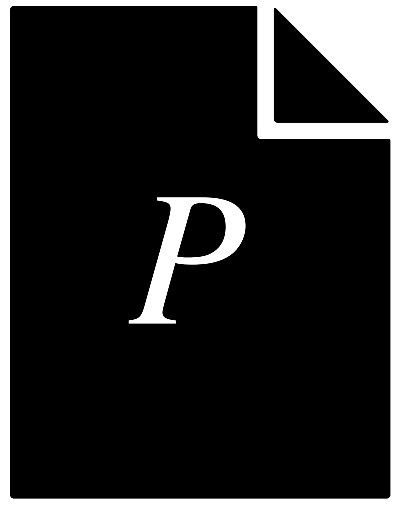
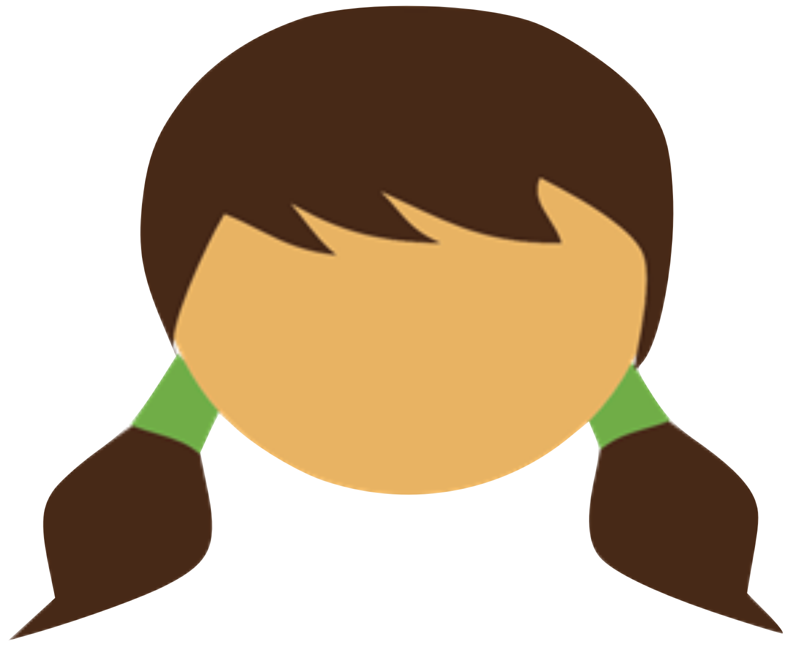
Efficient Arithmetic in Garbled Circuits

David Heath
University of Illinois Urbana-Champaign

Garbled Circuits







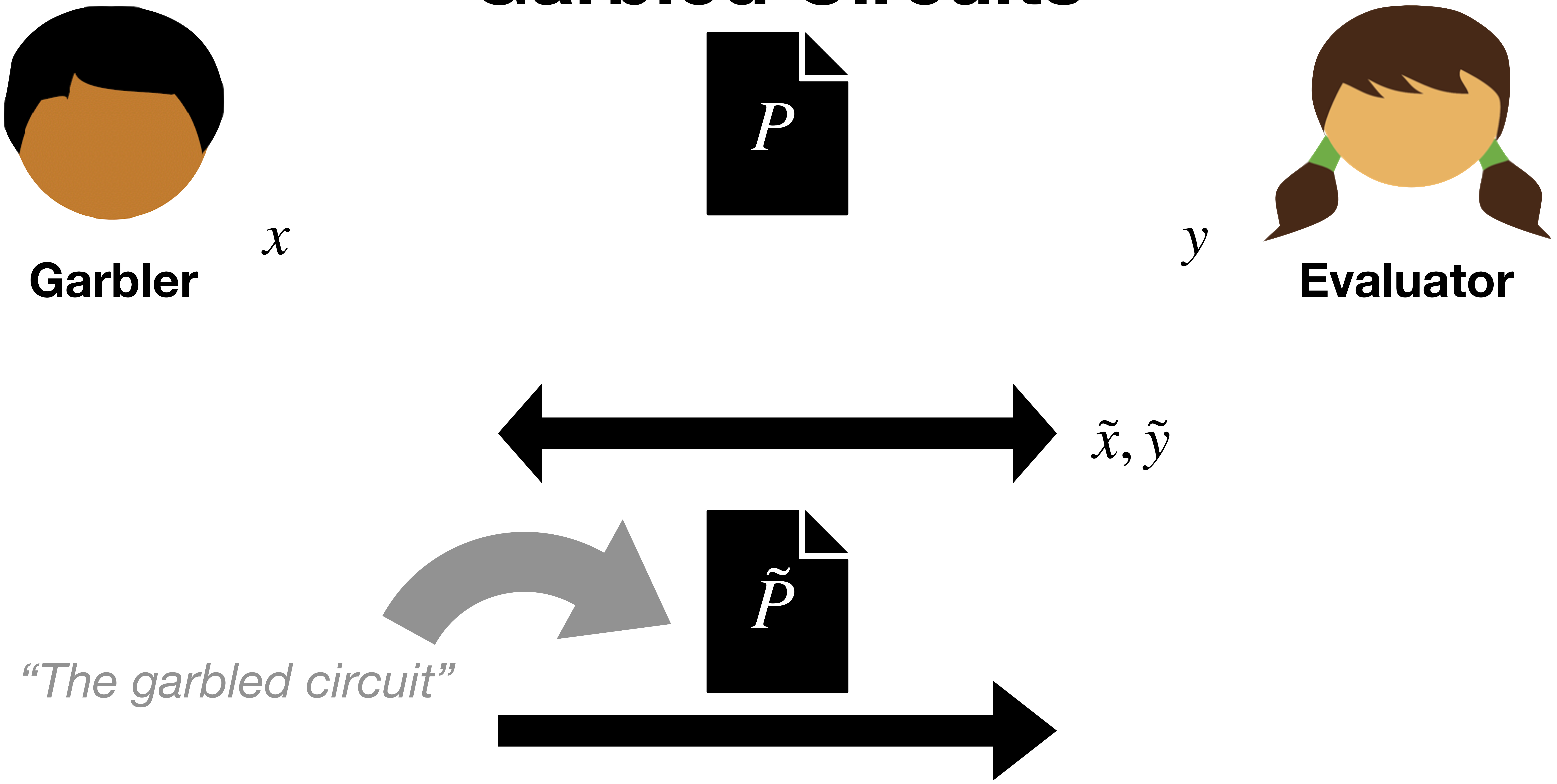
x

Garbler

y

Evaluator

Garbled Circuits



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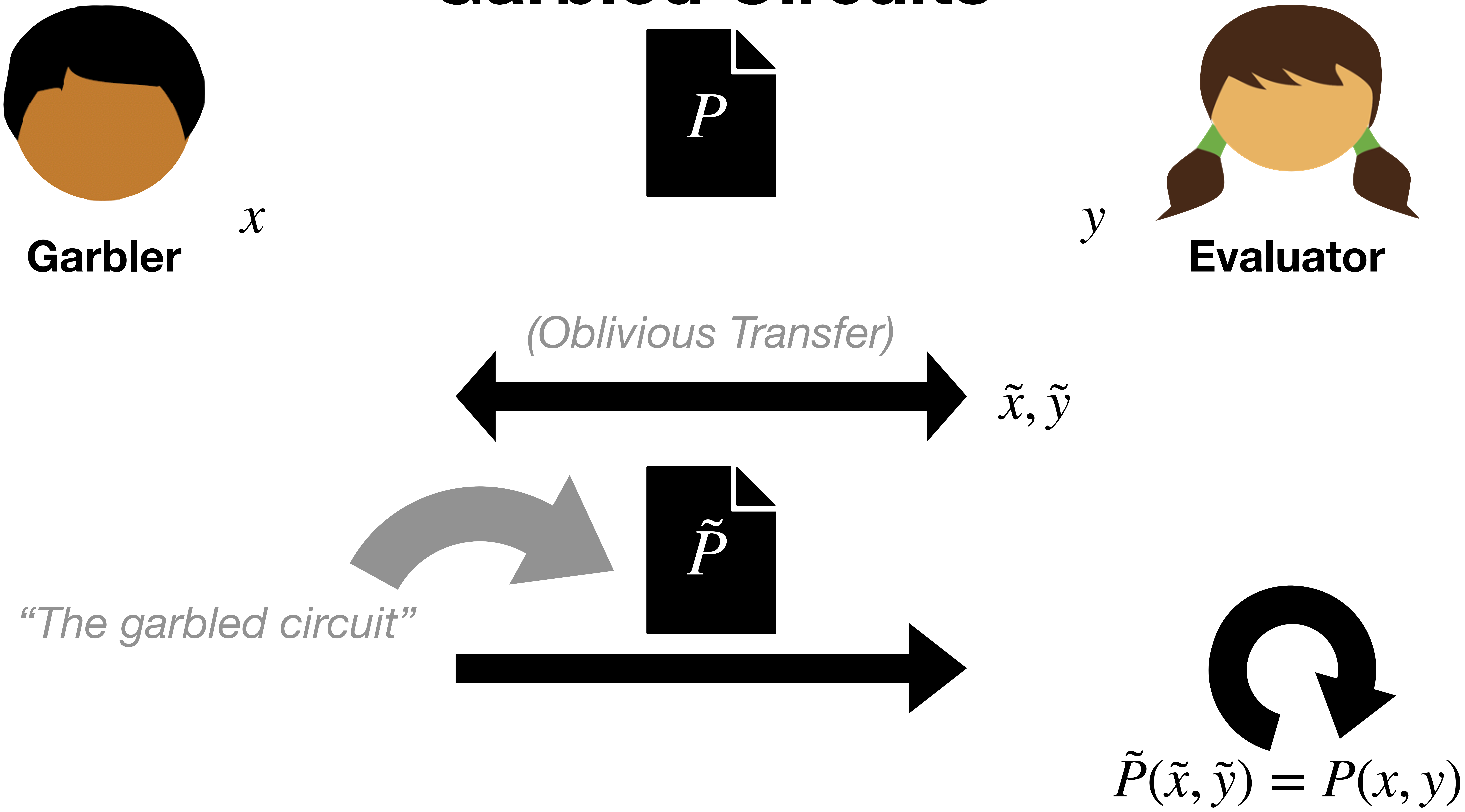
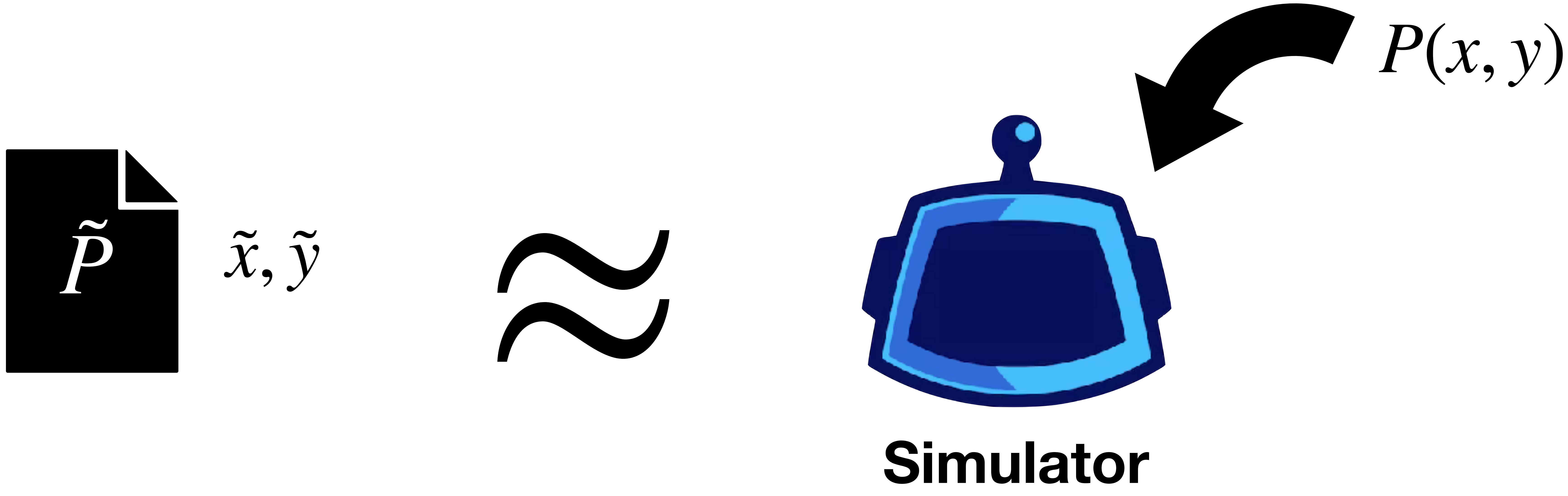


Garbled Circuits

Privacy:



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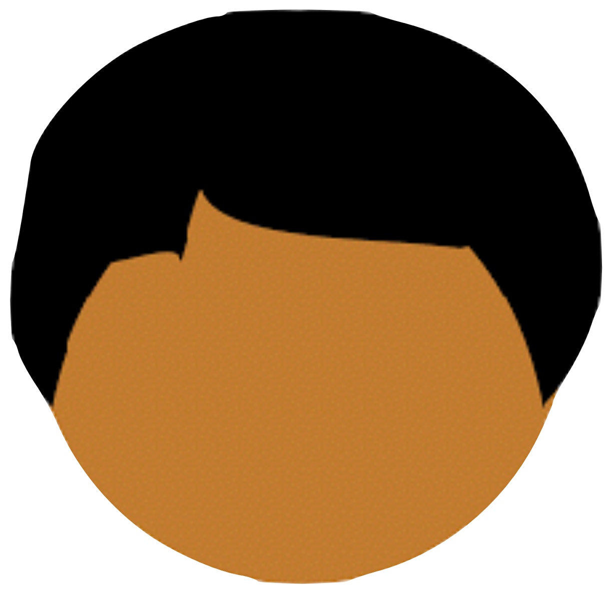
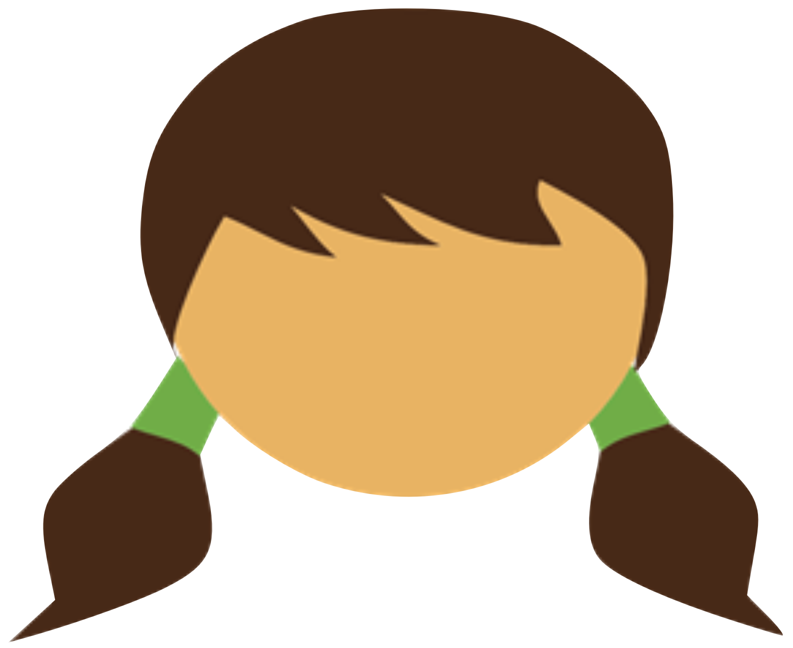
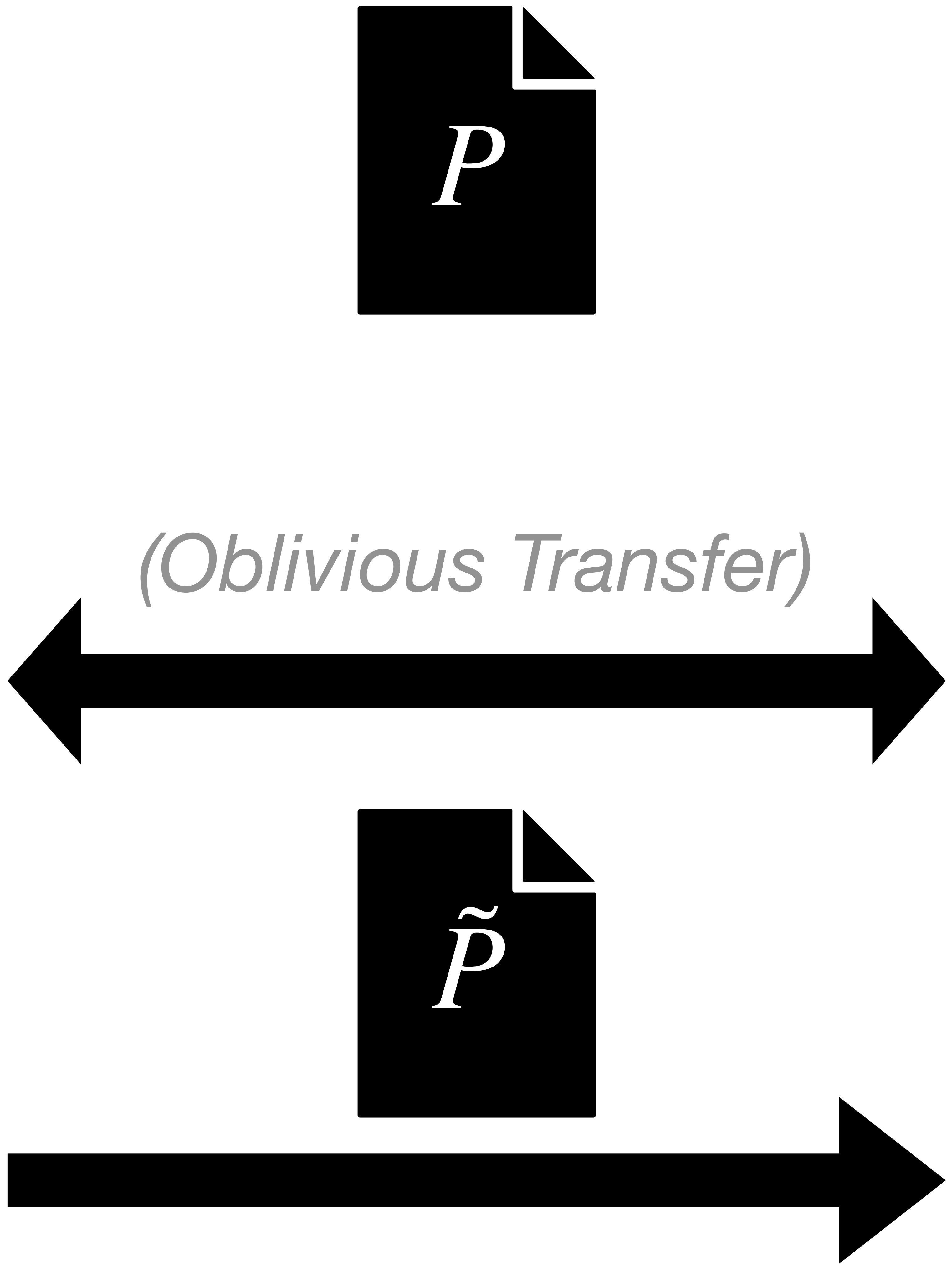


Garbler x



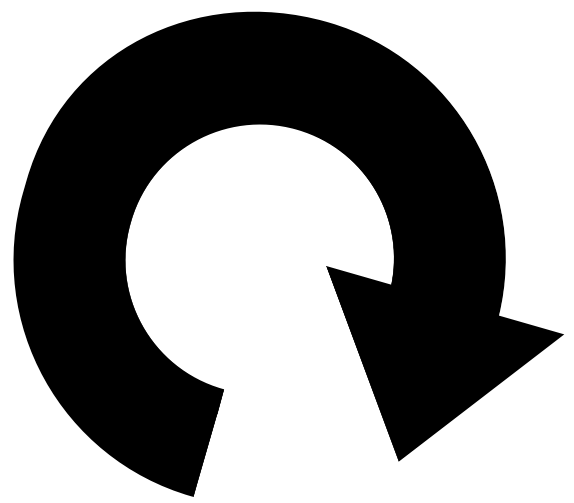
y Evaluator



$\tilde{x}, \tilde{y}$

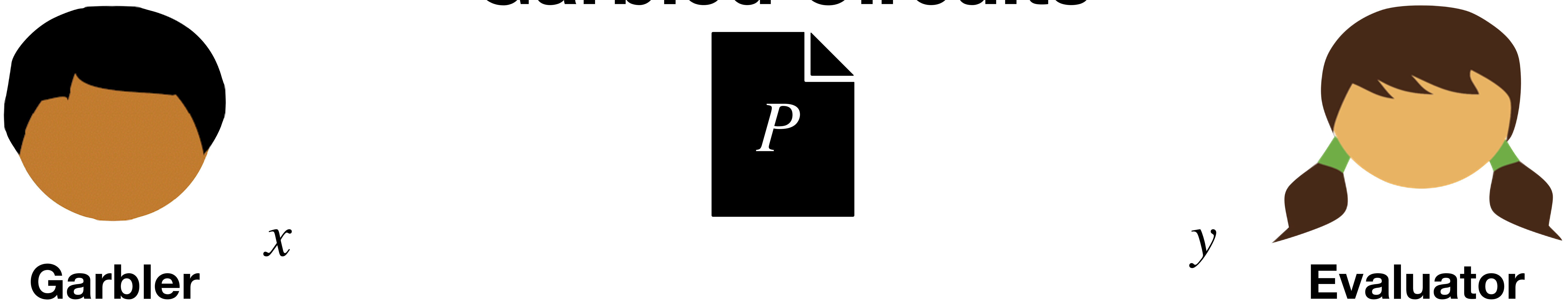
constant round protocols ✓

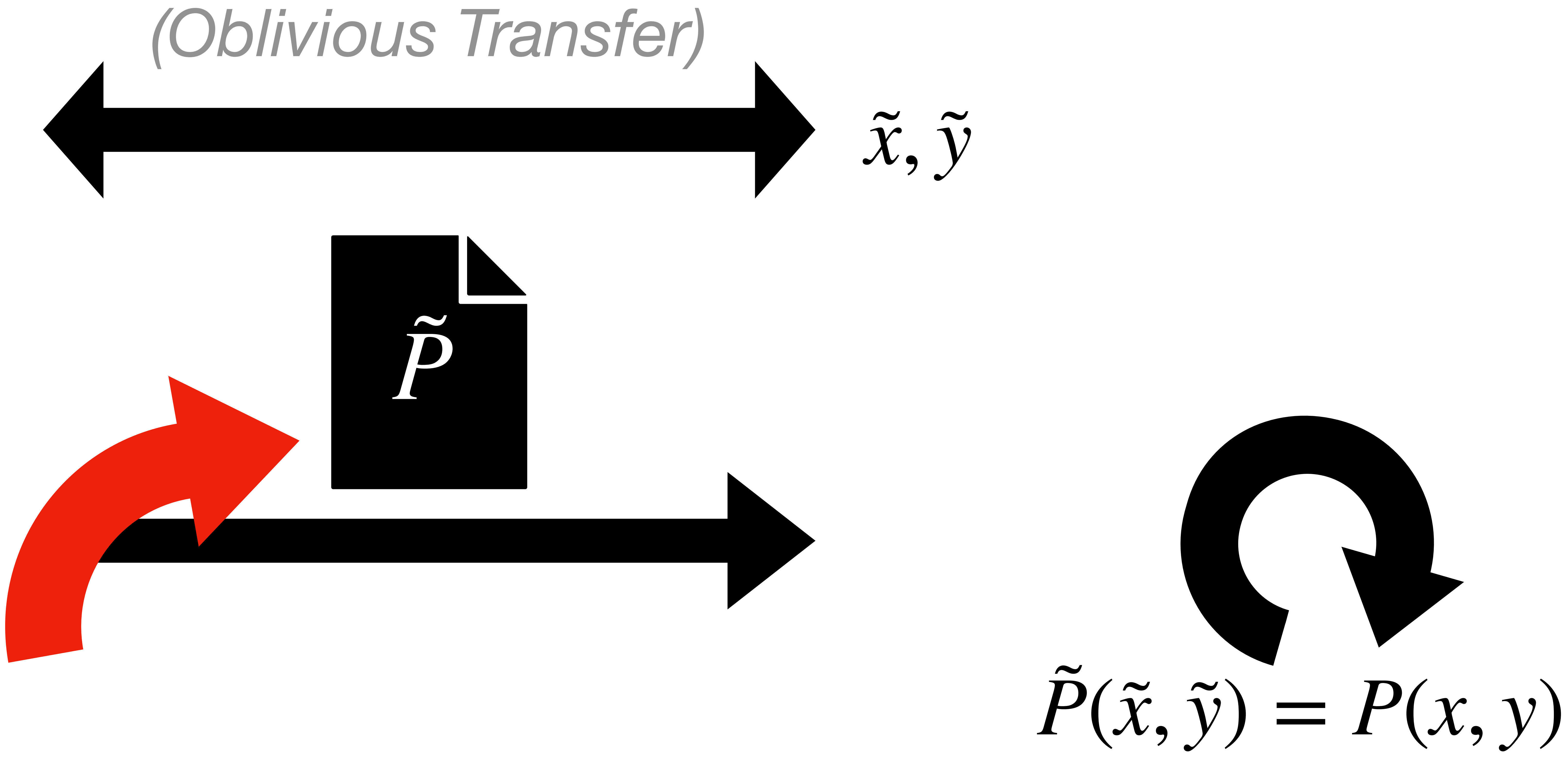
fast, symmetric-key primitives ✓



$$\tilde{P}(\tilde{x}, \tilde{y}) = P(x, y)$$

Garbled Circuits



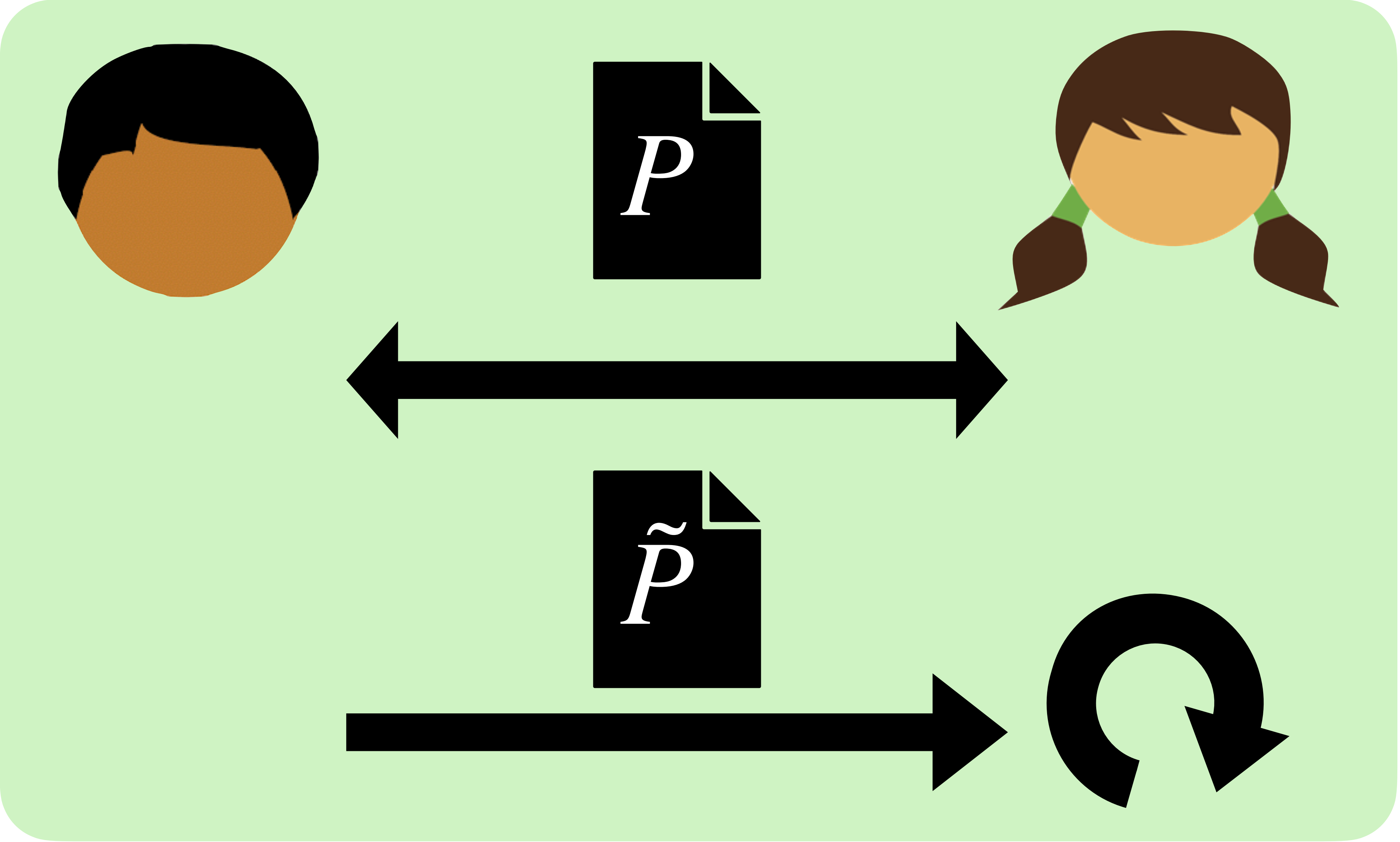


constant round protocols ✓

fast, symmetric-key primitives ✓

high bandwidth consumption ✗

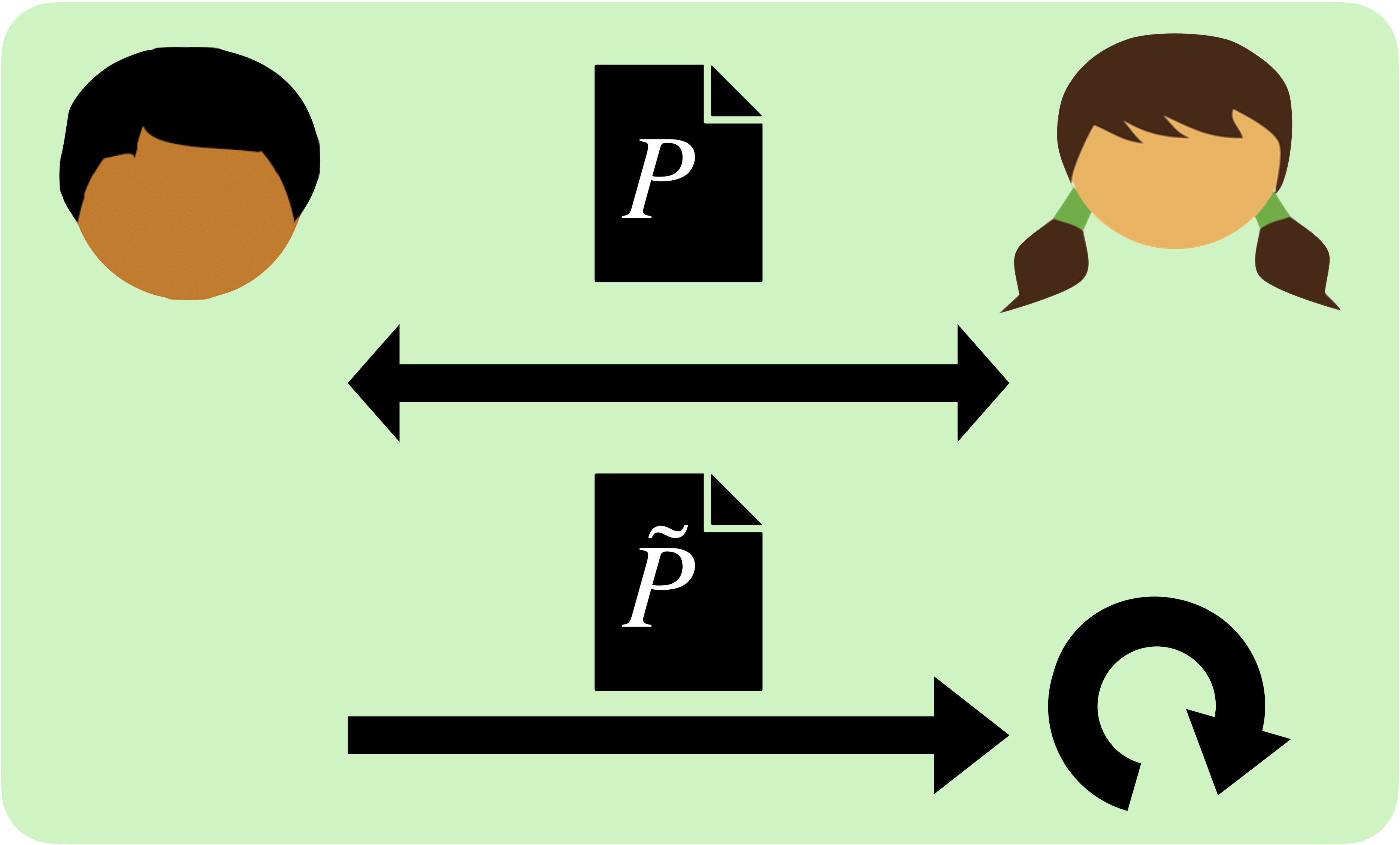
Arithmetic Garbling



Traditionally, P is a Boolean circuit

Garbling of n-gate Boolean circuit has size $O(n \cdot \lambda)$ bits

Arithmetic Garbling



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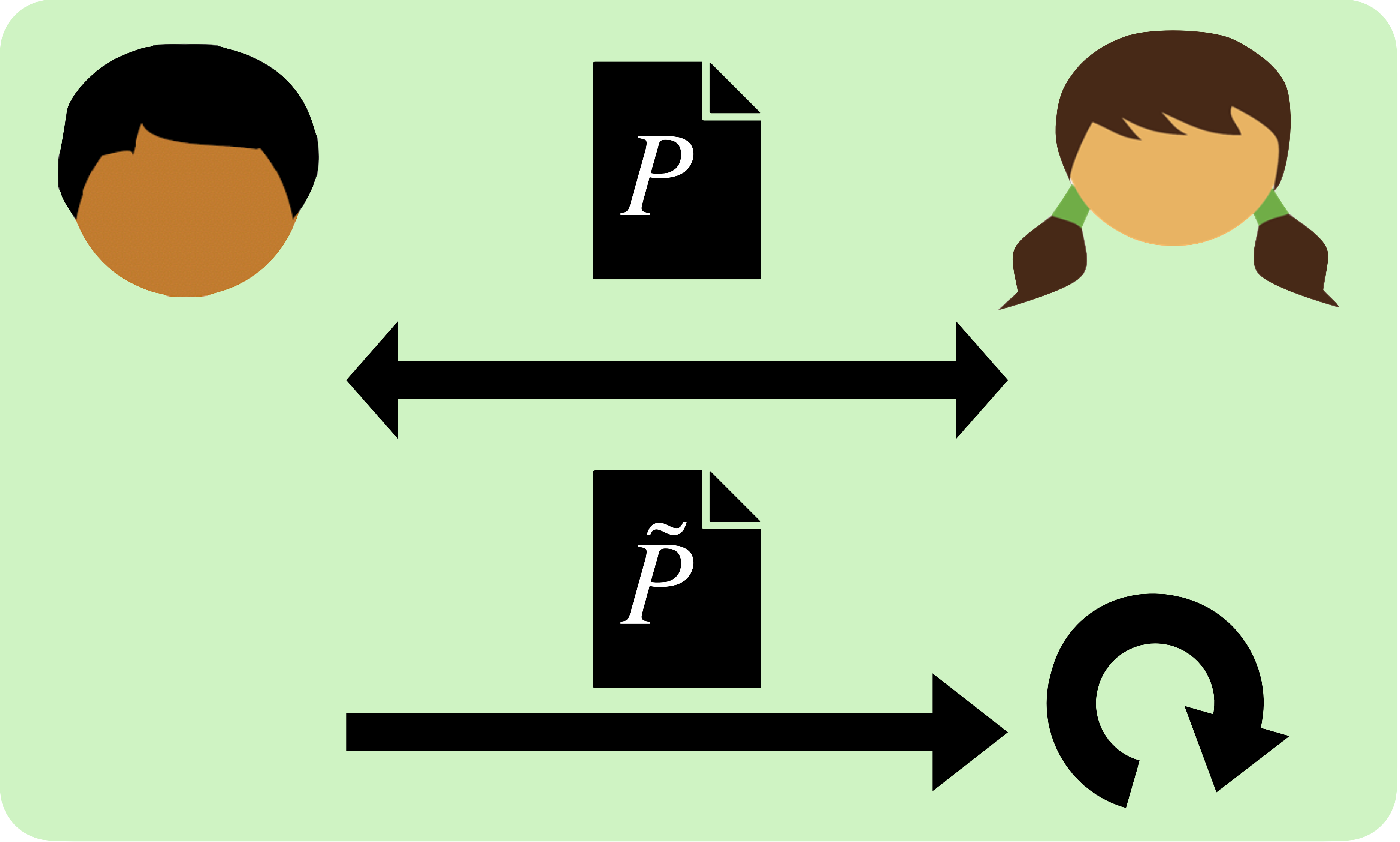
Desirable to garble **arithmetic circuits**

E.g., privacy-preserving machine learning

Garbling arithmetic gates was a challenge

Arithmetic Garbling



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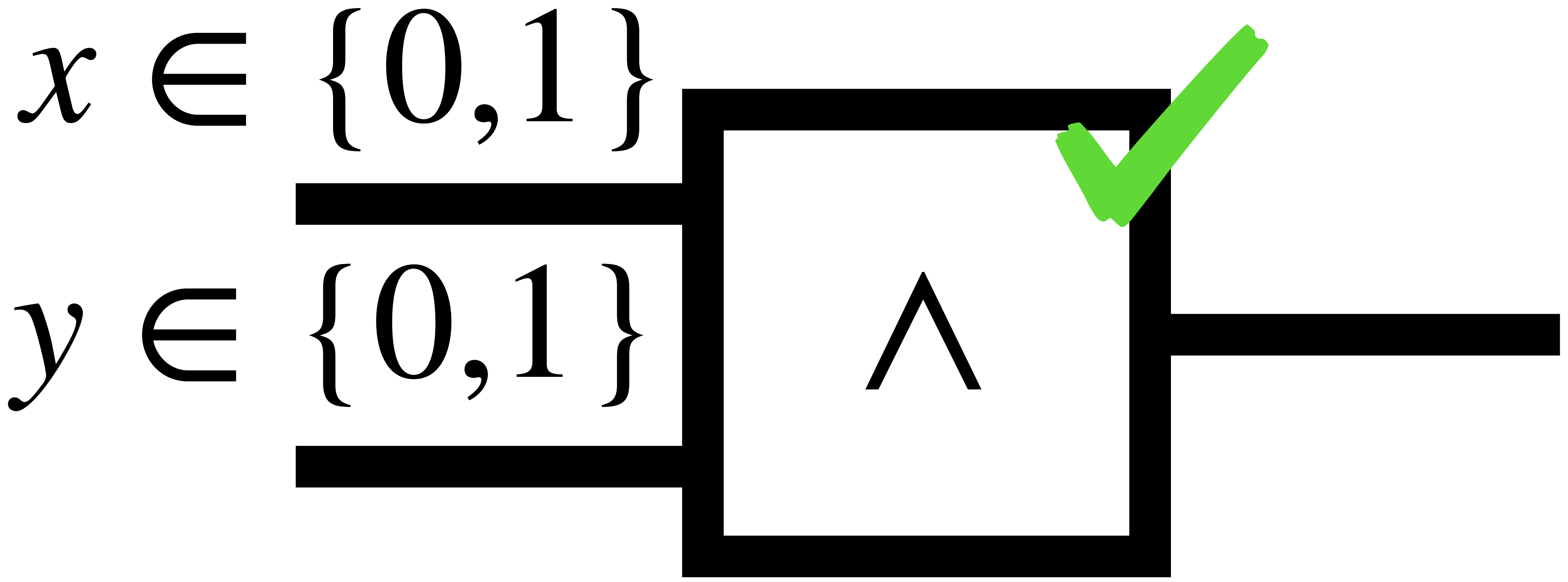

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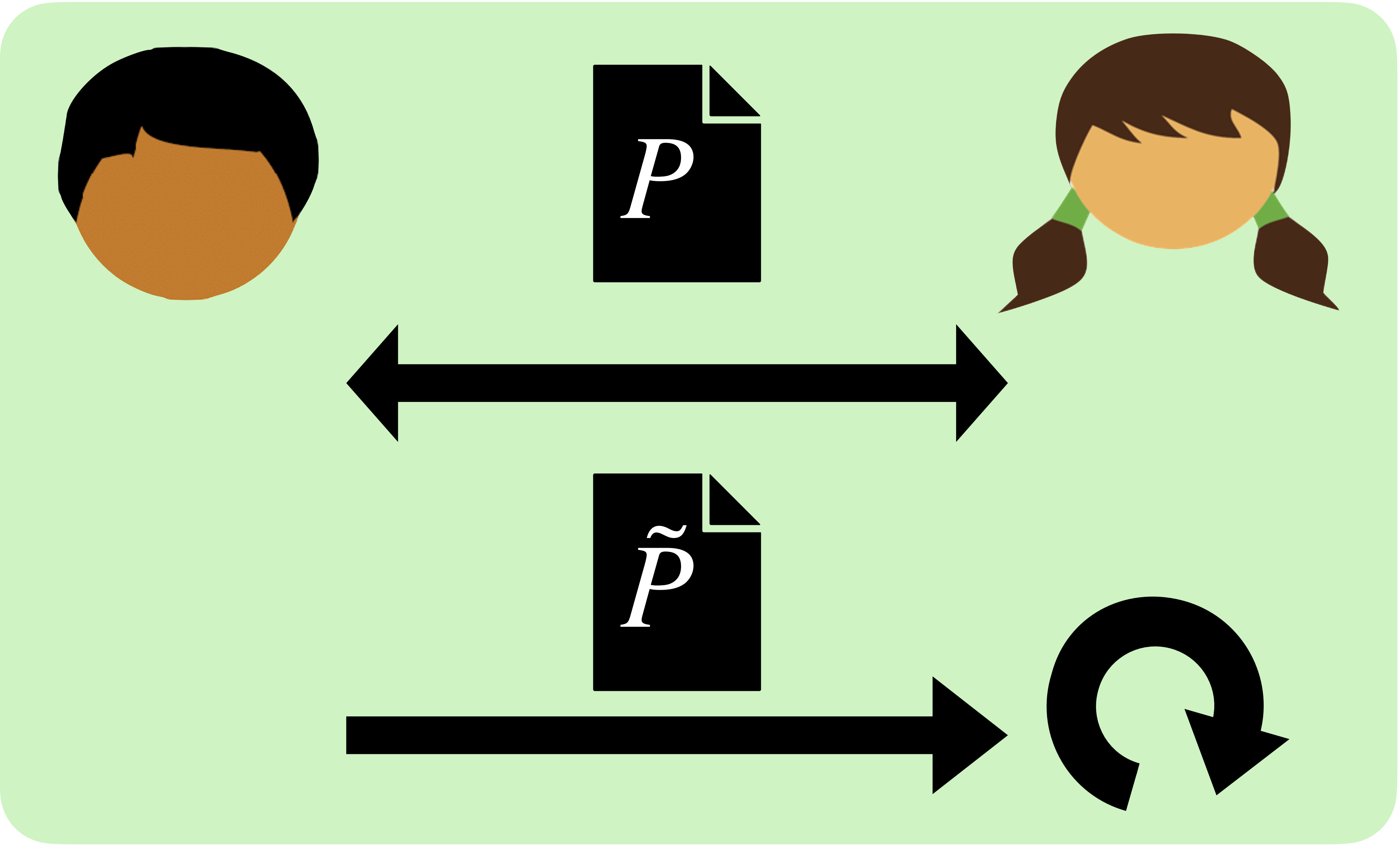

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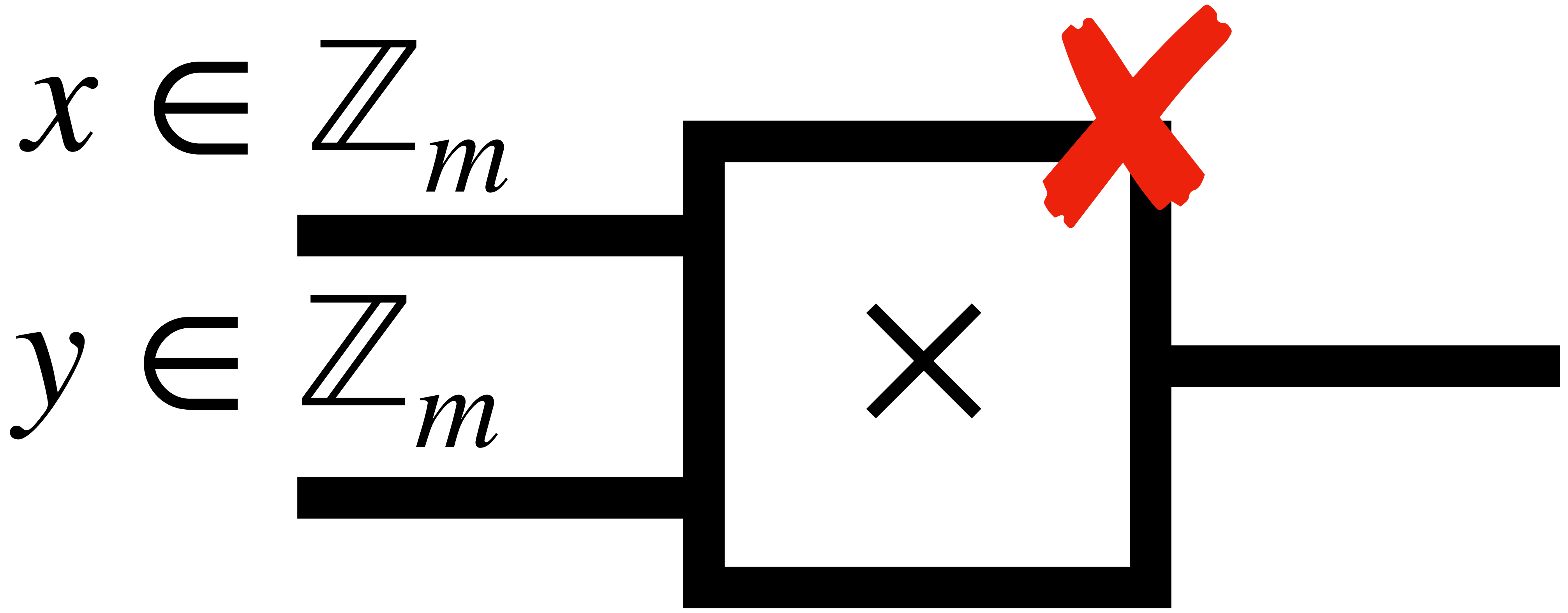
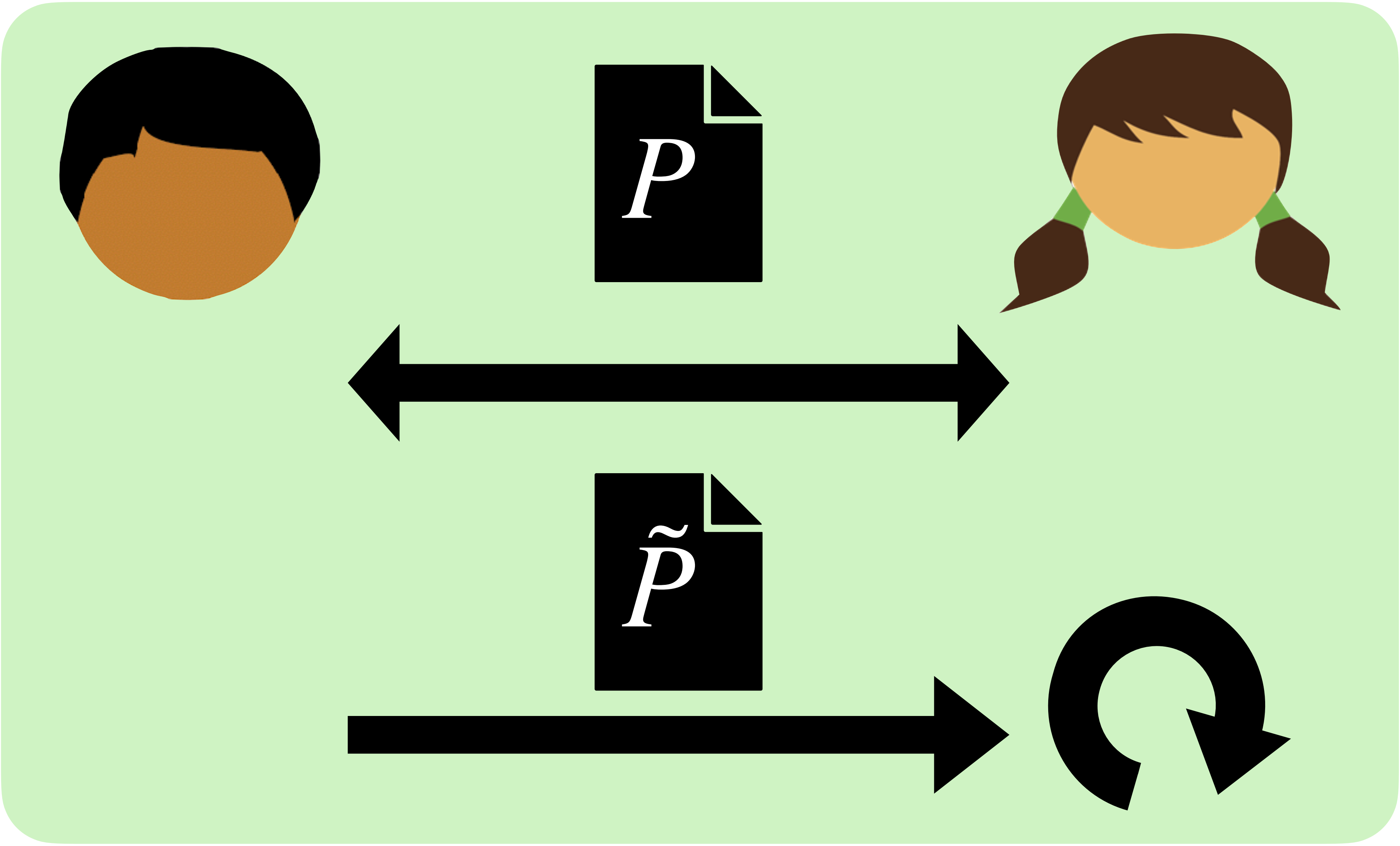


This Work

Consider: P is an n-gate arithmetic circuit over ℓ-bit integers

Goal: Generate small encoding $\tilde{P}$



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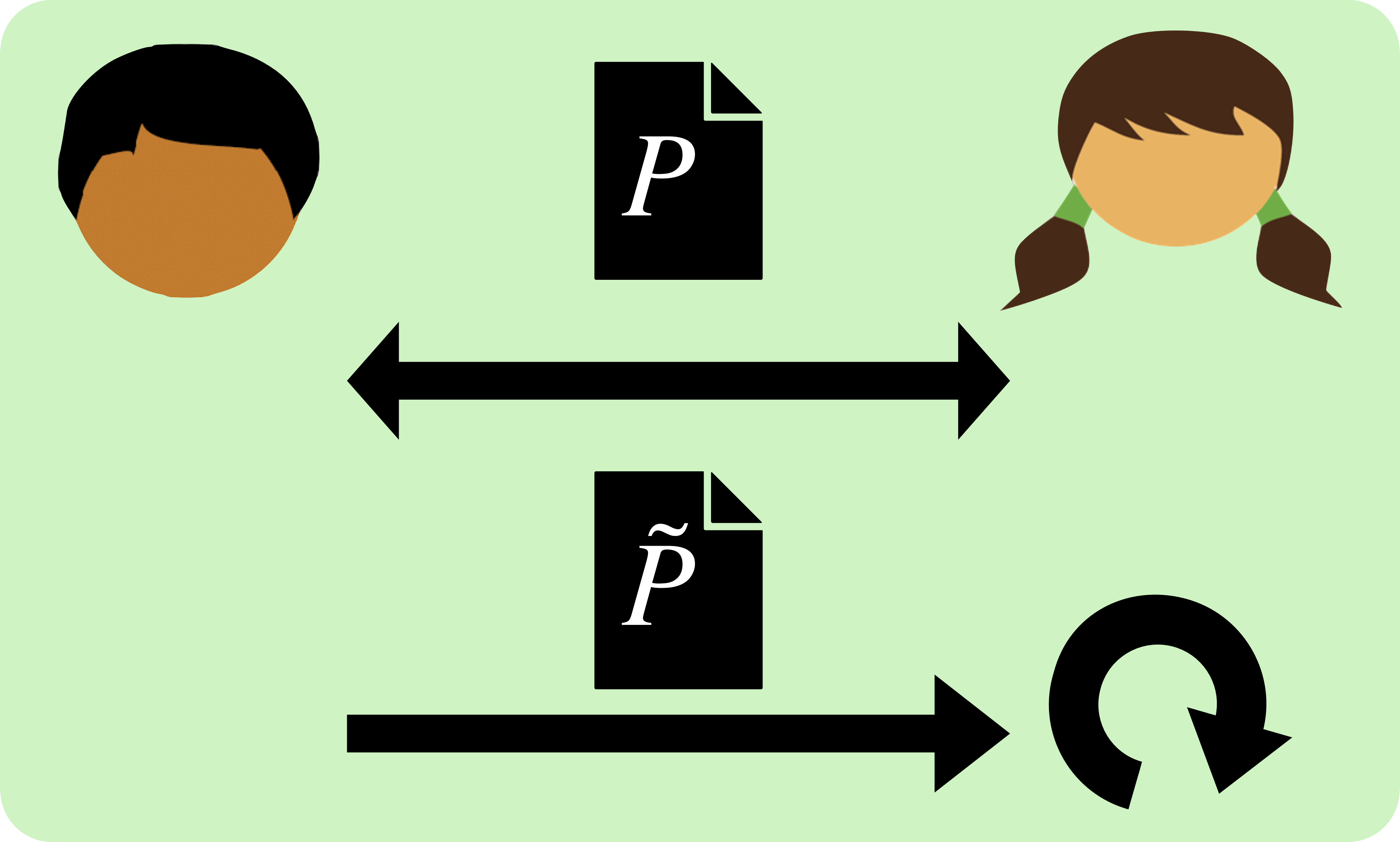

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Main Result: $\tilde{P}$ is $O(n \cdot \ell \cdot \lambda)$ bits long

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- Assumes circular-correlation robust hashes (common in practical symmetric-key GC)

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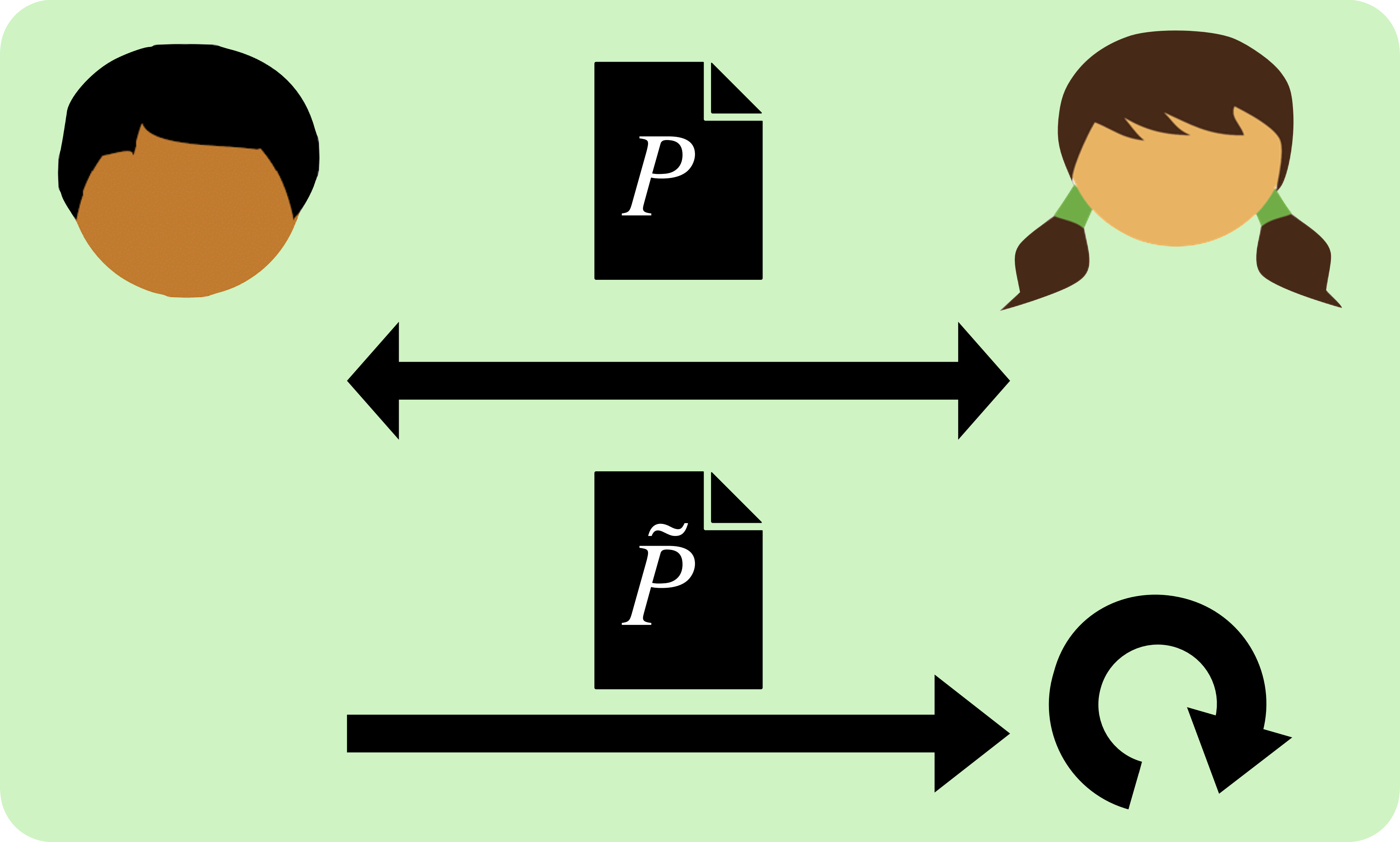

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Surprise Factor: ℓ-bit multiplication at cost $O(\ell \cdot \lambda)$

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Schoolbook	$O(\ell \cdot \lambda)$	$O(\ell^2 \cdot \lambda)$	
Karatsuba	$O(\ell \cdot \lambda)$	$O(\ell^{1.585} \cdot \lambda)$	

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This Work

Goal

Integer
Arithmetic

This Work

Goal

Long Integer Arithmetic

Short Integer Arithmetic

Chinese Remainder Theorem

This Work

Goal

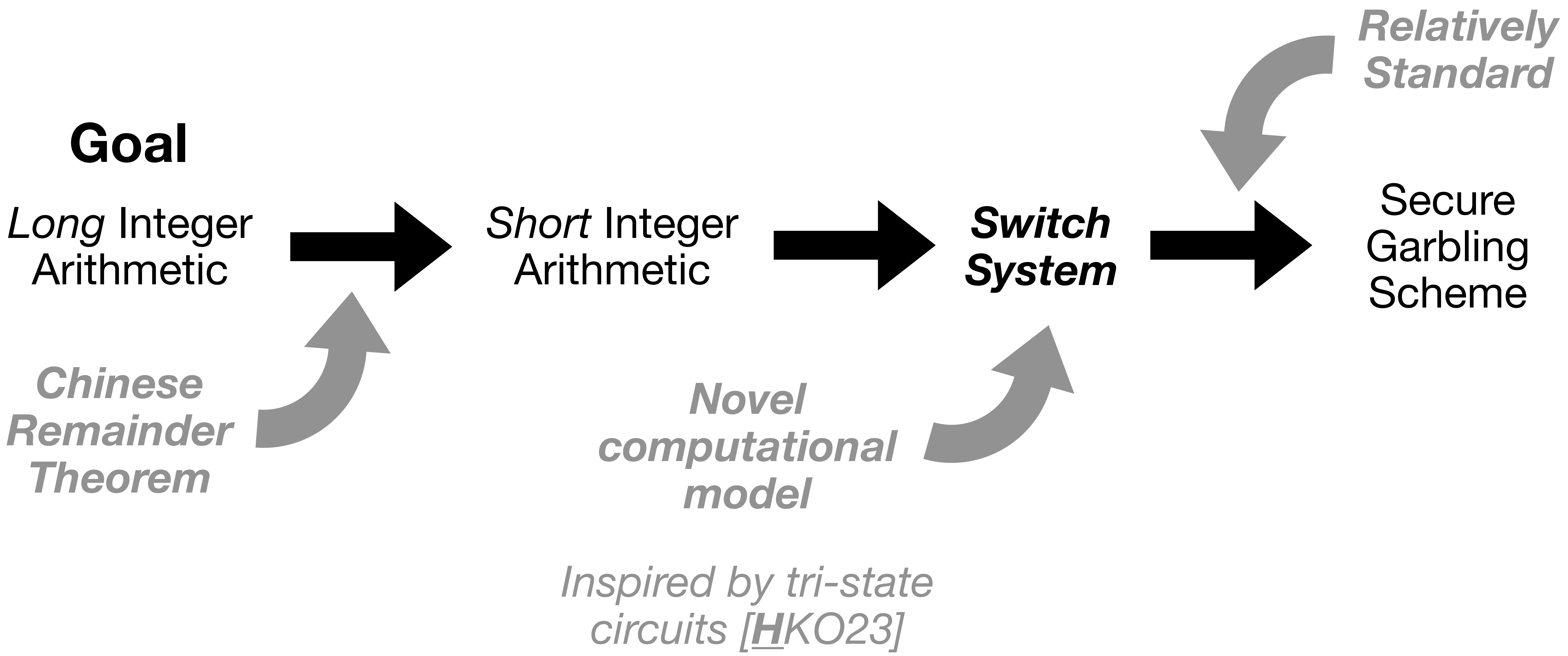
Long Integer Arithmetic → *Short* Integer Arithmetic → ***Switch System***

Chinese Remainder Theorem

Novel computational model

*Inspired by tri-state circuits [**H**KO23]*

This Work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Long Integer Arithmetic
Chinese Remainder Theorem
Short Integer Arithmetic
Novel computational model
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Switch System
Relatively Standard
Secure Garbling Scheme

This Work

Goal

Long Integer Arithmetic

Chinese Remainder Theorem

Today's Focus

Short Integer Arithmetic

Switch System

Novel computational model

*Inspired by tri-state circuits [<u>**H**</u>KO23]*

Relatively Standard

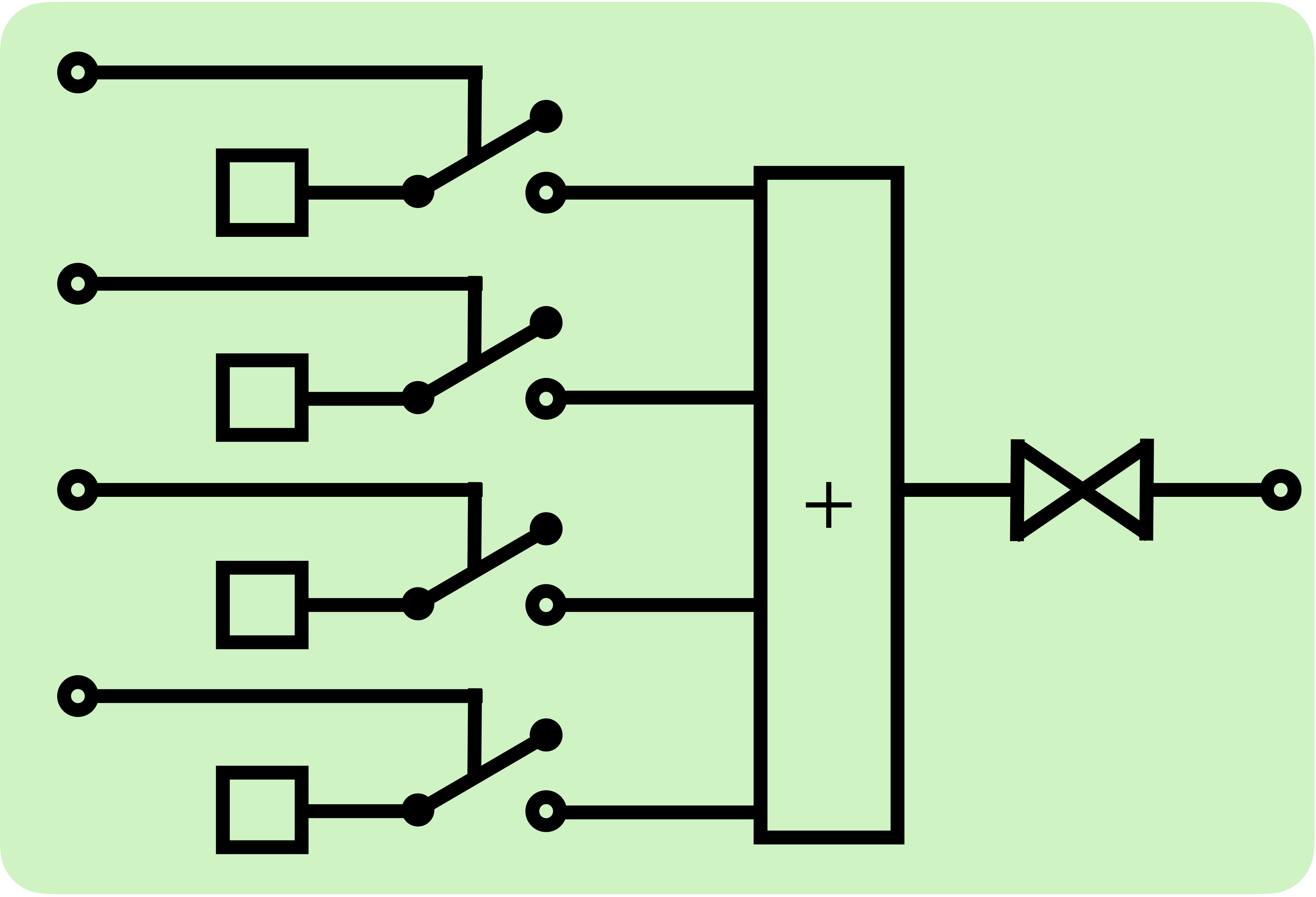
Secure Garbling Scheme

Switch Systems

Alternative to Boolean Circuits

Relatively Straightforward to Garble

Models Computation as a **Constraint System** that the evaluator will solve



See Paper

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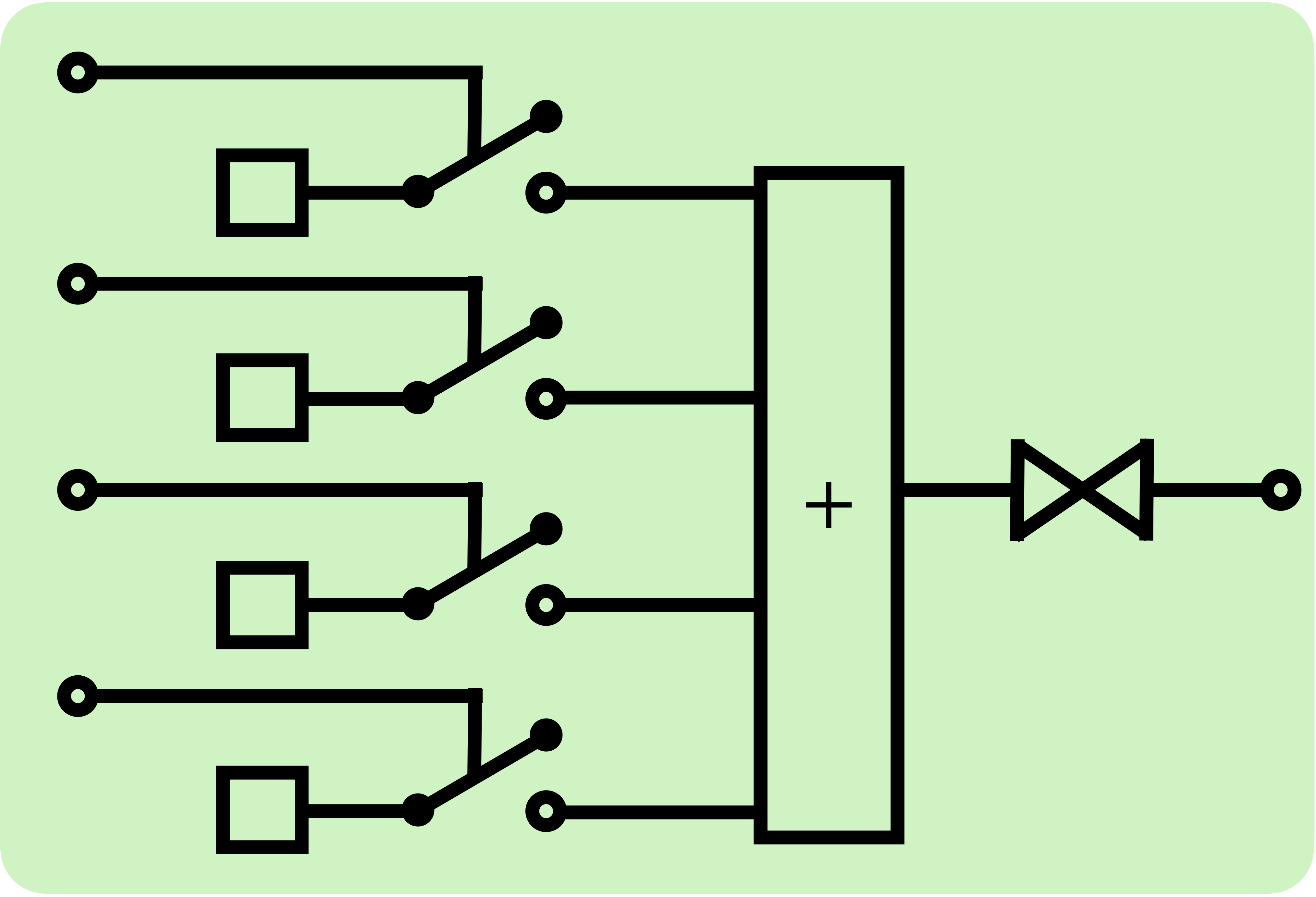

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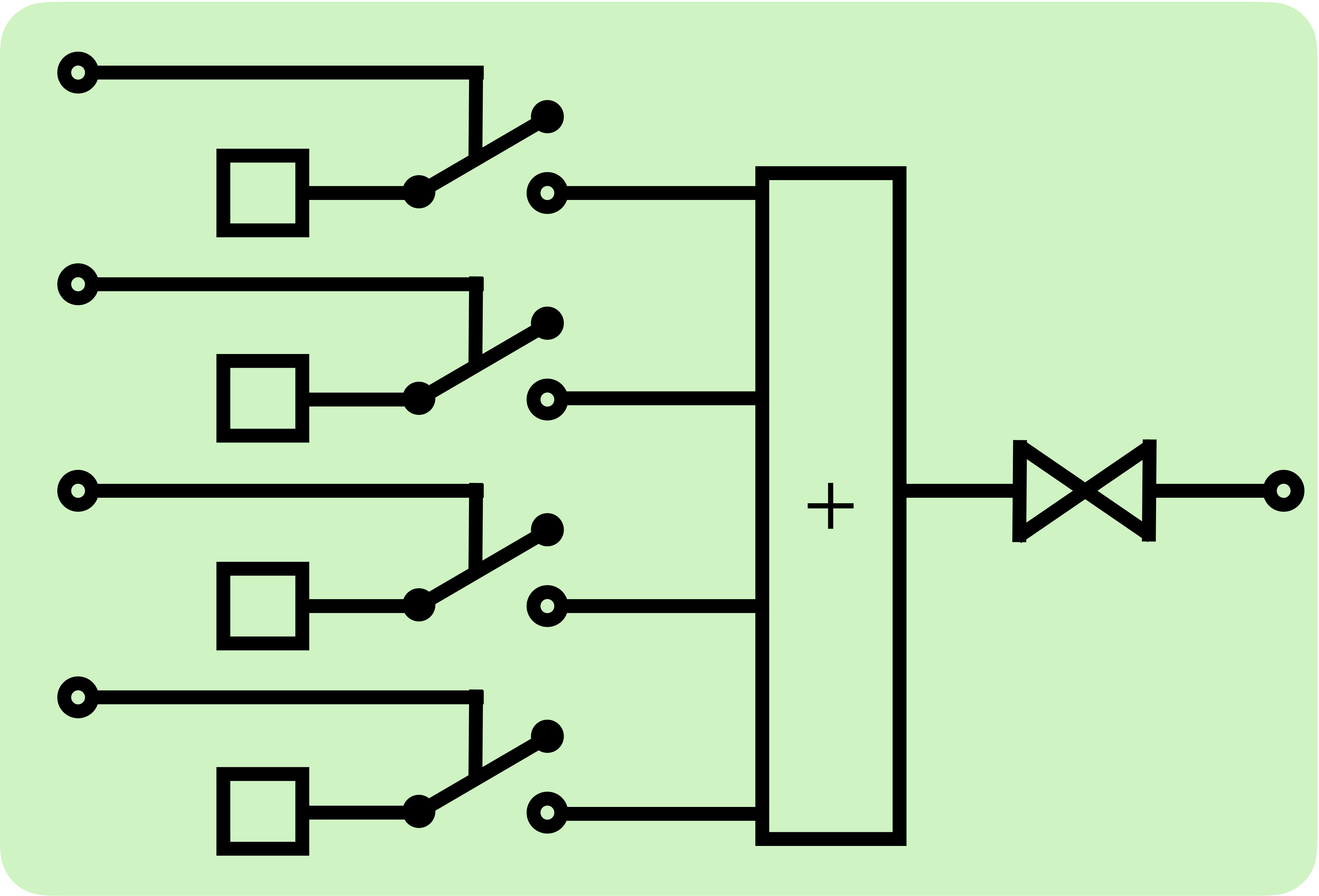
Switch Systems



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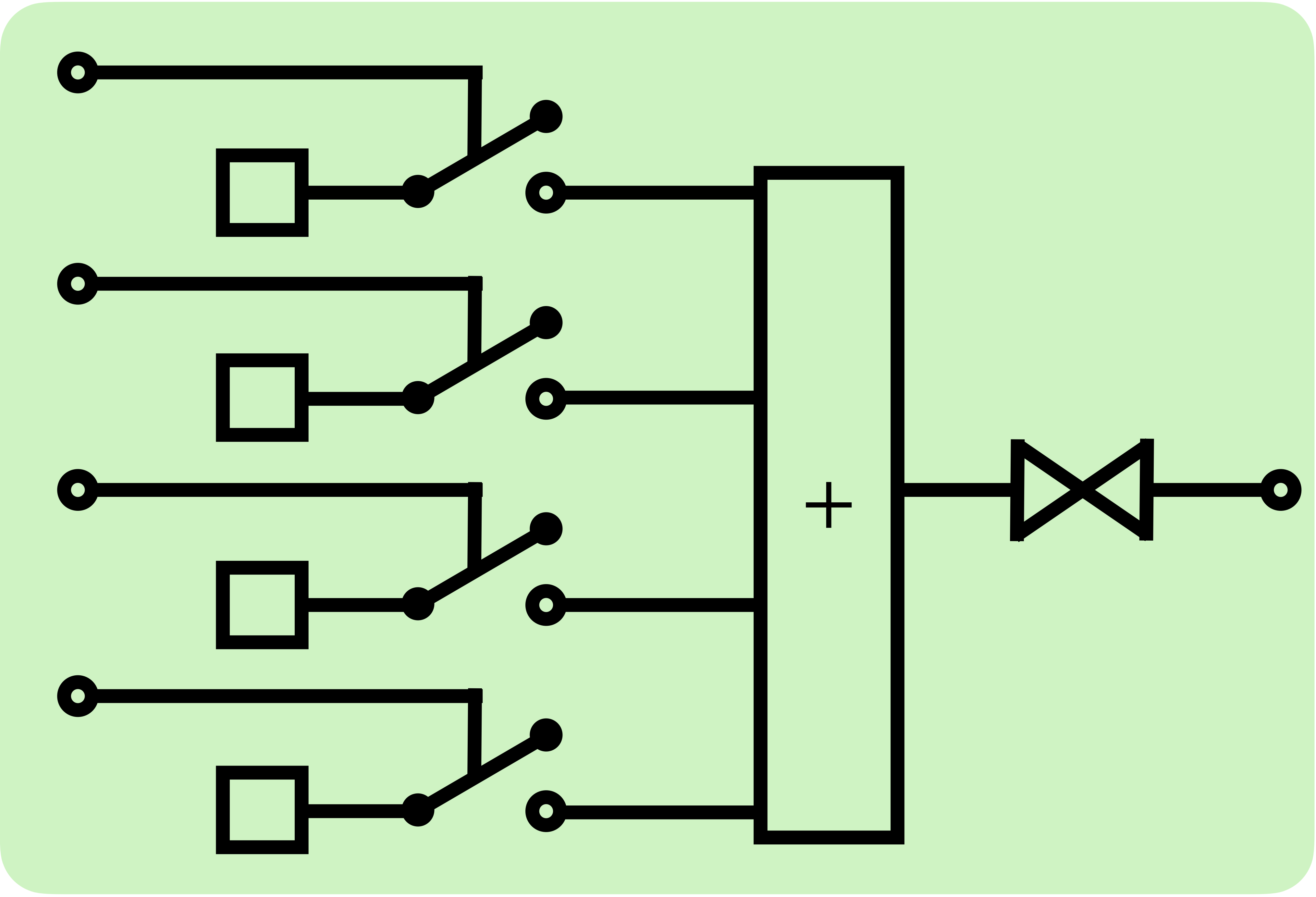
Switch Systems

Alternative to Boolean Circuits

Relatively Straightforward to Garble

Models Computation as a **Constraint System** that the evaluator will solve

Captures much of the state-of-the-art in symmetric-key garbling

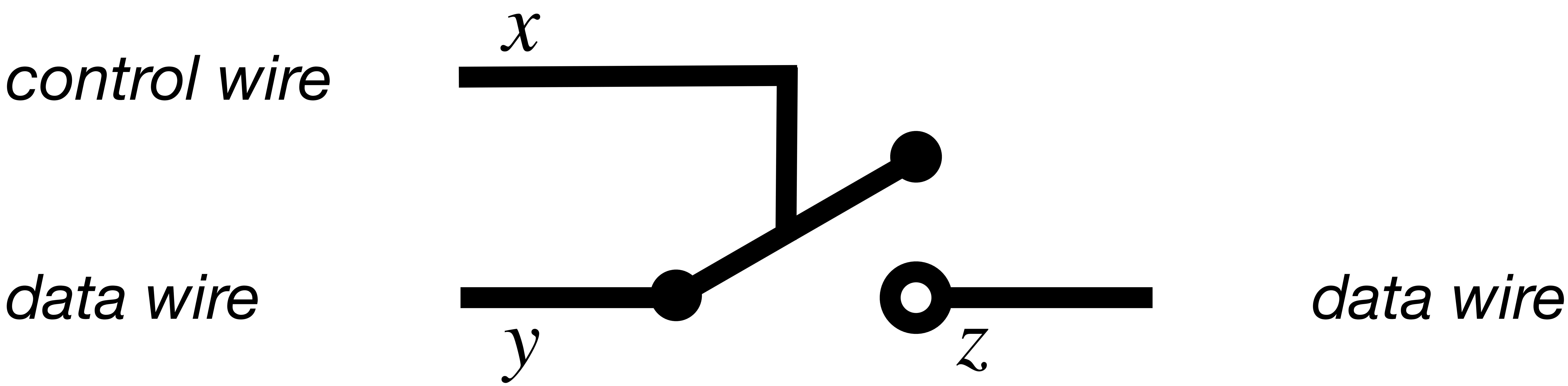


See Paper

Free XOR, Half-AND Gates, Garbled RAM, One-Hot Garbling, ***Arithmetic Computations***

Switch Systems

Switch



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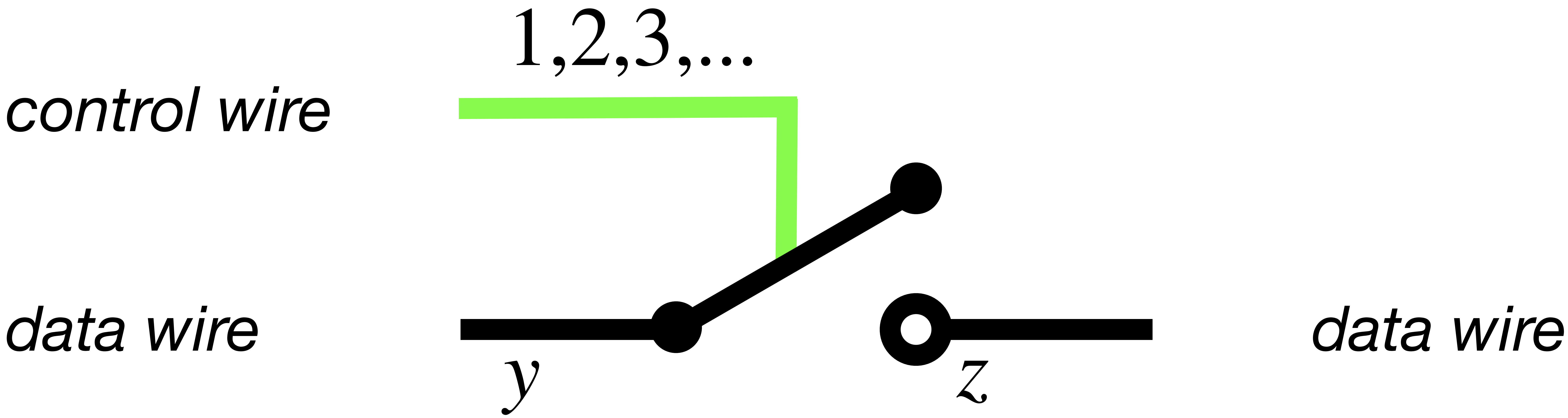

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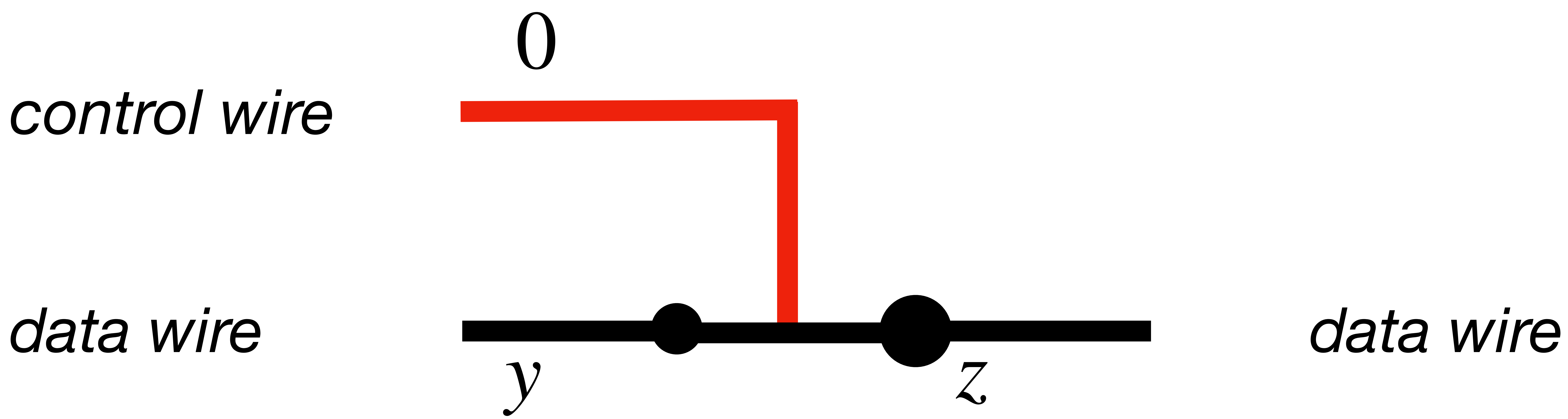

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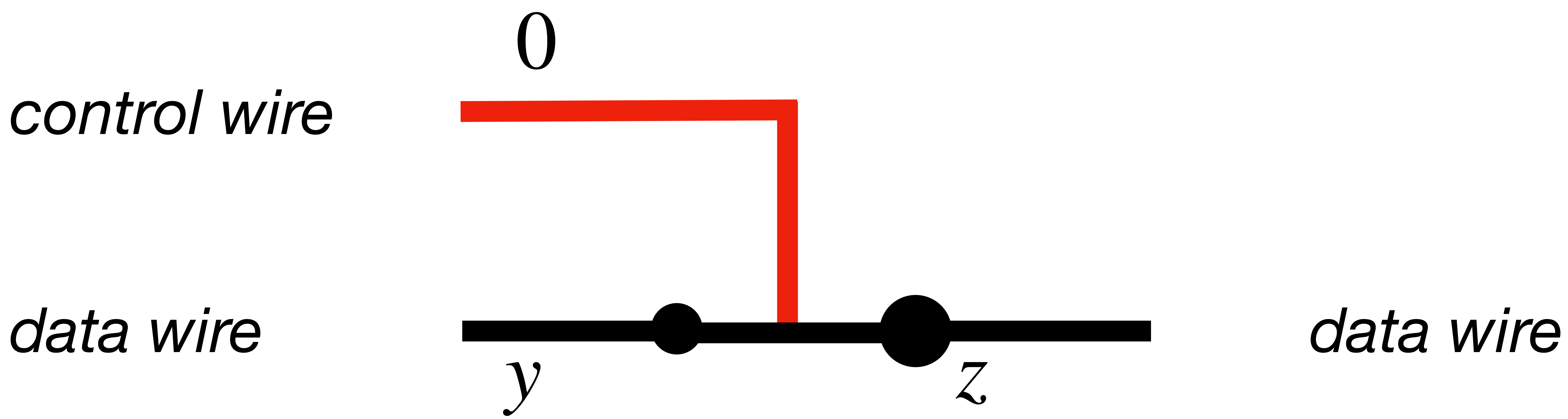


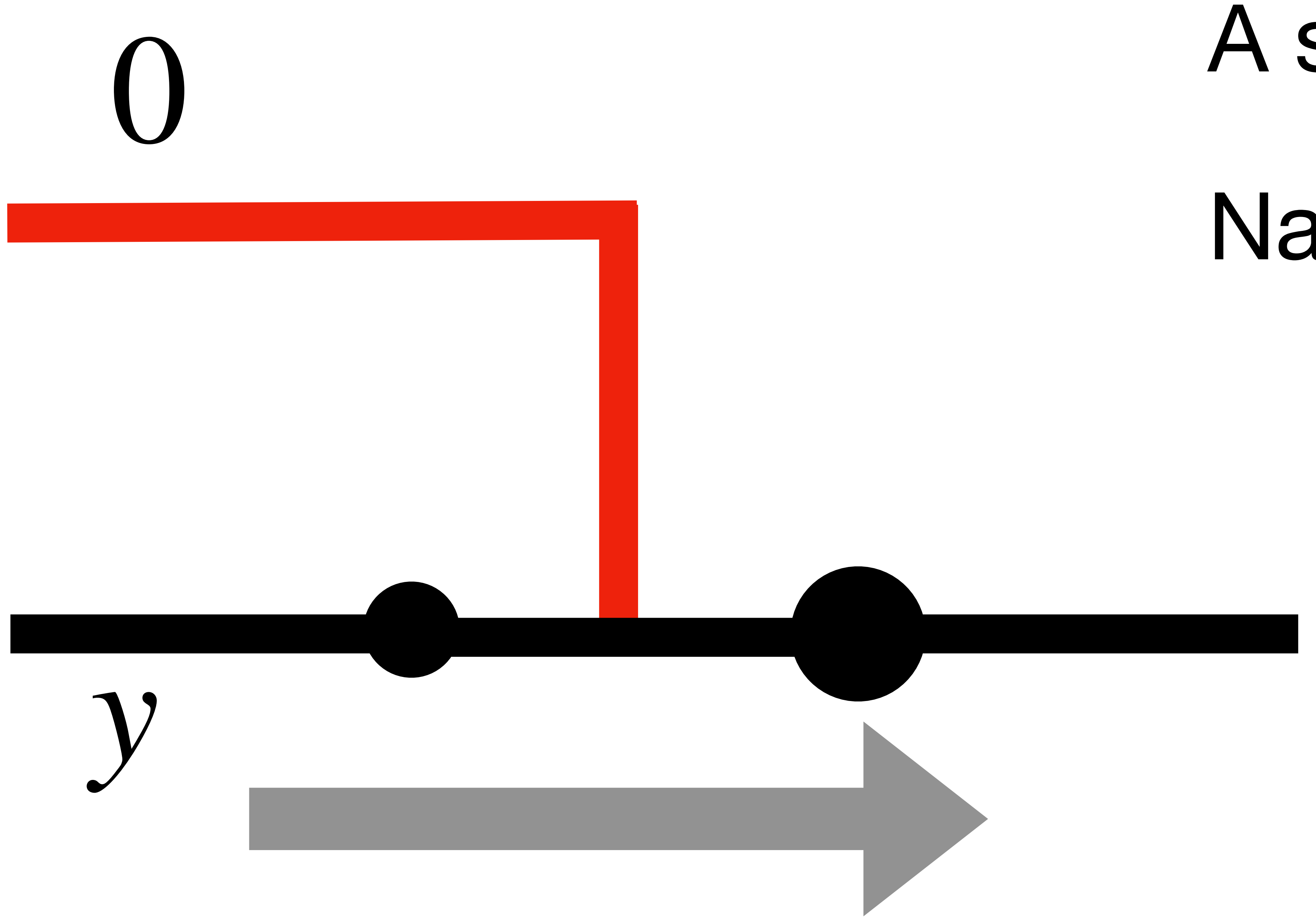
$$x = 0 \implies y = z$$

Switch Systems

Switch

A switch enforces a *constraint*

Namely, it is *bidirectional*



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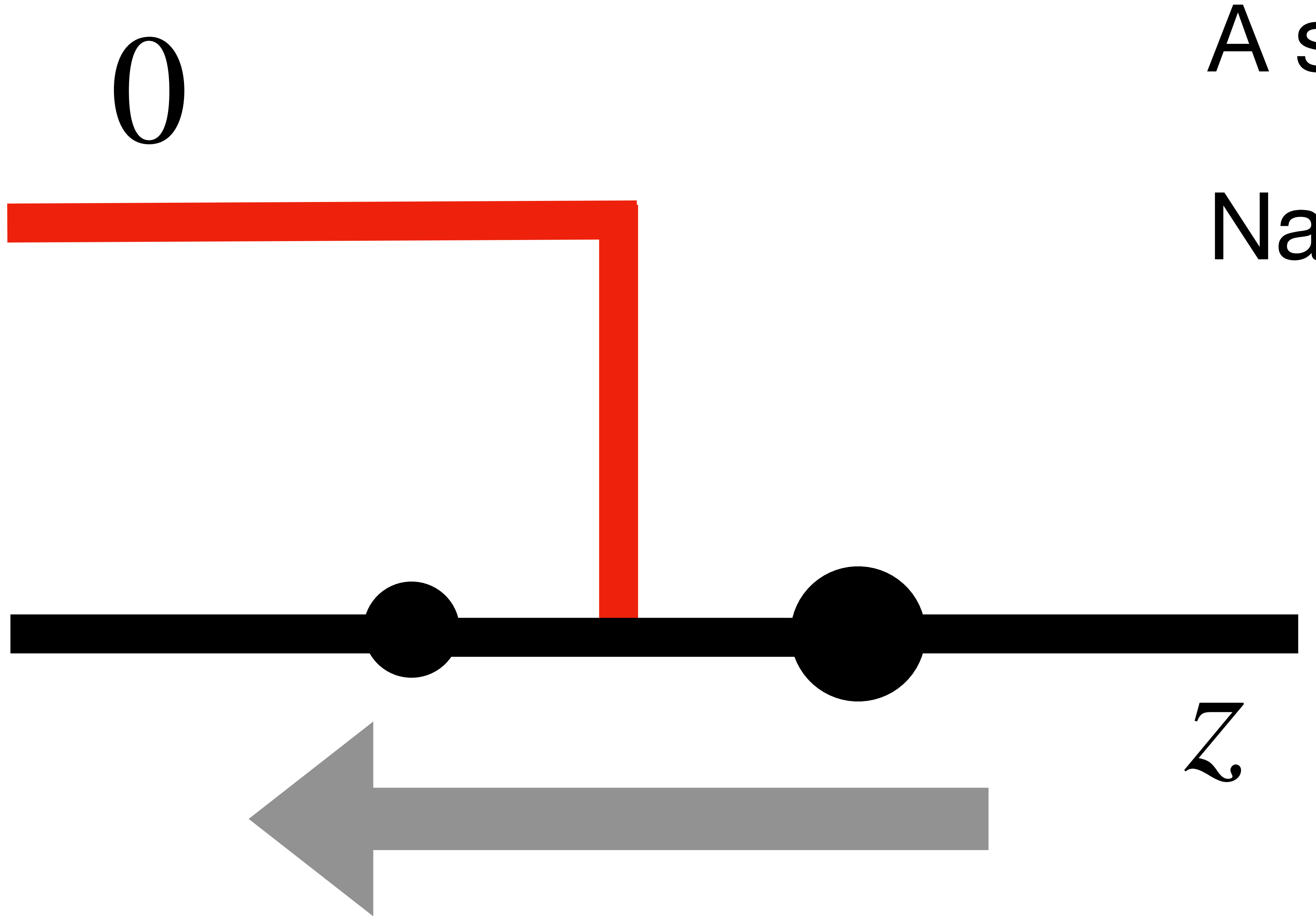

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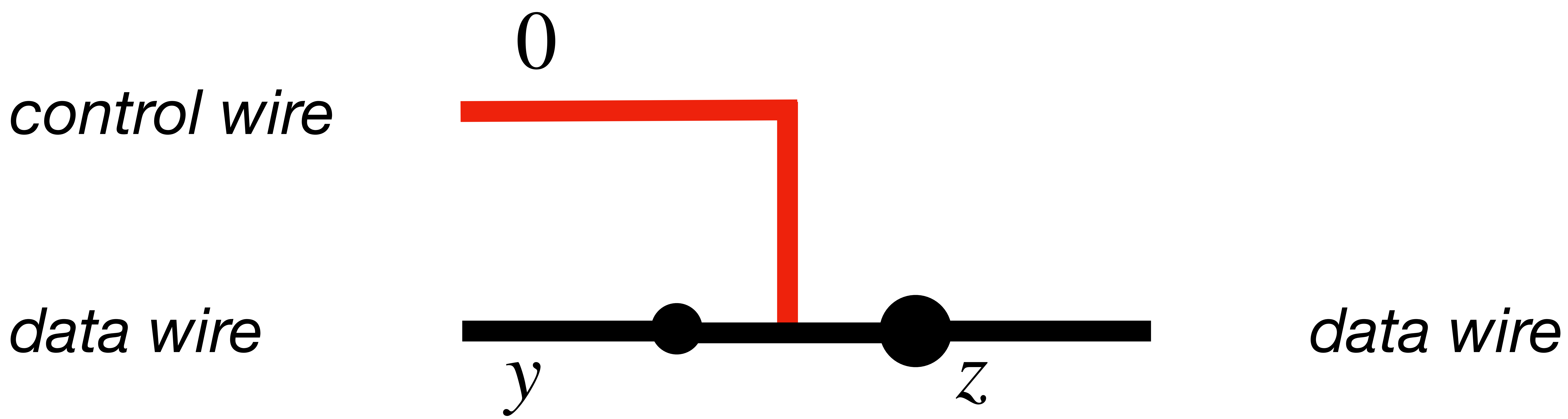


$$x = 0 \implies y = z$$

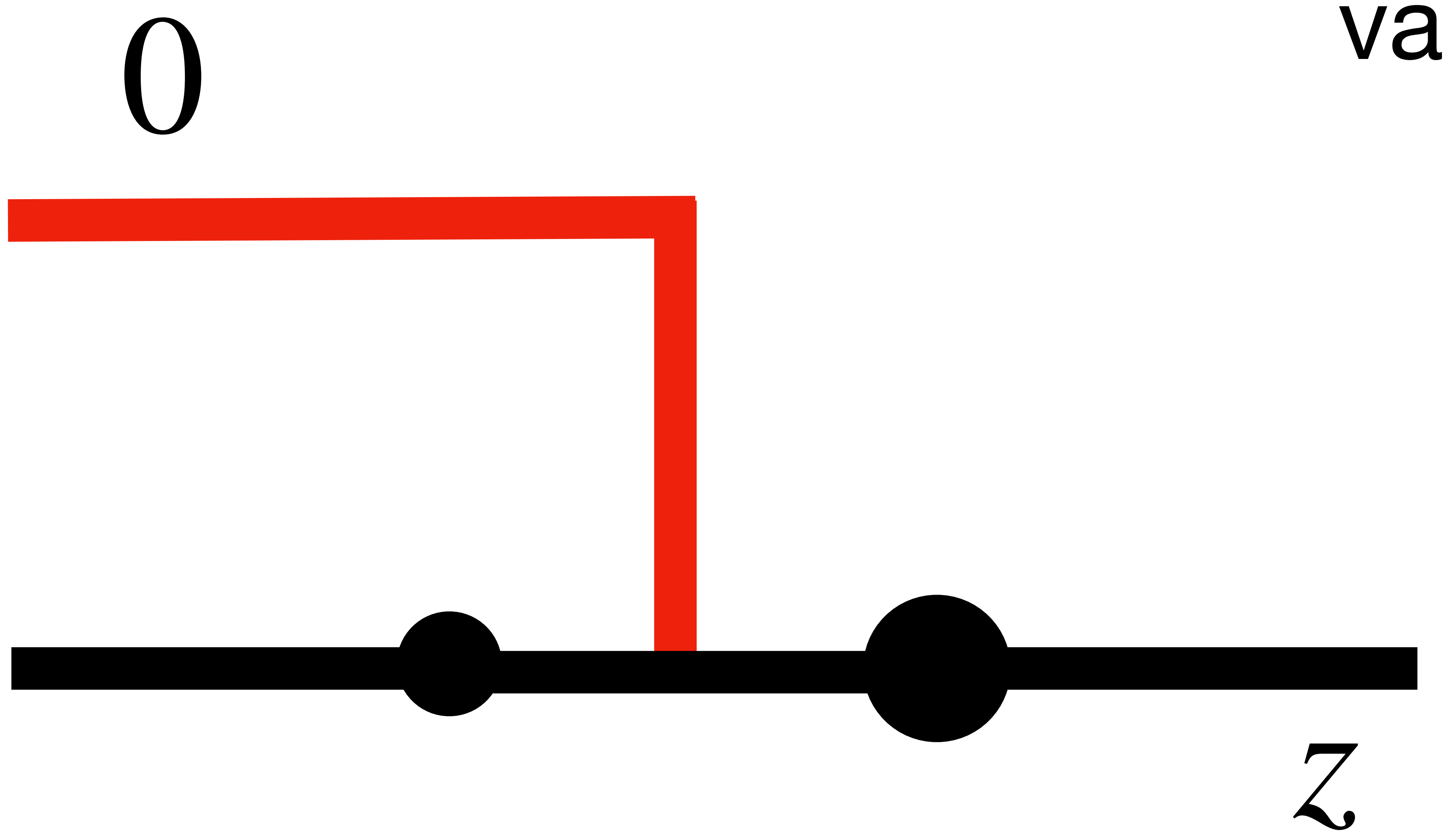
Insight: Garbler chooses one key per value per wire.

Difference between keys on data wires is equal to the hash of the zero control key

Switch Systems

Switch

GC Evaluator will learn value of every control wire



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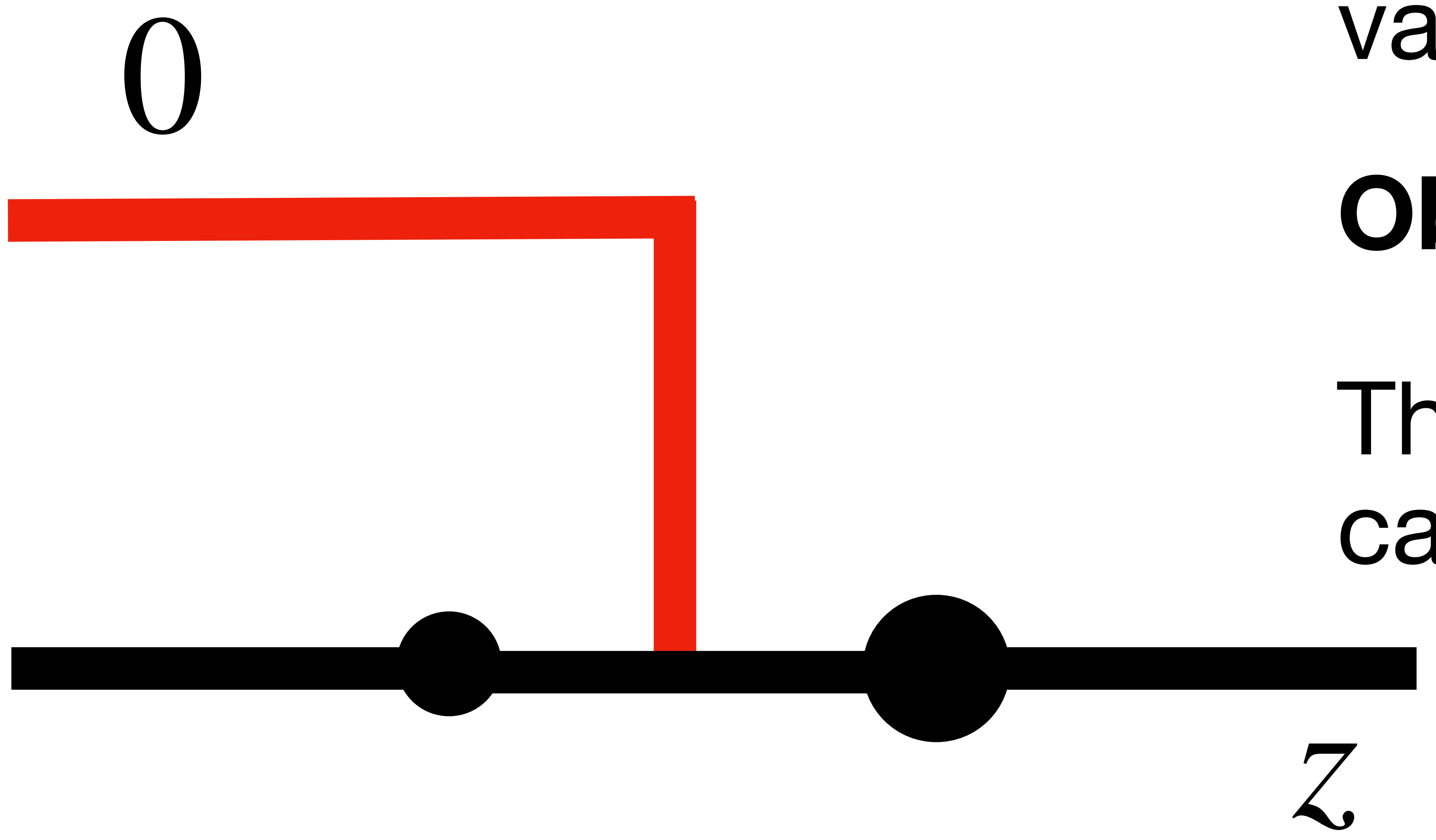

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Oblivious switch system:

The control wire values can be simulated

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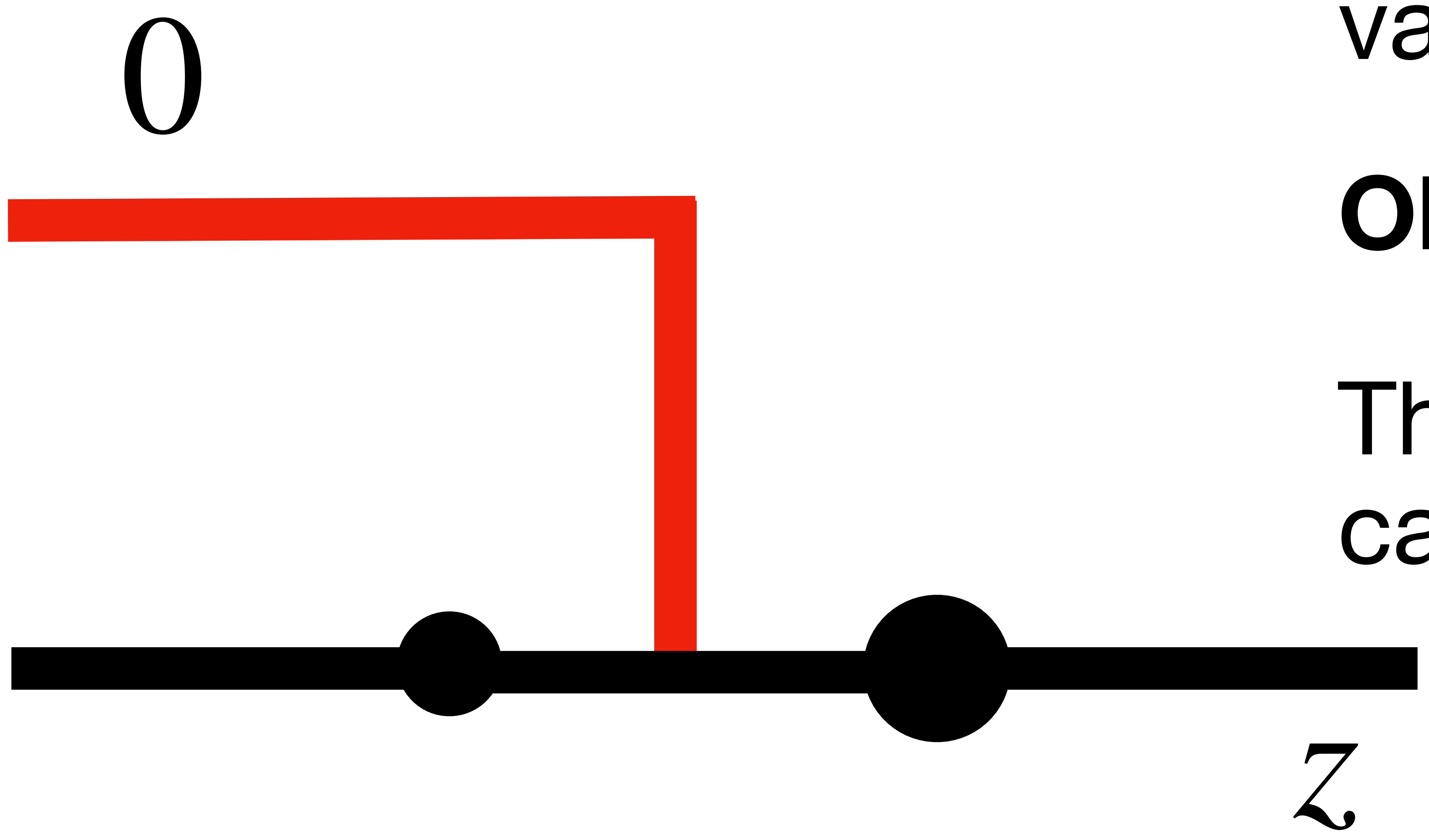

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GC Evaluator will learn value of every control wire

Oblivious switch system:

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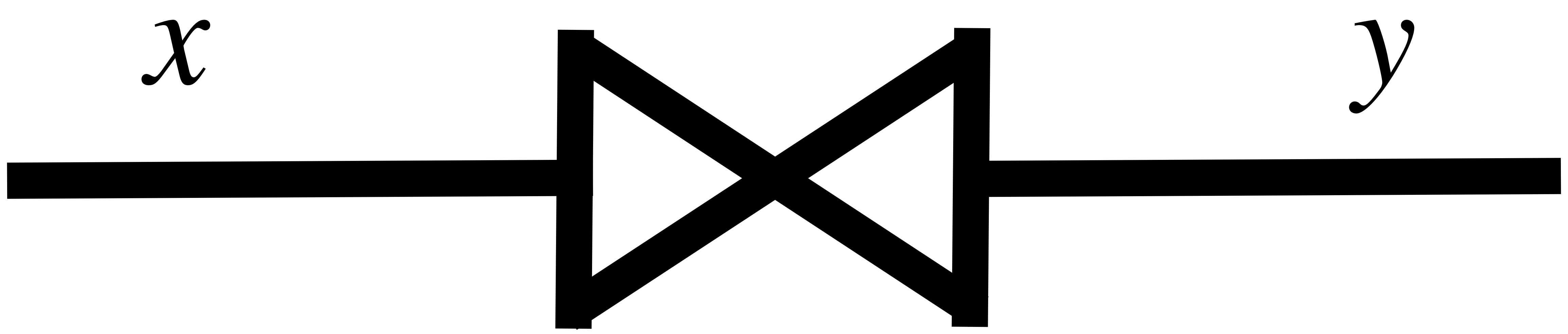
Insight: Garbler can introduce one-time-pad masks that allow to safely reveal control values

Switch Systems

Join



$$x = y$$

Switch Systems

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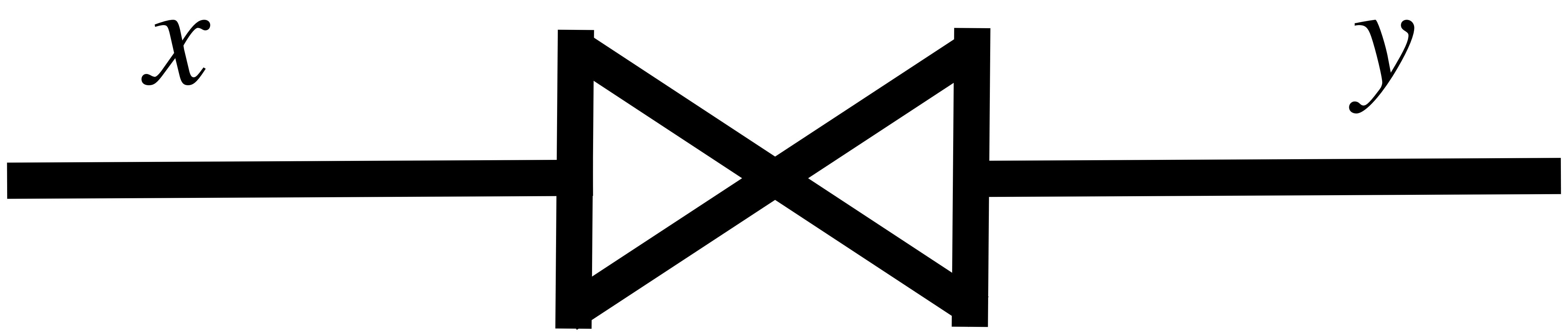


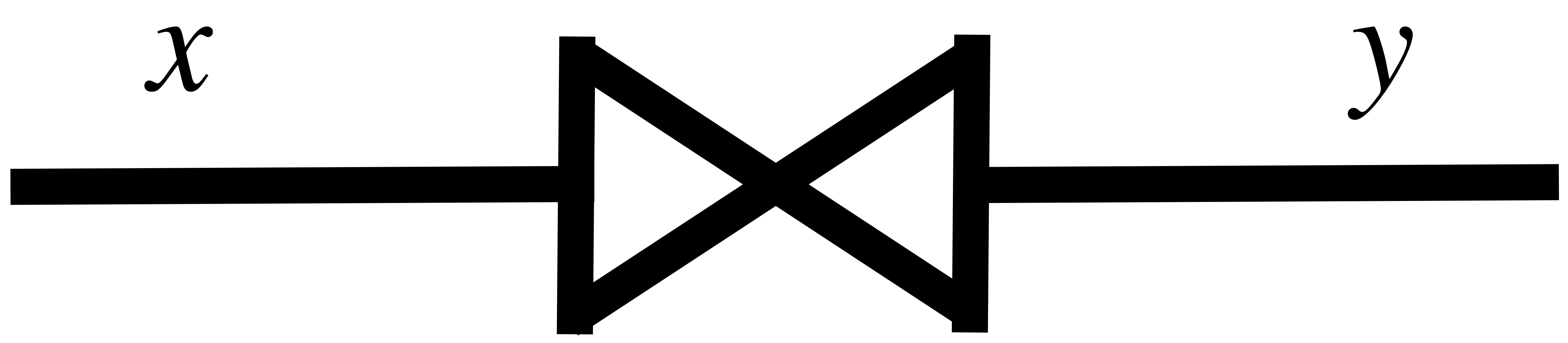
$$x = y$$

NOTE! The only gates that contribute to the size of a garbled circuit are joins!

Switch Systems

Join

Switch systems evaluate as a system of constraints, but they must be set up as a circuit



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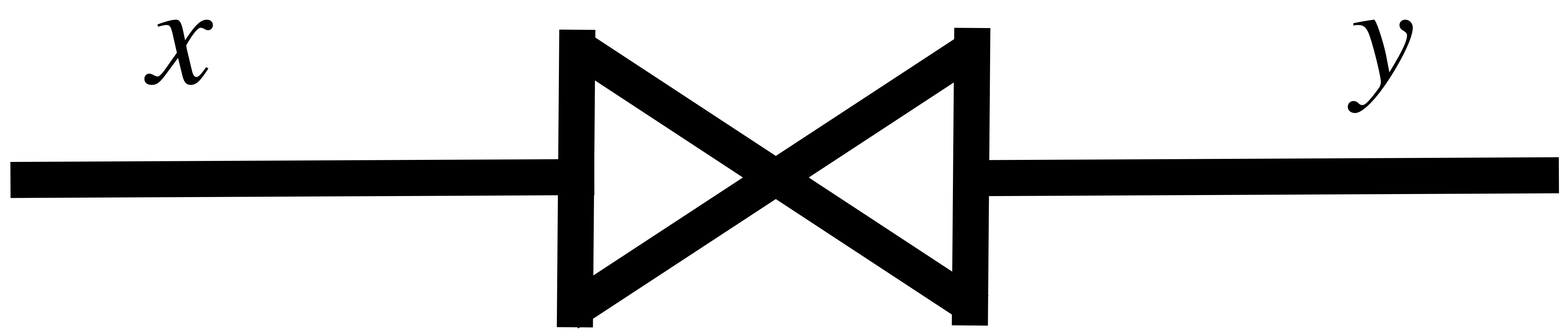

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Insight: Garbler encrypts system in circuit order, evaluator solves constraints

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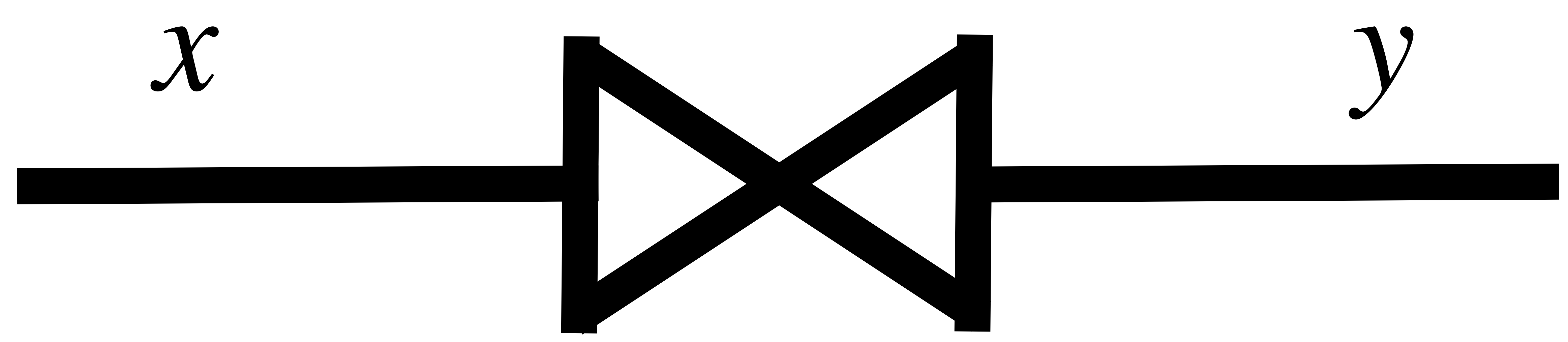

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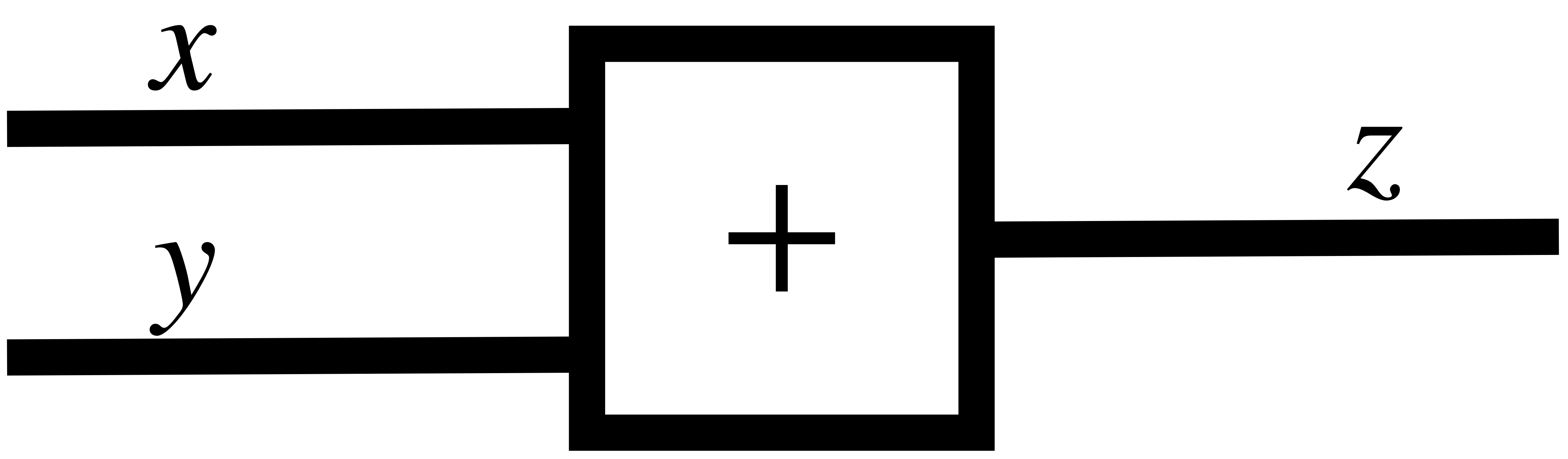
Insight: Garbler encrypts system in circuit order, evaluator solves constraints

NOTE! The only gates that contribute to the size of a garbled circuit are joins!

To improve GC handling, reduce the number of joins!

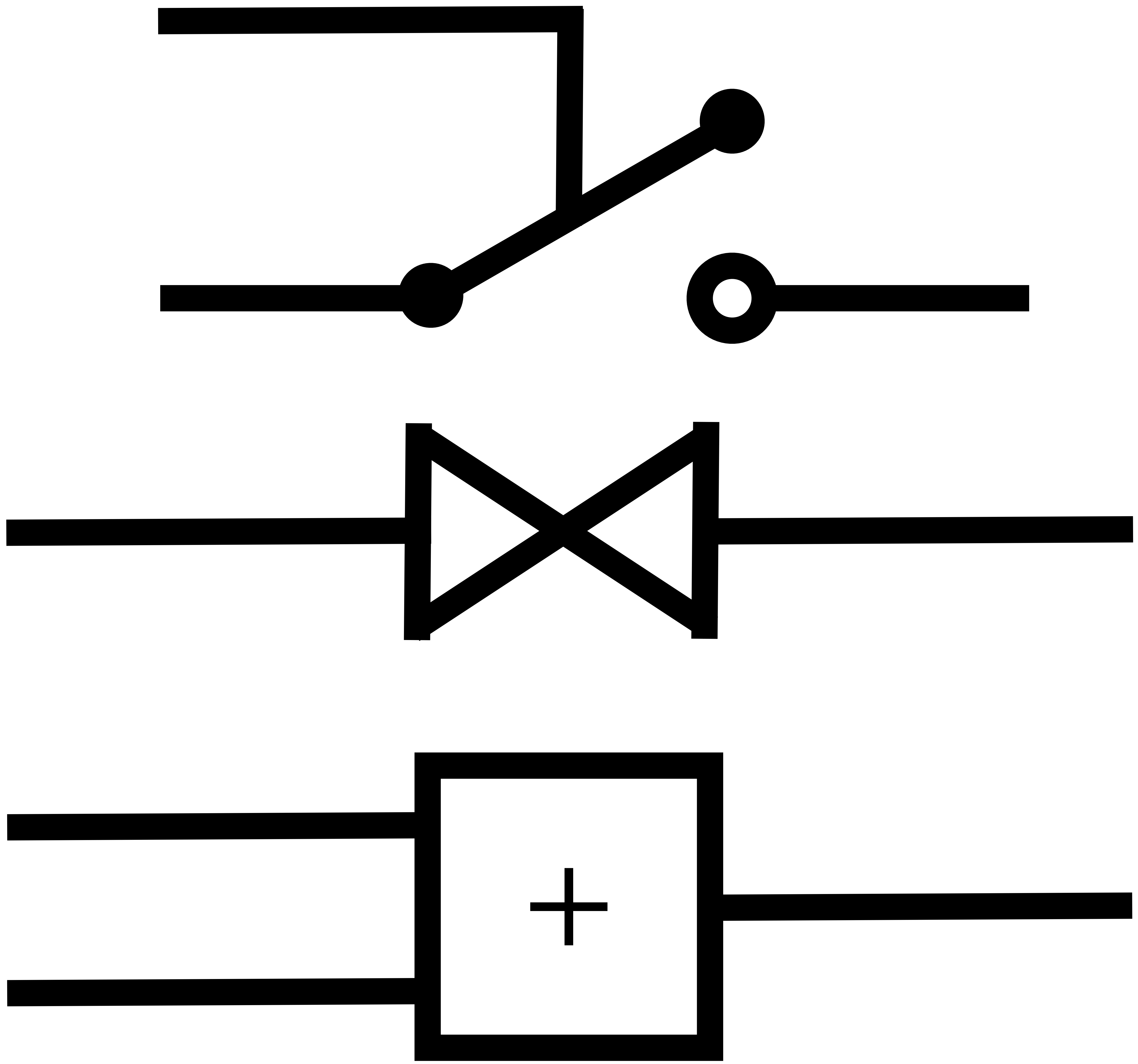
Switch Systems

ADD



$$x + y = z$$

Switch Systems



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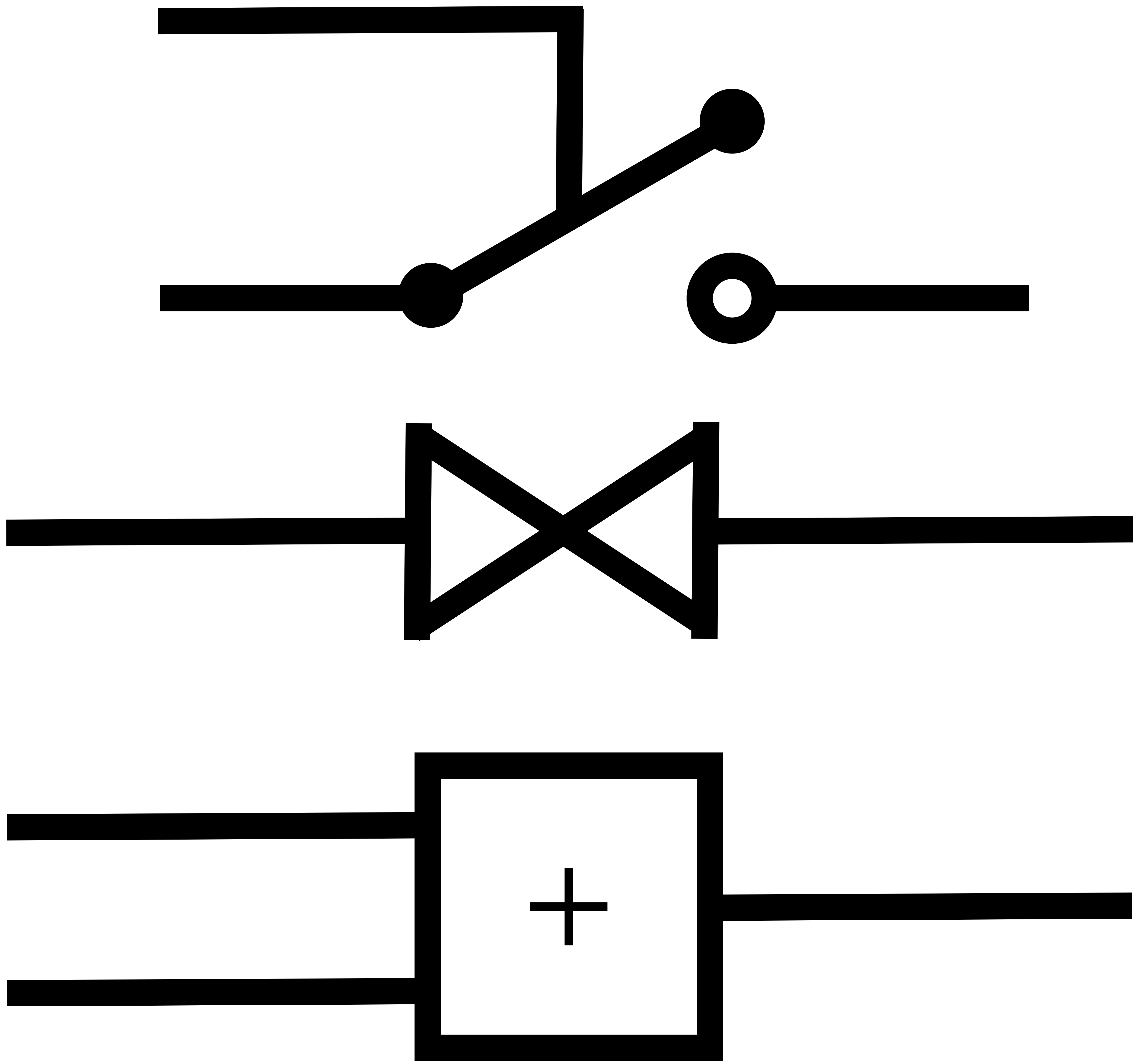
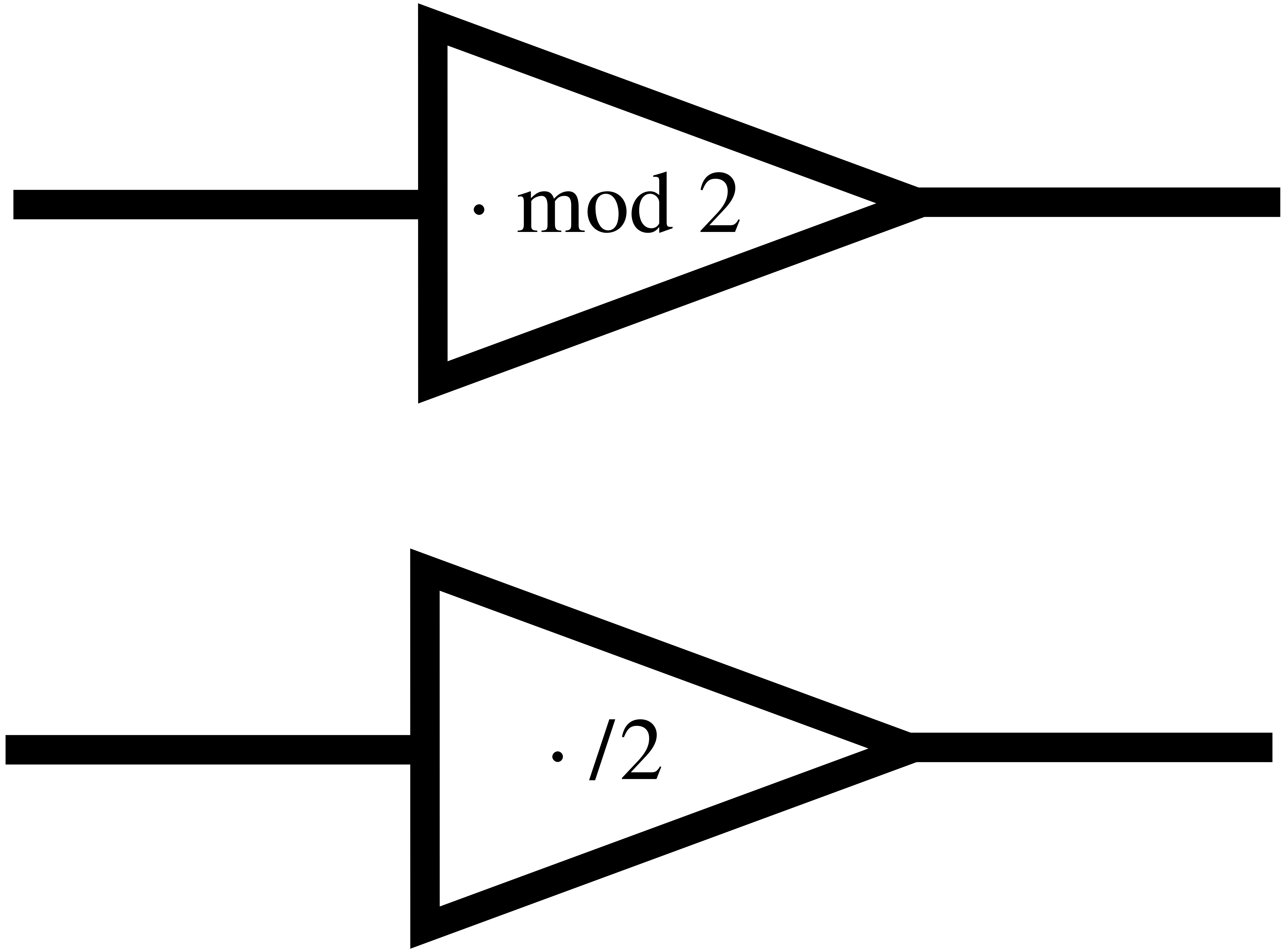
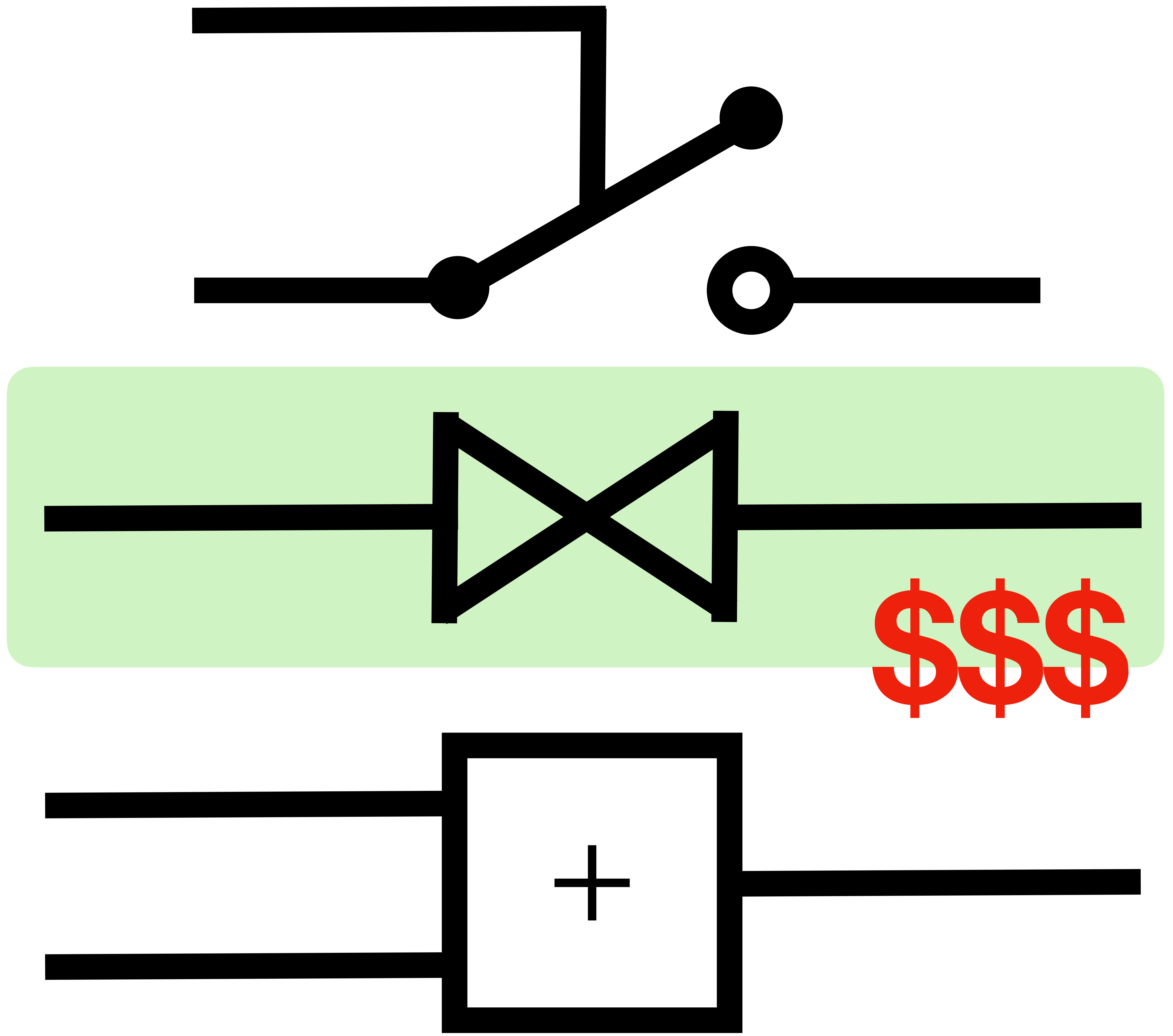
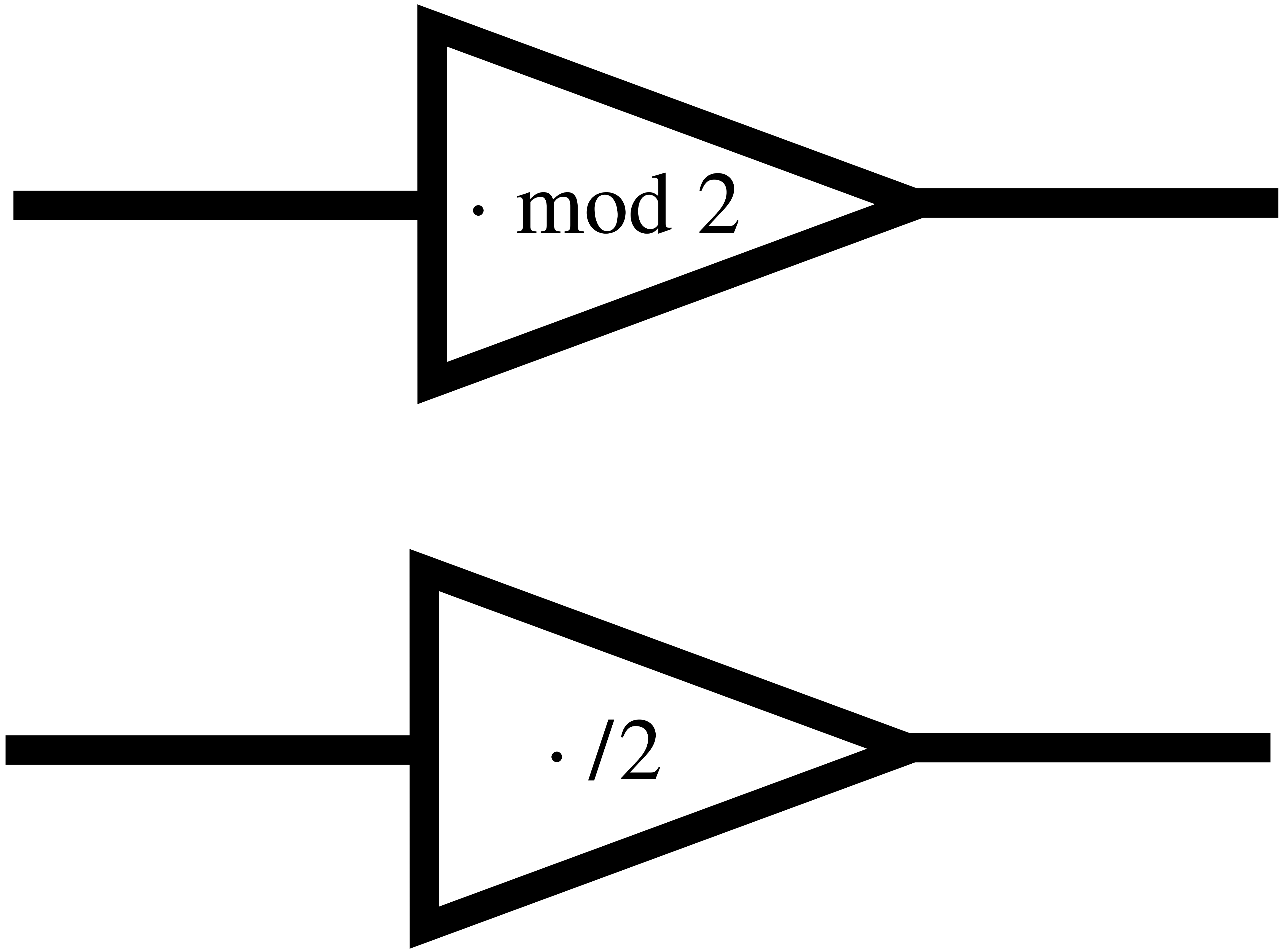




See paper

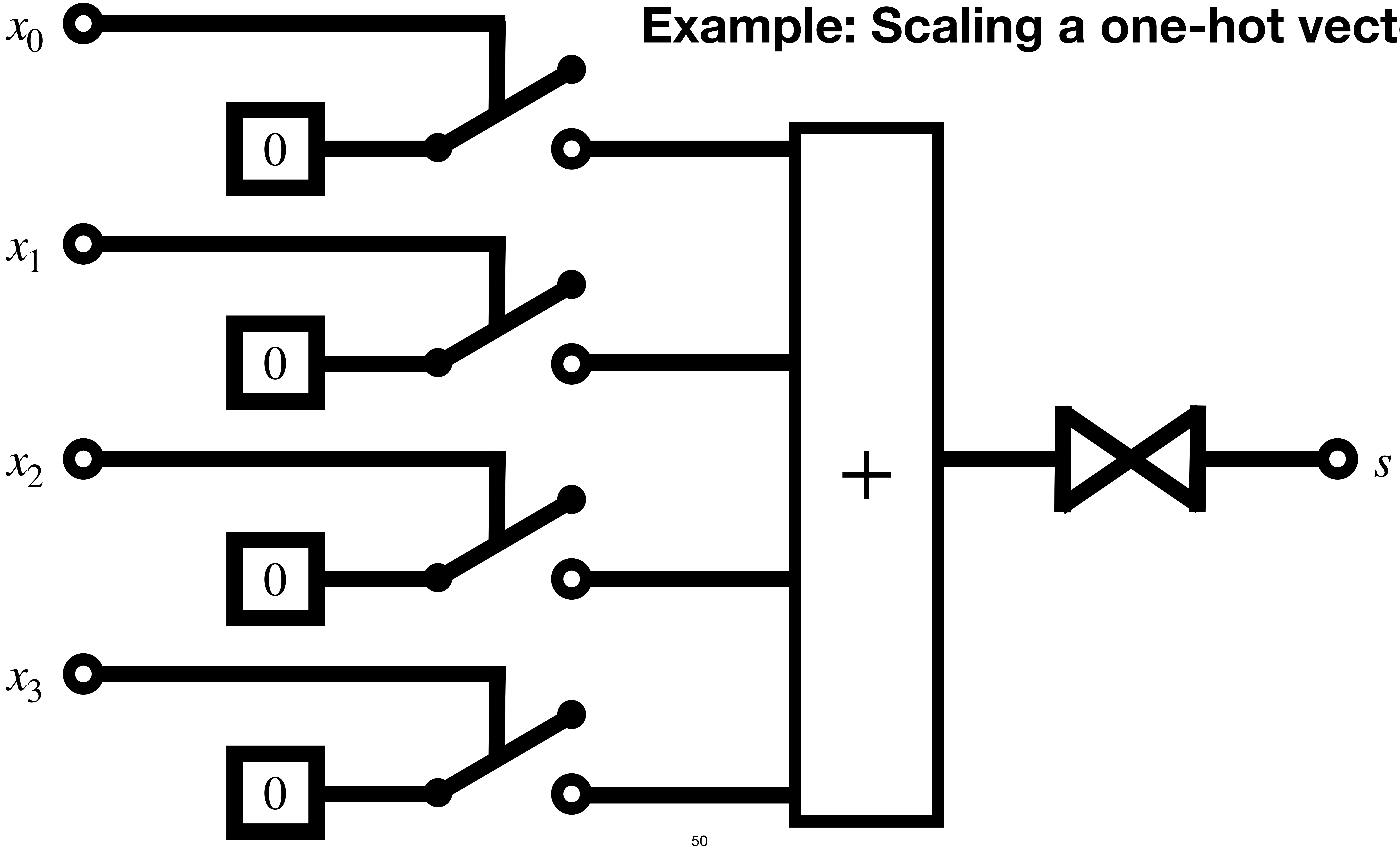
Switch Systems





See paper

Example: Scaling a one-hot vector



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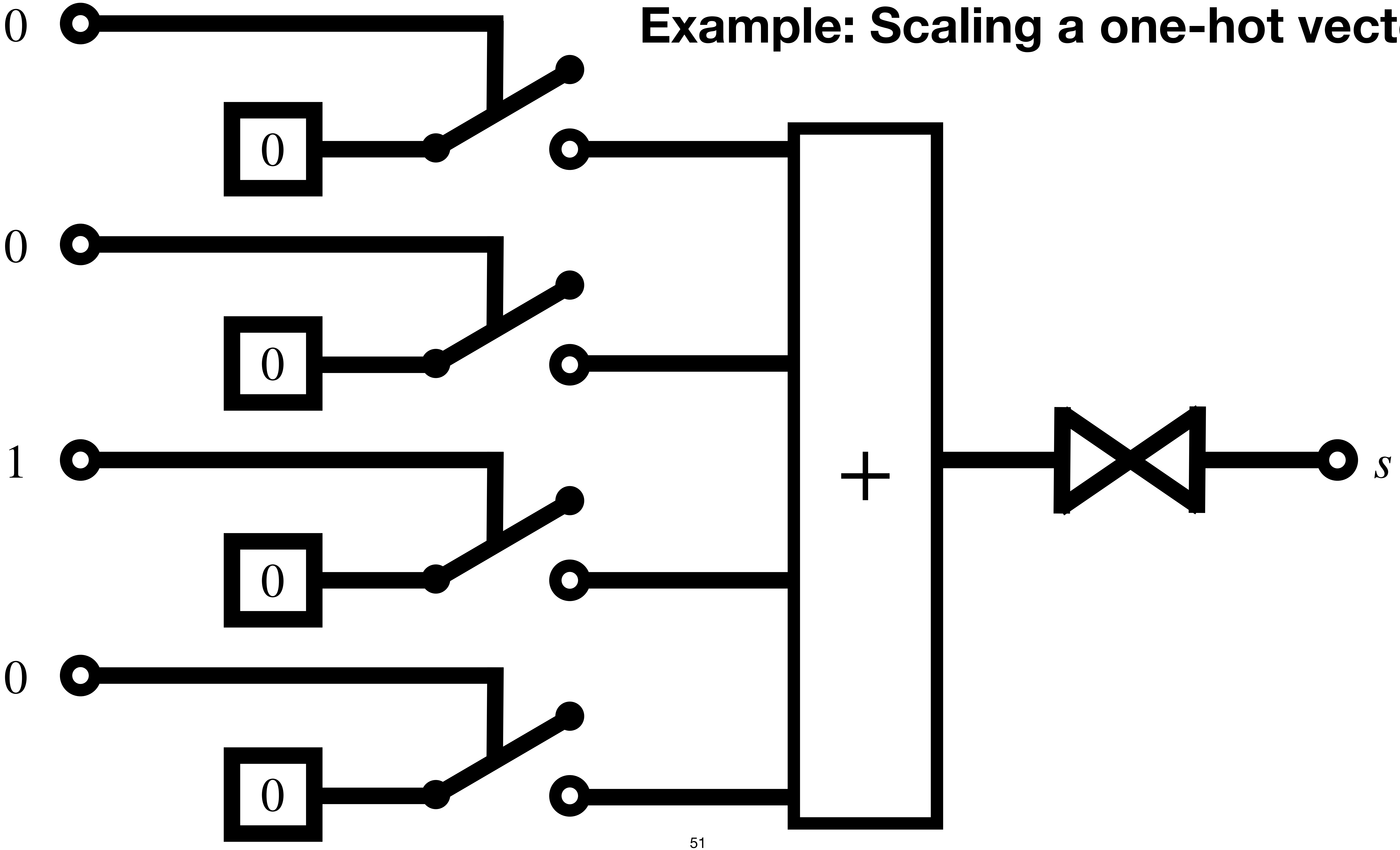

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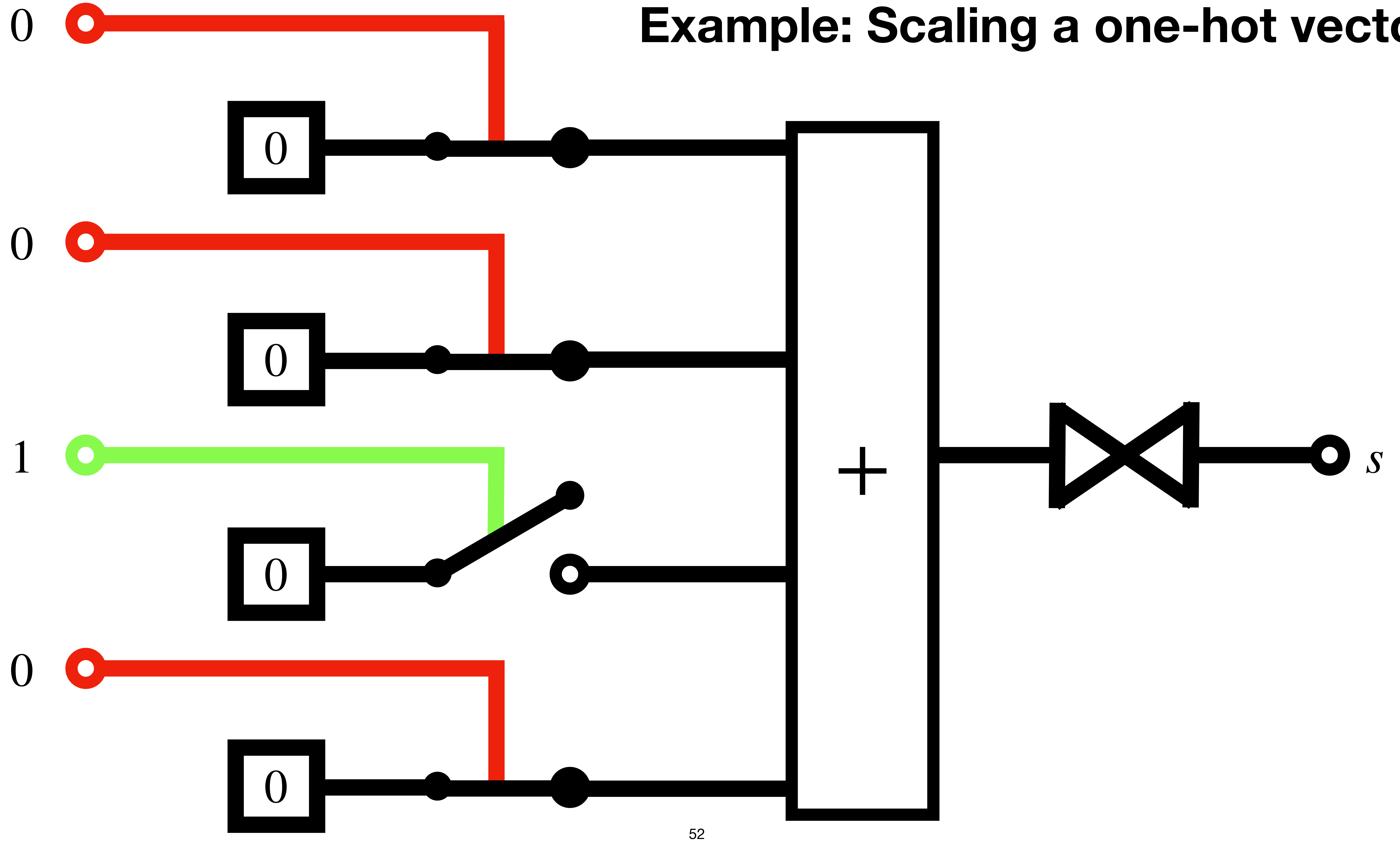
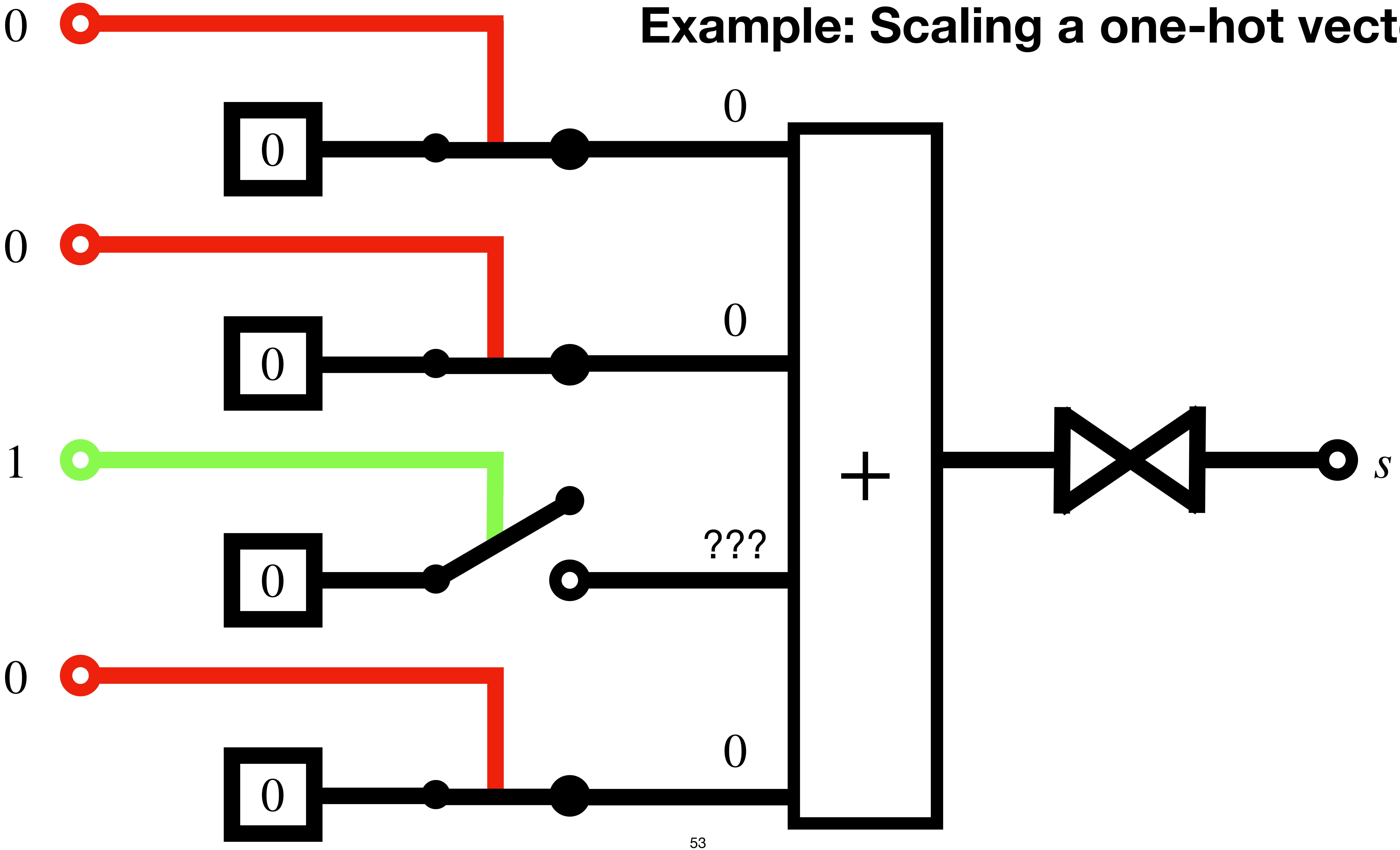


Example: Scaling a one-hot vector



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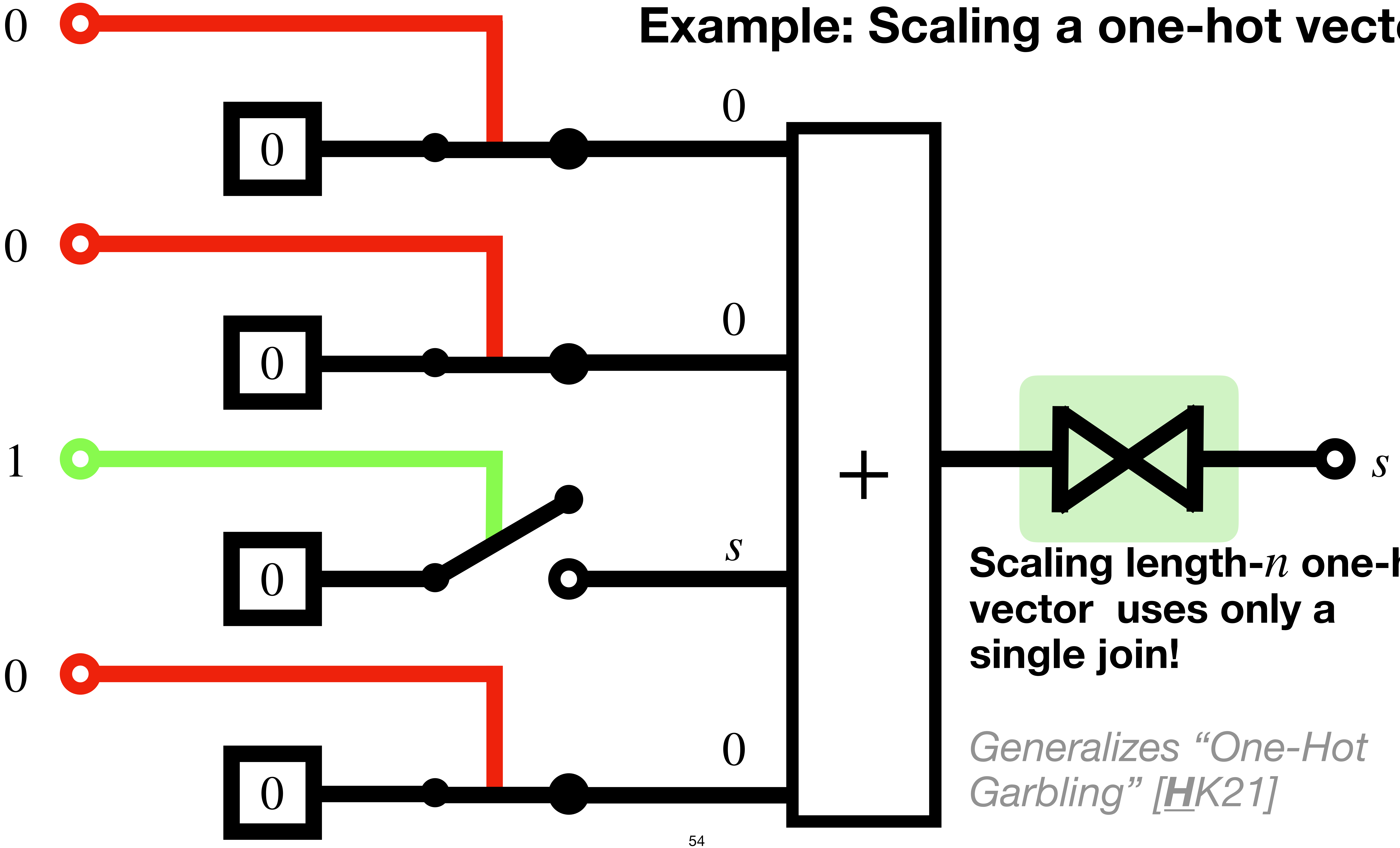

Scaling length-n one-hot vector uses only a single join!

Generalizes "One-Hot Garbling" [HK21]

How to Multiply Short Integers

Let $x, y \in \mathbb{Z}_{2^k}$

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Suppose we have one-hot encoding of x and a single wire that encodes y

$$\begin{array}{r c} & \\ & \\ x\text{-th position} & \\ & \\ & \left[\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{matrix}\right] \end{array}$$

For sake of example, let $x = 4$

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$$\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix} \xrightarrow{\text{scale by } y} \begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix}^\intercal \cdot \begin{bmatrix}0\\0\\0\\0\\y\\0\\0\\0\end{bmatrix}$$

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Via ADD gates

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Single wire that encodes $x \cdot y$

Via ADD gates

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Single wire that encodes $x \cdot y$

Uses only a single join gate $\Rightarrow$ garbled encoding of size $k \cdot \lambda$ bits

How to Multiply Short Integers

$$\mathrm{hot}(x), y \mapsto x \cdot y$$

To multiply again, we need to convert single wire to one-hot encoding

This is achieved by another (complex) switch system, somewhat similar to one-hot scaling

See paper

This Work

First symmetric-key garbling scheme for arithmetic circuits that achieves linear-cost multiplication

switch systems generalize much of the garbled circuit literature

Opens possibility of new custom arithmetic garbled "gates"

See Paper For

More details on (oblivious) switch systems

How to garble switch systems

Switch system that converts between one-hot representation and arithmetic representation

Long integer handling, based on Chinese Remainder Theorem