ASYMPTOTICS AND IMPROVEMENTS OF SIEVING FOR CODES

May 2024, Eurocrypt 2024

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based on joint work with: Léo Ducas (CWI, LEI),
Andre Esser (TII), and Elena Kirshanova (TII)

Motivation: Sieving is a well-known and widely used technique for attacking decoding problems in **lattice-based** cryptography.

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Goal: Adapt sieving techniques to the **code-based setting** and make them competitive with state-of-the-art algorithms.

SIEVING FOR CODES

Problem definition

Decoding problem, DP(n, k, w)

Given an $[n, k]$ binary linear code $\mathcal{C}$ and a weight w, find a codeword of Hamming weight[1] w.

[1] **Hamming weight,** $|\cdot|$: The number of non-zero entries of a vector.

Information Set Decoding (ISD)

Information set decoding algorithms are the best known generic[2] attacks for the decoding problem.

[2] For certain parameter ranges, statistical decoding performs better. (Kevin Carrier et al. Statistical Decoding 2.0: Reducing Decoding to LPN. Cryptology ePrint Archive, Paper 2022/1000. 2022)

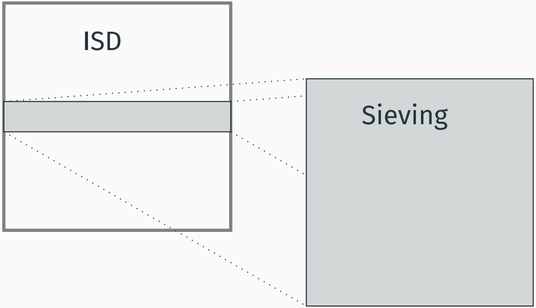
Information Set Decoding (ISD)

Information set decoding algorithms are the best known generic attacks for the decoding problem.

Recently, a new ISD algorithm based using **sieving** as a **subroutine** was proposed in [GJN23][2].

[2] Qian Guo, Thomas Johansson, and Vu Nguyen. A New Sieving-Style Information-Set Decoding Algorithm. 2023.

Inspiration Part I: [GJN23] approach[3]



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Provides slight improvements in asymptotic runtime over the baseline algorithm due to Prange[4].

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Inspiration Part I: [GJN23] approach[5]

Provides slight improvements in asymptotic runtime over the baseline algorithm due to Prange[4].

Gives very good time-memory trade-offs.

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Inspiration Part II: Near-Neighbor Search

In the lattice-based setting, **sieving** was successfully combined with **near-neighbor search**[6][7][8].

[6]Thijs Laarhoven. Sieving for shortest vectors in lattices using angular locality-sensitive hashing. Cryptology ePrint Archive, Paper 2014/744. 2014.

[7]Anja Becker, Nicolas Gama, and Antoine Joux. Speeding-up lattice sieving without increasing the memory, using sub-quadratic nearest neighbor search. Cryptology ePrint Archive, Paper 2015/522. 2015.

[8]Anja Becker, Léo Ducas, et al. New directions in nearest neighbor searching with applications to lattice sieving. 2015.

Inspiration Part II: Near-Neighbor Search

In the lattice-based setting, **sieving** was successfully combined with **near-neighbor search**.

[MO15][6], [BM18][7], etc. and Kévin Carrier's thesis[8] explored **near-neighbor search** in the coding setting.

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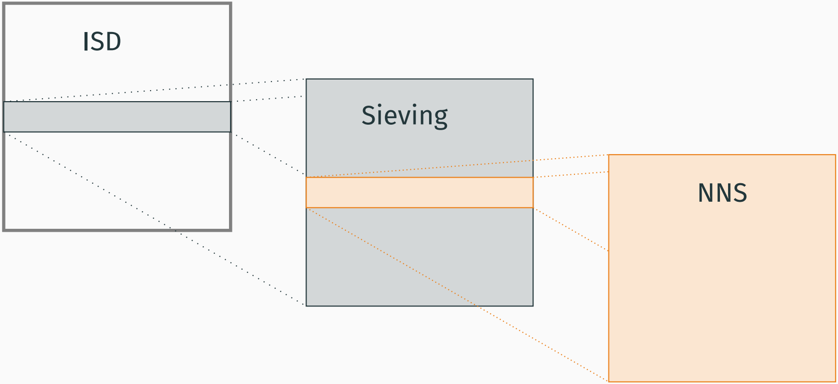
[6] Alexander May and Ilya Ozerov. "On Computing Nearest Neighbors with Applications to Decoding of Binary Linear Codes". In: 2015.

[7] Leif Both and Alexander May. "Decoding Linear Codes with High Error Rate and Its Impact for LPN Security". In: ed. by Tanja Lange and Rainer Steinwandt. 2018.

[8] Kévin Carrier. "Recherche de Presque-Collisions pour le Décodage et la Reconnaissance de Codes Correcteurs. (Near-collisions finding problem for decoding and recognition of error correcting codes)". PhD thesis. 2020.

Our generalization



NEAR-NEIGHBOR SEARCH

DEFINITIONS AND NOTATION

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where $+$ denotes bitwise XOR.

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where $\wedge$ denotes bitwise AND.

Near-Neighbor Search Problem

Near-Neighbor Search (NNS), NNS($\mathcal{L}$, p)

Given a target weight p and an input list $\mathcal{L} \subseteq \mathcal{S}_p^m$, find all pairs $(\mathbf{x}, \mathbf{y}) \in \mathcal{L} \times \mathcal{L}$ satisfying $|\mathbf{x} + \mathbf{y}| = p$.

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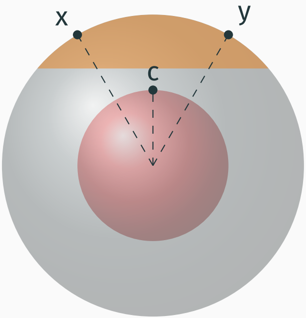
$\rightarrow$ Brute-force search runtime: $\tilde{\mathcal{O}}(|\mathcal{L}|^2)$.

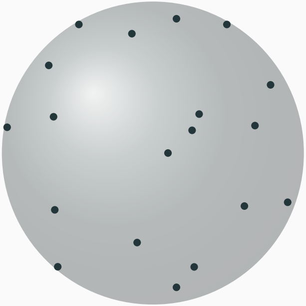
Localized search

For a suitable choice of α, if $|\mathbf{x} \wedge \mathbf{c}| = |\mathbf{y} \wedge \mathbf{c}| = \alpha$, **x** and **y** are likely **near neighbors**.



Locality-Sensitive Filtering (LSF)[9]

Given $\mathcal{L} \subseteq \mathcal{S}_p^m$



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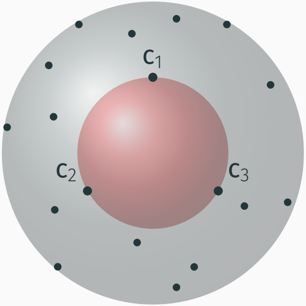

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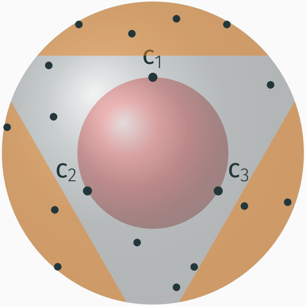
Given $\mathcal{L} \subseteq \mathcal{S}_p^m$, set of centers $\mathcal{C}_f$ and parameter α, perform

- **bucketing phase**: for each element $\mathbf{x} \in \mathcal{L}$, if $|\mathbf{x} \wedge \mathbf{c}| = \alpha$, assign $\mathbf{x}$ to a bucket $\mathcal{B}_\alpha(\mathbf{c})$,



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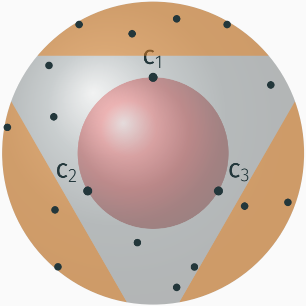
- **checking phase**: for each $\mathbf{c} \in \mathcal{C}_f$, check which $(\mathbf{x}, \mathbf{y}) \in \mathcal{B}_\alpha(\mathbf{c}) \times \mathcal{B}_\alpha(\mathbf{c})$ are near neighbors and add them to the output list.



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Algorithm NNS using locality-sensitive filtering

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for x ∈ L do
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        Add x to B_α(c).
```

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for c ∈ C_f do
    for (x, y) ∈ B_α(c) × B_α(c) do
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return L'
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Basic idea: For any $(\mathbf{x}, \mathbf{y}) \in \mathcal{S}_p^m \times \mathcal{S}_p^m$ satisfying $|\mathbf{x} \wedge \mathbf{y}| = p/2$, there exists a unique $\mathbf{c} \in \mathcal{S}_{p/2}^m$ such that $|\mathbf{x} \wedge \mathbf{c}| = |\mathbf{y} \wedge \mathbf{c}| = p/2$.

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Parameters:

$$\mathcal{C}_f = \mathcal{S}_{p/2}^m, \quad \alpha = p/2.$$

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$\to$ FINDVALIDCENTERS subroutine needs to perform efficient decoding.

Random Product Codes approach (RPC)

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Parameters:

$$\mathcal{C}_{\mathcal{H}}^{(i)} \subseteq \mathcal{S}_{v/t}^{m/t}, \quad \mathcal{C}_{\mathcal{H}} = \mathcal{C}_{\mathcal{H}}^{(1)} \times \cdots \times \mathcal{C}_{\mathcal{H}}^{(t)}, \quad \alpha, v \leq p/2 \text{ - to be optimized,}$$

where t is chosen to guarantee random behavior of the $\mathcal{C}_{\mathcal{H}}$ and an efficient decodability.

Memory optimal versions (HASH and RPC memo-opt)

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We interleave the bucketing and the checking phase.

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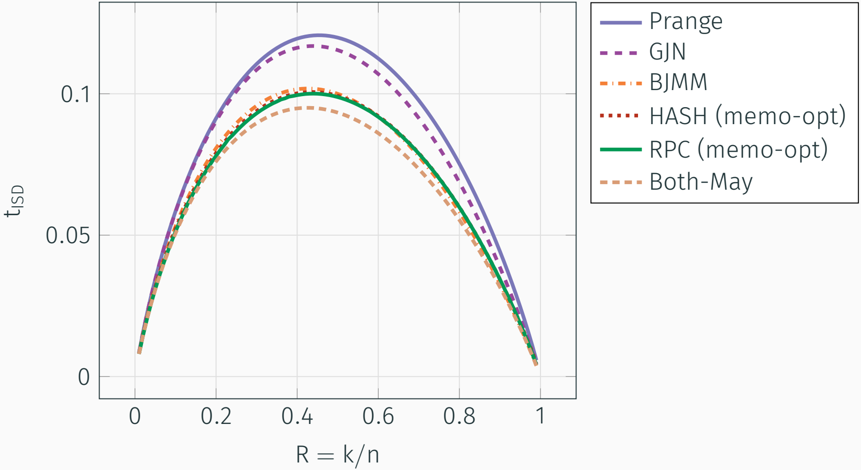
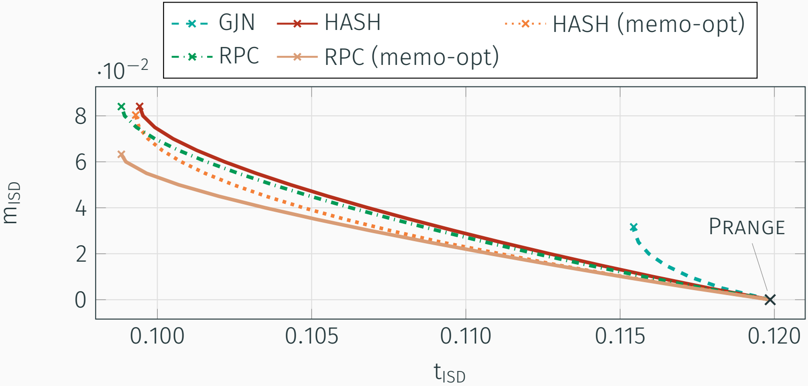
Memory optimal approach

The initial set of filters contains $|\mathcal{C}_f|/d$ centers but we repeat the algorithm d times.

COMPARISONS AND CONCLUSIONS



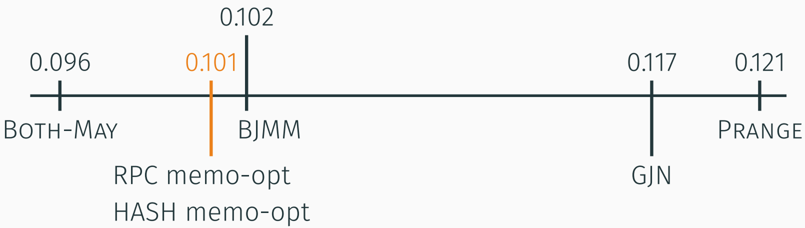
Asymptotic runtime exponent for different ISD variants for the unique-decoding regime.



Time-memory trade-off curves of different SievingISD instantiations.

Summa summarum

We introduce **sieving-based ISD** algorithms whose asymptotic runtime and memory are close to those of the state-of-the-art.



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eprint GitHub repo

Thank you for your attention!