Isogeny Problems with Level Structure

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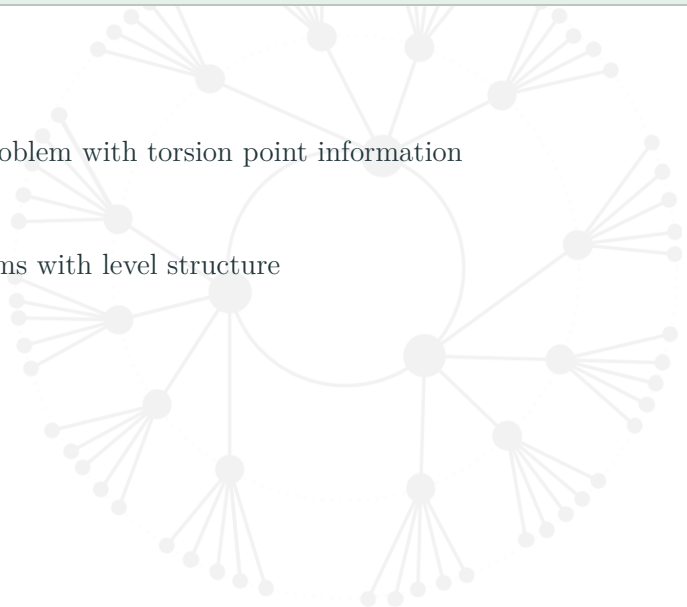
Outline



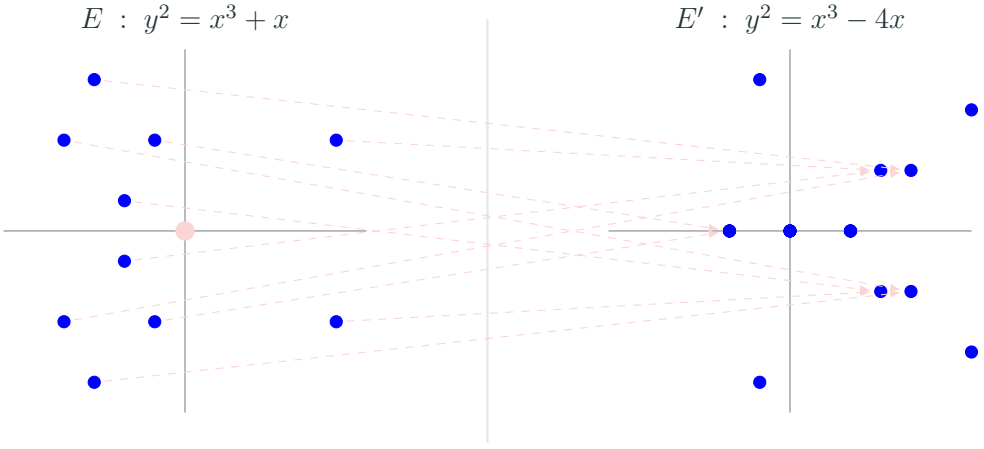
The isogeny problem with torsion point information

Isogeny problems with level structure

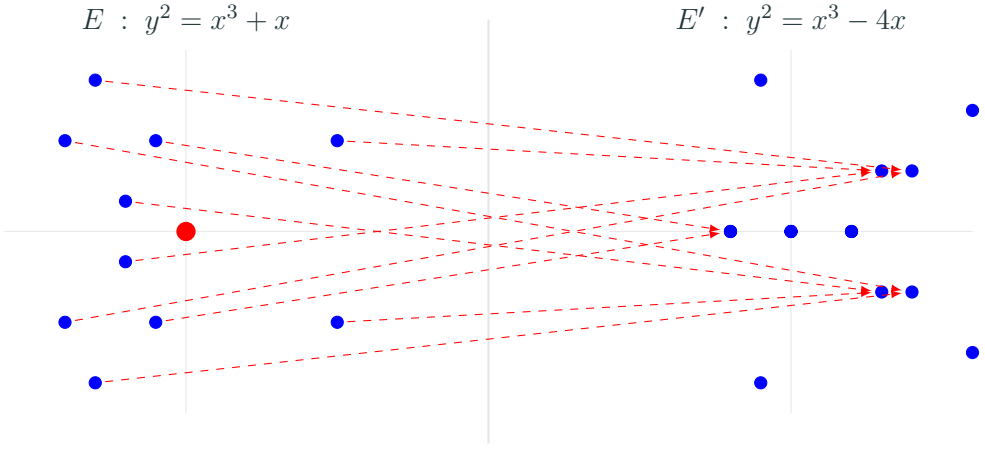
A reduction

The isogeny problem with torsion point information

Isogenies



Isogenies



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.

Pure isogeny problem

Pure isogeny problem: *given two (isogenous) elliptic curves E' and E', compute an isogeny $\phi : E \to E'$.*

The genesis of torsion point information

First isogeny-based key exchange (CRS[1]): class group actions on ordinary curves.

Isogeny group actions are subject to quantum sub-exponential attacks.

Jao and De Feo (SIDH 2011): use supersingular isogenies.

Caveat: supersingular isogenies do not commute in general.

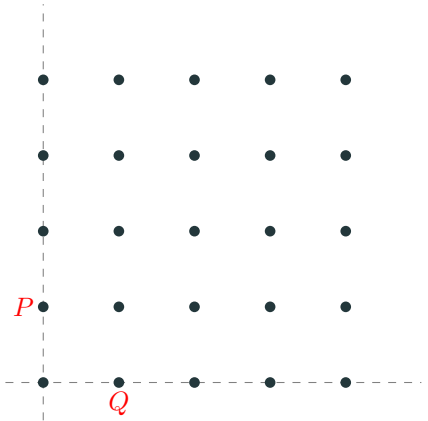
Solution: reveal the images of some torsion points.

[1] Couveignes 1997, Rostovtsev and Stolbunov 2006.

Torsion

Over an algebraically closed field, for any N coprime to the characteristic:

$$E[N] \simeq \mathbf{Z}/N \times \mathbf{Z}/N$$

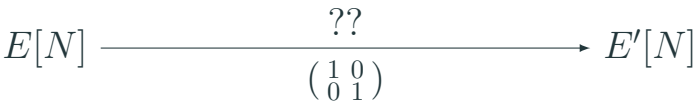


Isogeny problem with torsion point information (SIDH)

$$E, P, Q \xrightarrow{\quad ?? \quad} E', P', Q'$$

$E[N] = \langle P, Q \rangle$, $P' := \phi(P), Q' := \phi(Q)$

Isogeny problem with torsion point information (SIDH)

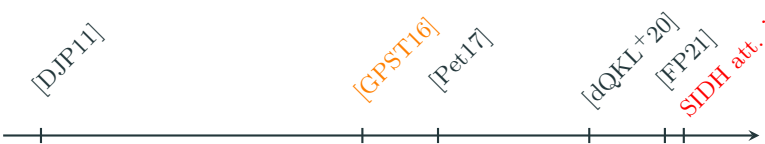


Fixing $\langle P, Q \rangle = E[N]$ and $\langle \phi(P), \phi(Q) \rangle = E'[N]$

Torsion point attacks

Non exhaustive list: BdQL+ 2019, ...

Torsion point attacks



Torsion point information!!

SIDH att. '22: [CD22, MMP+23, Rob22]

Torsion point attacks

Torsion point information!!

SIDH att. '22: [CD22, MMP+23, Rob22]

SIDH attacks (Robert's version)

Let E, E' be elliptic curves, let $\phi : E \to E'$ be an isogeny of degree d and let N be a smooth integer coprime to d such that $N^2 > d$.

There exists a polynomial time algorithm that, given E, E', d, N, a basis (P, Q) of $E[N]$ and its image $(\phi(P), \phi(Q))$ under ϕ, computes ϕ.

Isogeny problems with level structure

Γ-SIDH problems (isogeny problems with level structure)

Level structure = basis of $E[N]$ up to linear transformations $\Gamma \subset \mathsf{GL}_2(\mathbb{Z}/N)$

$$E[N] \xrightarrow[\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)]{??} E'[N]$$

Fixing $\langle P, Q\rangle = E[N]$ and $\langle \phi(P), \phi(Q)\rangle = E'[N]$

Γ-SIDH problems (isogeny problems with level structure)

Level structure = basis of $E[N]$ up to linear transformations $\Gamma \subset \mathsf{GL}_2(\mathbb{Z}/N)$

$$E[N] \xrightarrow[\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) \cdot \Gamma]{??} E'[N]$$

Fixing $\langle P, Q\rangle = E[N]$ and $\langle \phi(P), \phi(Q)\rangle = E'[N]$

Some examples of level structures

$\Gamma = \{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\}$: A basis (P, Q) of $E[N]$, plain SIDH.

$\Gamma = \{\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}\}$: Image of a basis (P, Q) of $E[N]$ up to scalar; M-SIDH.

$\Gamma = \{\begin{pmatrix} * & \\ & * \end{pmatrix}\}$: Images of two cyclic subgroups $\langle P \rangle$ and $\langle Q \rangle$ of order N; isogeny group actions (CSIDH et al., SCALLOP et al., ...), binSIDH, terSIDH, ...

$\Gamma_1 = \{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\}$: Image of a point P of order N.

$\Gamma_0 = \{\begin{pmatrix} * & * \\ & * \end{pmatrix}\}$: Images of a cyclic group $\langle P \rangle$ of order N; SIDH signatures.

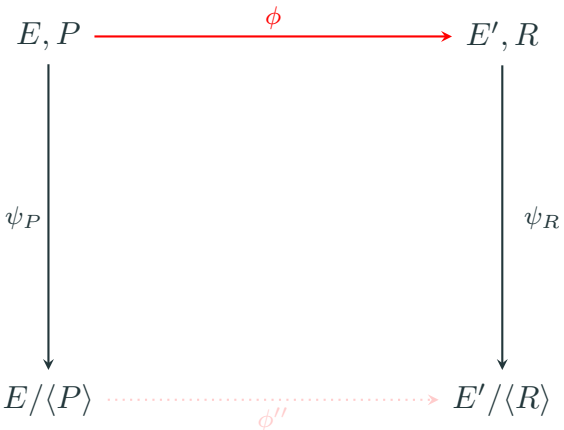
A reduction

A reduction

The Γ_1-SIDH problem (of level N)

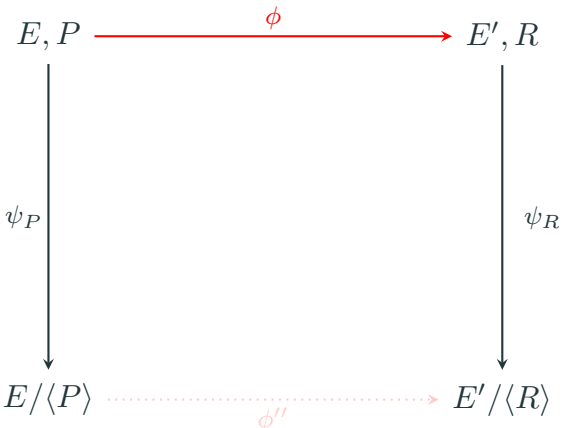
$$E, P \xrightarrow{\phi} E', R = \phi(P)$$

A reduction



A reduction

Now, let us assume that $N = \ell^2$ is a square.



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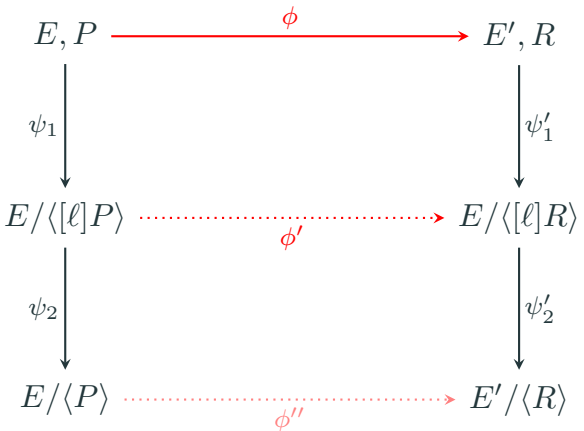

A reduction

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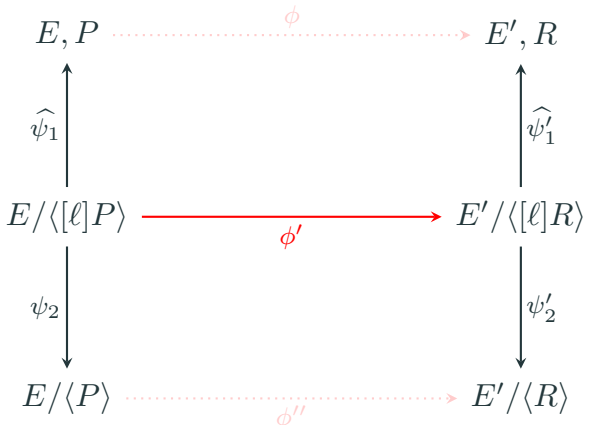
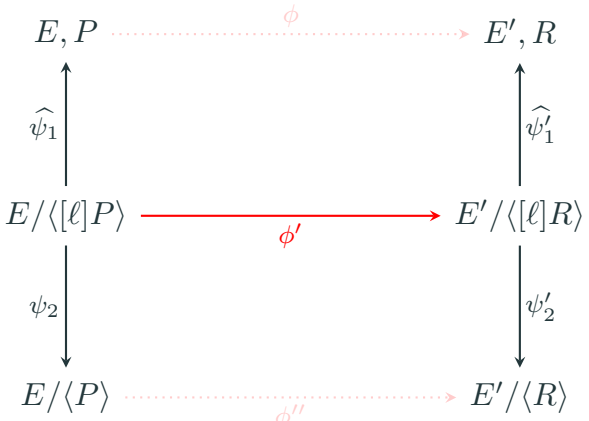


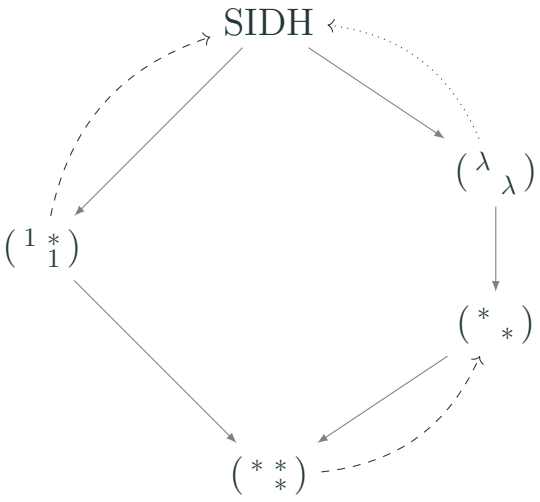
You recover the action of ϕ' on the ℓ-torsion using pairings and DLPs.

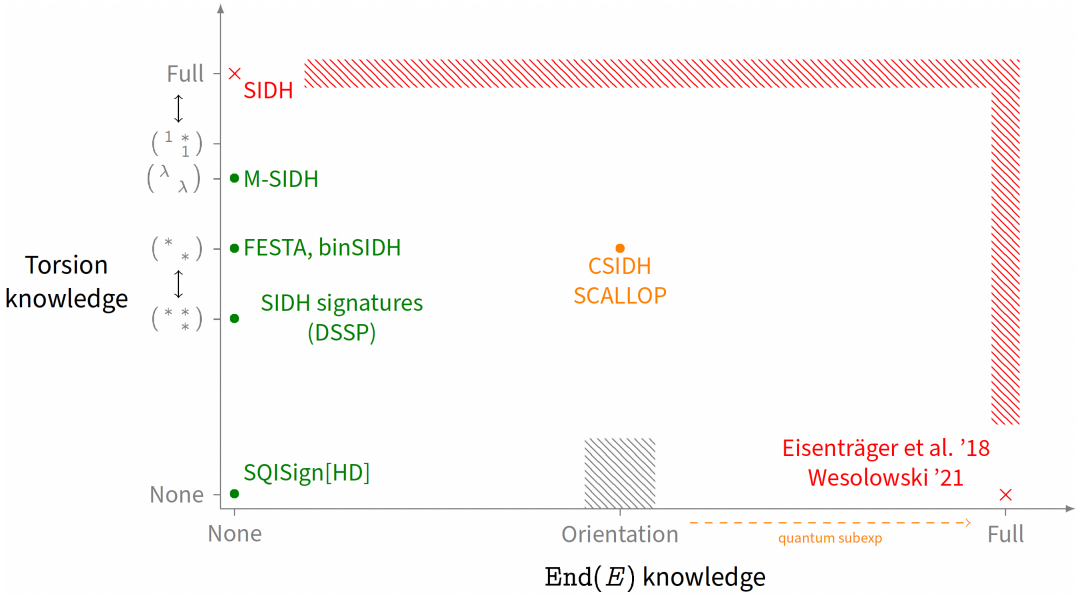
A reduction

The (plain) SIDH problem (of level $\ell = \sqrt{N}$)



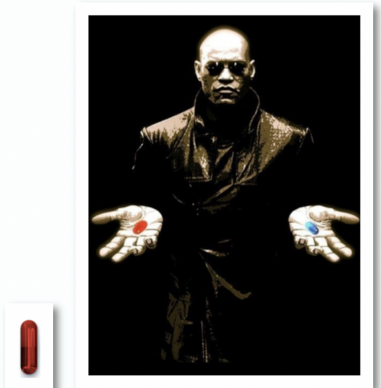
Some reductions




Full
$\begin{pmatrix} 1 & * \\ 1 & \end{pmatrix}$
$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$
$\begin{pmatrix} * & \\ & * \end{pmatrix}$
$\begin{pmatrix} * & * \\ & * \end{pmatrix}$
None
Torsion knowledge
SIDH
M-SIDH
FESTA, binSIDH
SIDH signatures (DSSP)
SQISign[HD]
CSIDH
SCALLOP
Eisenträger et al. ’18
Wesolowski ’21
None
Orientation
quantum subexp
Full
End(E) knowledge

Let's stop here and have lunch.

But before that, you have to choose your pill!

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