Cryptanalysis of rank-2 module-LIP in Totally Real Number Fields

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Hawk (Ducas, Postlethwaite, Pulles, van Woerden 2022)[1]

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2. based on module-LIP over cyclotomic fields
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Does not break Hawk!

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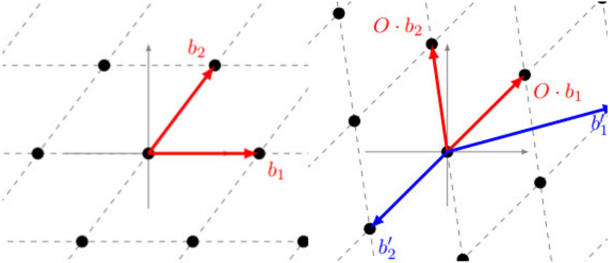
The module-Lattice Isomorphism Problem

- LIP: Find an isometry (distance preserving map) sending $\mathcal{L}_1$ on $\mathcal{L}_2$.



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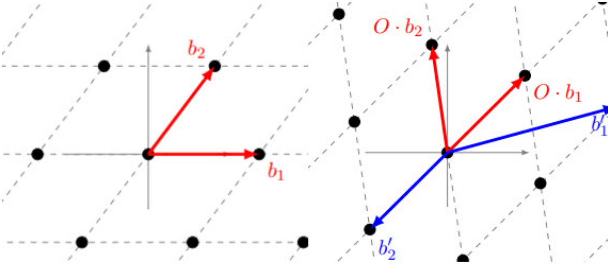

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State of the art: need to compute many short vectors in $\mathcal{L}_1$ and $\mathcal{L}_2$ (SVP, hard problem)

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- **module lattices** are finitely generated modules over $\mathcal{O}_K$ (K a number field).

Examples. $K = \mathbb{Q}[X]/(X^{2^k} + 1)$ and $\mathcal{O}_K = \mathbb{Z}[X]/(X^{2^k} + 1)$ (or $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$).

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1. fractional ideals of K (rank one)
2. $\mathcal{O}_K \oplus \mathcal{O}_K$ (rank two)
3. in general: $M = \mathfrak{a}_1 v_1 \oplus \cdots \oplus \mathfrak{a}_\ell v_\ell$ (rank ℓ, $v_i \in K^\ell$, $\mathfrak{a}_i \subset K$)

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State of the art: embed module lattices to lattices $\subset \mathbb{R}^{d\ell}$ and solve LIP instance.

The module-Lattice Isomorphism Problem

Motivating example. K any number field and $M = \mathcal{O}_K \oplus \mathcal{O}_K$ (as in Hawk).

Notation: $X^* := \overline{X}^T$, for any $X \in M_2(K)$.

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Move to **quadratic forms**:

$$B \longmapsto G = B^* B \;\; ; \;\; B' \longmapsto G' = B'^* B', \quad \text{Gram matrix / Humbert form.}$$

$$B' = OBU \quad \Longrightarrow \quad G' = U^* G U, \quad \textbf{congruent} \text{ to } G.$$

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Taking $B = G = I_2$, module-LIP with parameter K and I_2 is

module-LIP$_K^{I_2}$

Input: G' Gram matrix congruent to I_2
Goal: Compute **all** $U \in \mathrm{GL}_2(\mathcal{O}_K)$ s.t. $G' = U^* I_2 U = U^* U$.

Link with Hawk

Hawk : $K = \mathbb{Q}(\zeta_{2^k})$ **cyclotomic** number field (**not** totally real)
$U \in \mathrm{GL}_2(\mathcal{O}_K)$ (secret basis of $\mathcal{O}_K \oplus \mathcal{O}_K$)
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- Recovering U from G is a module-LIP$_K^{I_2}$ instance.
- Any solution $V^* V = G$ is a **key recovering** (up to automorphism).

The attack over totally real fields

Attack: Main idea

From now on K is **totally real** (*e.g.*, $K = \mathbb{Q}(\zeta + \zeta^{-1})$) and $U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K)$

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$$G = U^* U = \begin{pmatrix} a\bar{a} + b\bar{b} & \star \\ \star & c\bar{c} + d\bar{d} \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & \star \\ \star & c^2 + d^2 \end{pmatrix}$$

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Main idea: Solve relative norm equations to reconstruct U.

Attack: Solving norm equations

- Howgrave-Graham, Szydlo, "A Method to Solve Cyclotomic Norm Equations $f \star \bar{f}$"

NormEquation

Input: $q \in \mathcal{O}_K$, prime factorization of $|N_{K/\mathbb{Q}}(q)| \in \mathbb{N}$.
Output: all pairs $(x, y) \in \mathcal{O}_K \times \mathcal{O}_K$ such that $N_{L/K}(x + iy) = x^2 + y^2 = q$.

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It runs in time

$$\text{poly}\left(deg(K), (\log |N_{K/\mathbb{Q}}(q)|)^{\mathbf{r}}\right),$$

where **r** is the number of distinct prime factors of $q \cdot \mathcal{O}_K$.

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$\Rightarrow$ Get norm equations easy to solve.

Attack: Randomization step

GaussianGram

Input : G matrix, $s > 0$ sampling parameter.
Output : $(u, v) \in \mathcal{O}_K \oplus \mathcal{O}_K$ follows a discrete Gaussian distribution.

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Heuristic for the probability of success.
Numerical experiments + theoretical results (distribution of prime ideals, involves ρ_K, residue of ζ_K at 1).

Attack: Statement

Solving module-LIP for $\mathcal{O}_K \oplus \mathcal{O}_K$.

Suppose $K = \mathbb{Q}(\zeta_{2^k} + \zeta_{2^k}^{-1})$ and G a Gram matrix.
$\exists$ heuristic algorithm solving module-LIP$_K^{l_2}$ on input G in expected time

$$poly(\rho_K, deg(K), size(G)),$$

ρ_K residue of ζ_K at 1 (small in our experiments).

Numerical experiments

Full attack here: https://gitlab.inria.fr/capsule/code-for-module-lip

$(m, 2d)$	$(64, 32)$	$(128, 64)$	$(256, 128)$
Time	2	25	850

$(m, 2d)$	$(228, 72)$	$(276, 88)$	$(260, 96)$	$(232, 112)$	$(340, 128)$	$(296, 144)$
Time (s)	74	195	434	652	2980	4205

Table: Times in seconds for attacks over various maximal totally real subfields K of cyclotomic fields with conductors $m = 4k$, averaged over 5 instances. The degree d of K is $\varphi(m)/2$, and the lattices involved have dimension $2d$. The upper table are powers-of-two. Experiments performed on a MacBook Pro (Apple M2), with Sagemath 10.2 and Pari/GP 2.15.5.

More generally

- module-LIP defined for any number field, any module lattice $M \subset K^\ell$.

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- In general, can't hope for polynomial time complexity. Depends on the "Gram ideal" $\mathcal{G}(M) = \langle \, ||v||^2 \mid v \in M \rangle$. *e.g.*, $\mathcal{G}(\mathcal{O}_K \oplus \mathcal{O}_K) = \mathcal{O}_K$.

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Parameters: K totally real, $M \subset K^2$, with (pseudo-)basis B and $G = B^* B$.
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Input: G' (pseudo-)Gram matrix congruent to G.
$\exists$ heuristic algorithm finding all conguence matrices in expected time

$$\left(poly(\rho_K, \log \Delta_K, size(\mathbf{G'})) \right)^{\mathbf{r}+1} + T_{factor}(N_{K/\mathbb{Q}}(\mathcal{G}(M))),$$

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• Under some heuristic for the randomization, polynomial time (in many cases) algorithm solving module-LIP.

Open questions. • For modules with rank $\ell > 2$?
• Rank 2 over K cyclotomic ?

Thanks for your attention!

Full article here!