# Pseudorandom Isometries

Prabhanjan Ananth (UCSB),
Aditya Gulati (UCSB),
Fatih Kaleoglu (UCSB),
Yao-Ting Lin (UCSB)

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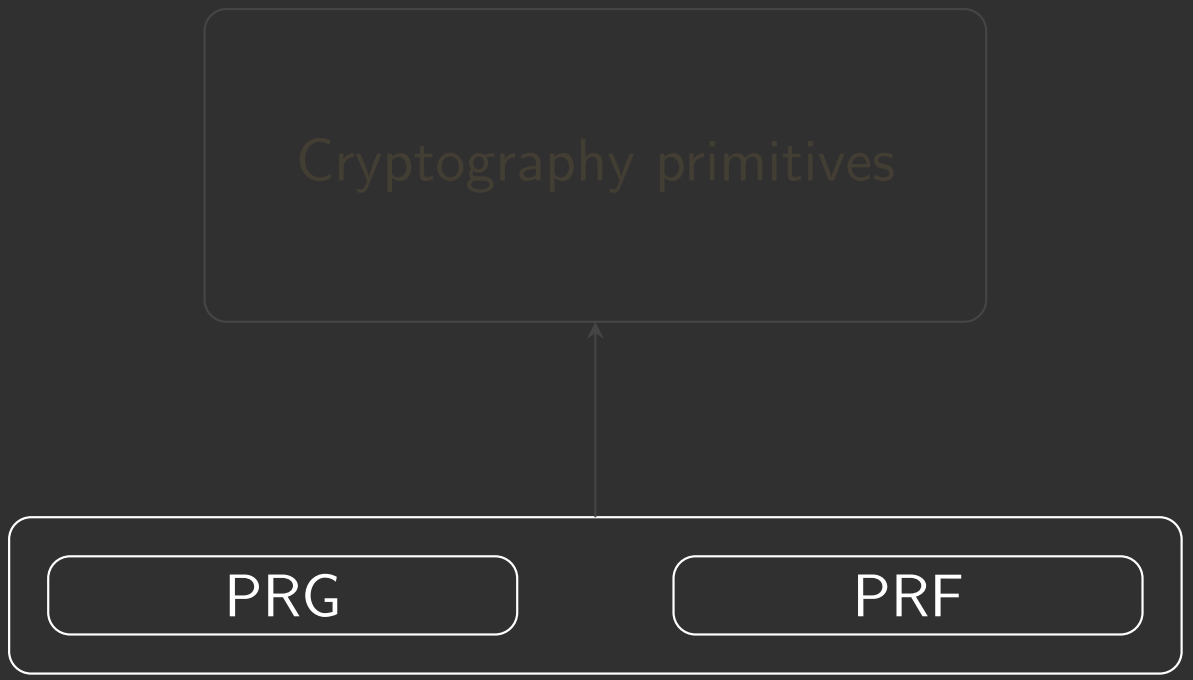
# Pseudorandomness

## Classical




## Quantum ?

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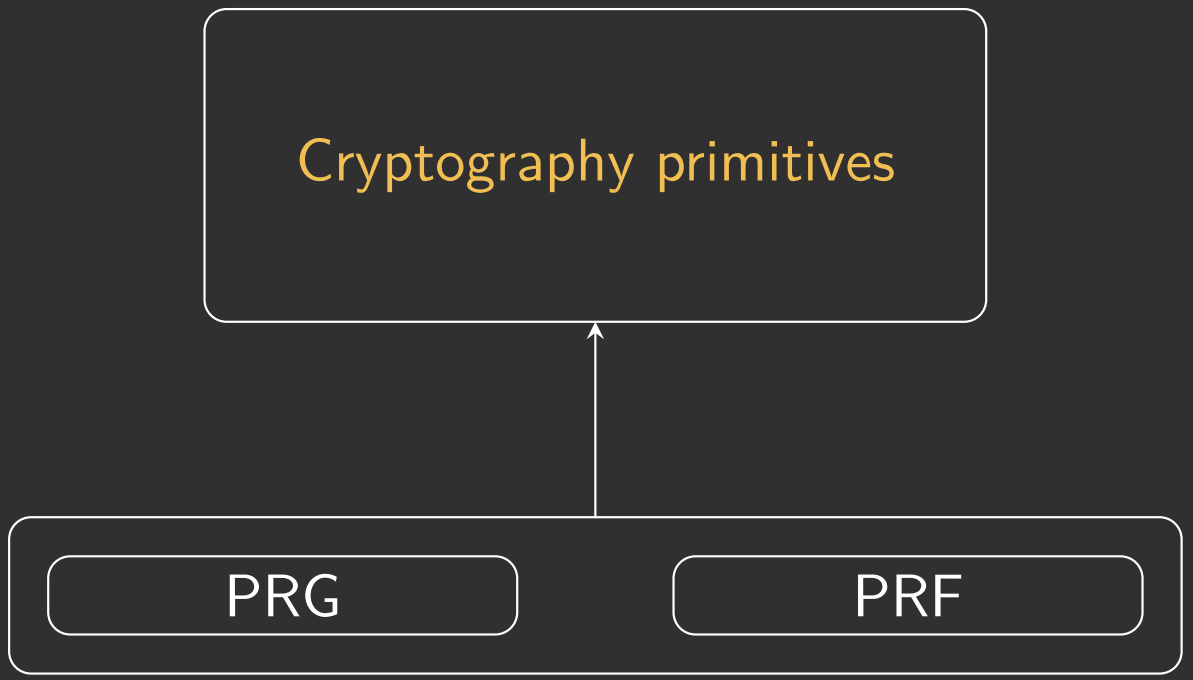


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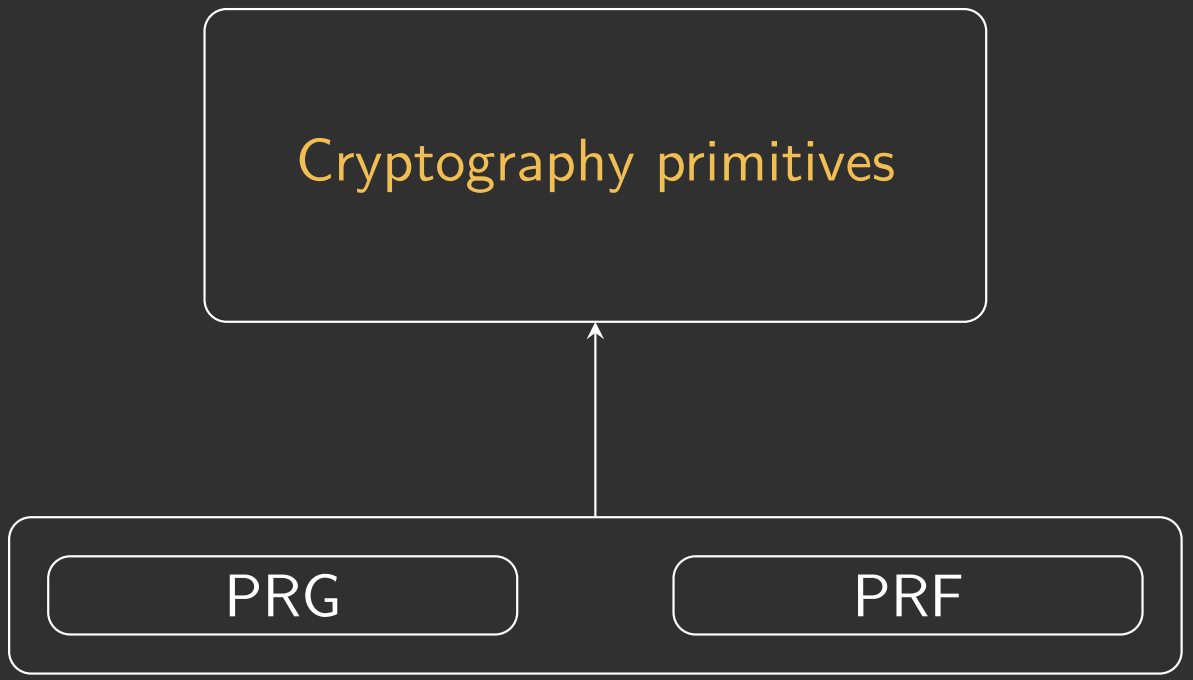
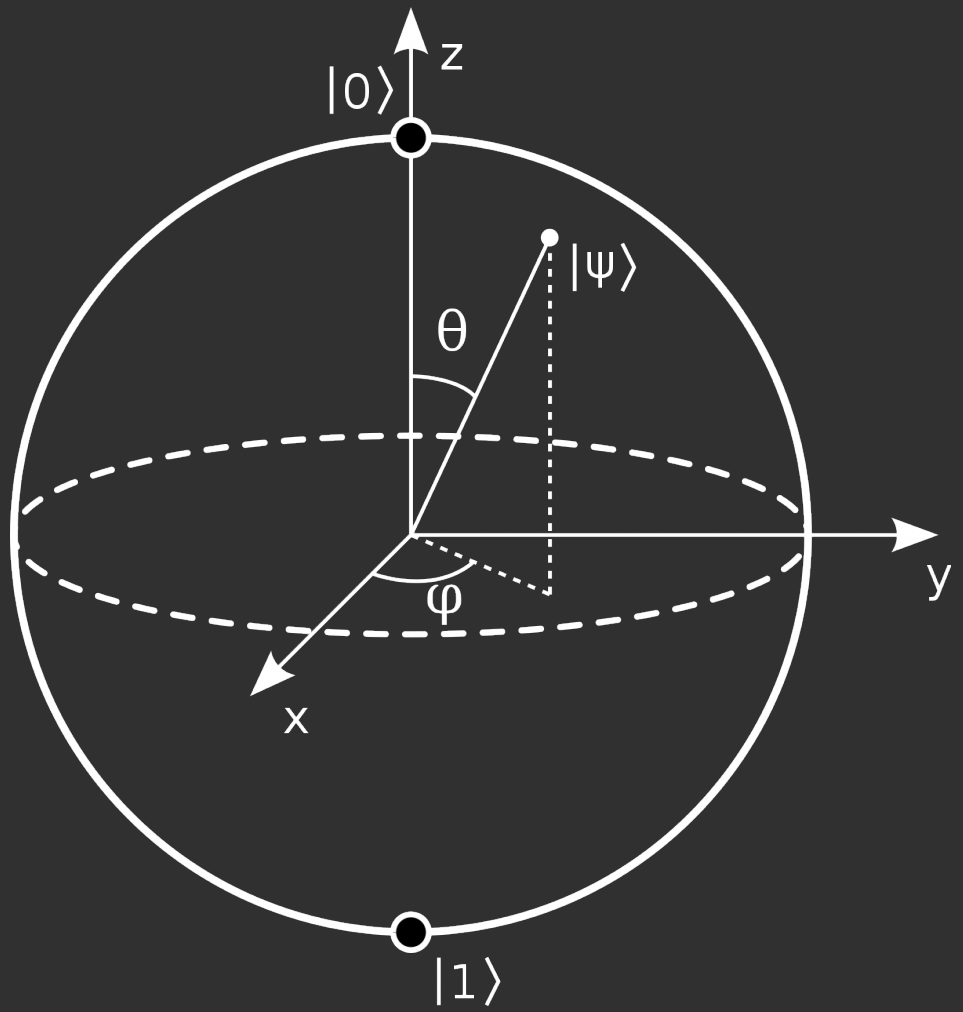



## Quantum ?

# Haar Random State




Haar random state looks like picking a random point on the bloch sphere.

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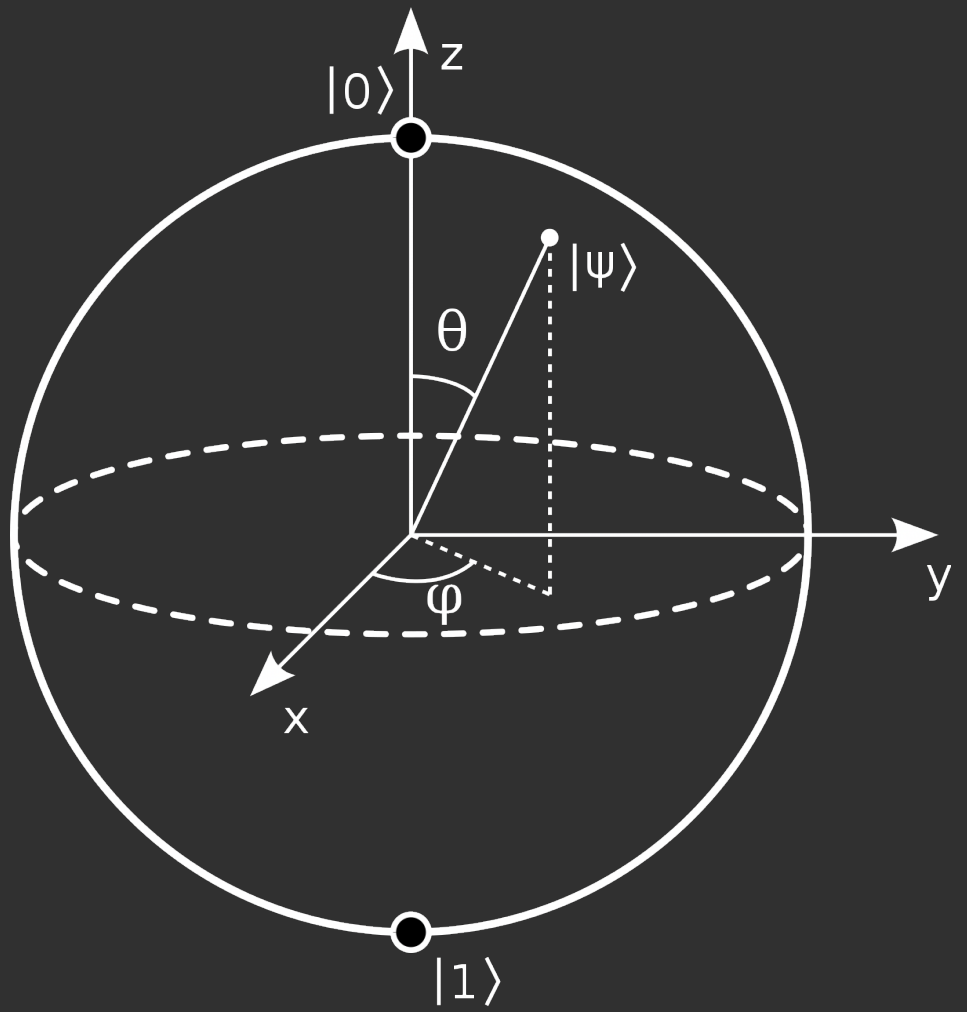


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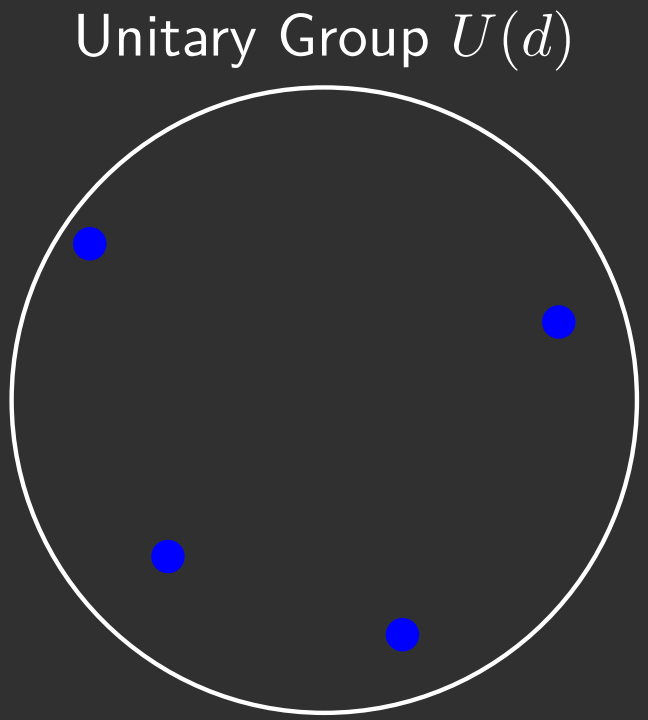
# Haar Random Unitaries

## Haar Random Unitaries



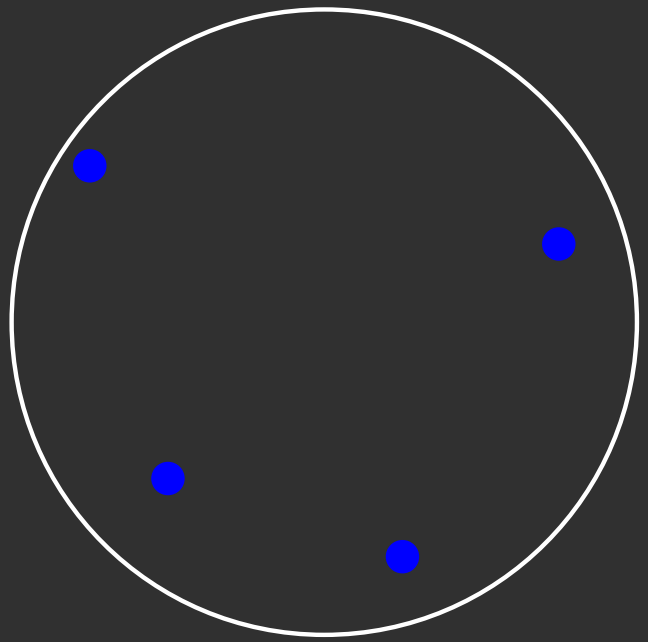

**Haar Random Unitary** "looks like" it is picked uniformly from the unitary group $U(d)$.

Figure: Haar measure on $U(d)$

# Haar Random Unitaries

## Haar Random Unitaries

Unitary Group $U(d)$



Uniform Distribution

**Property: Invariance**

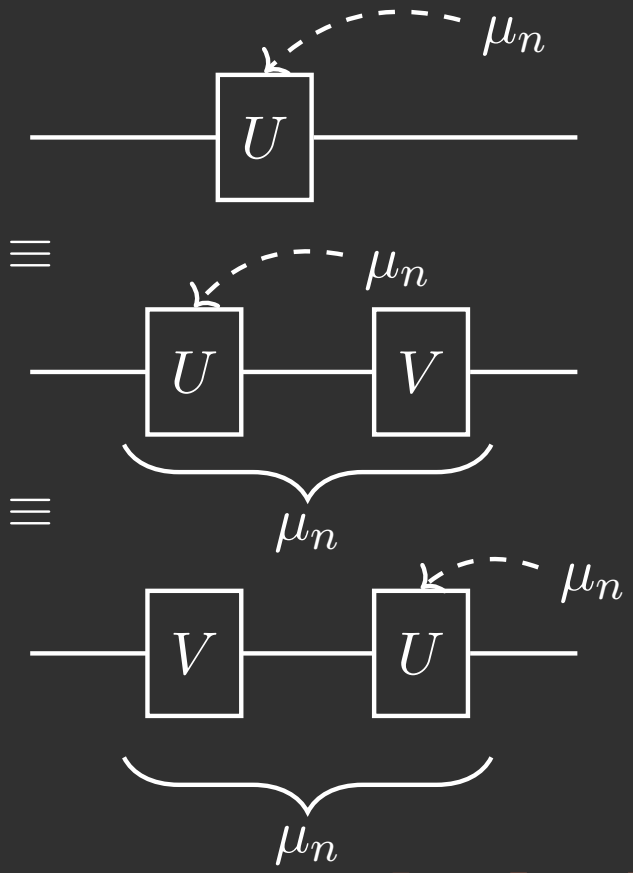



Figure: Haar measure on $U(d)$

# Quantum Pseudorandomness

## States

Pseudorandom States (PRS)

(JLS18,BS19,BS20,...)

Pseudorandom Function-like States (PRFS)

(AQY22,AGQY22,BBSS23,...)

## Operations

Pseudorandom Unitaries (PRU)

(JLS18,HBK23,MPSY24,...)

Pseudorandom State Scramblers (PRSS)

(LQS+23,...)

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**Applications:**

- Crypto from Minimal Assumptions

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Can we find a systematic method to analyse these primitives?

Pseudorandom Function-like States (PRFS)

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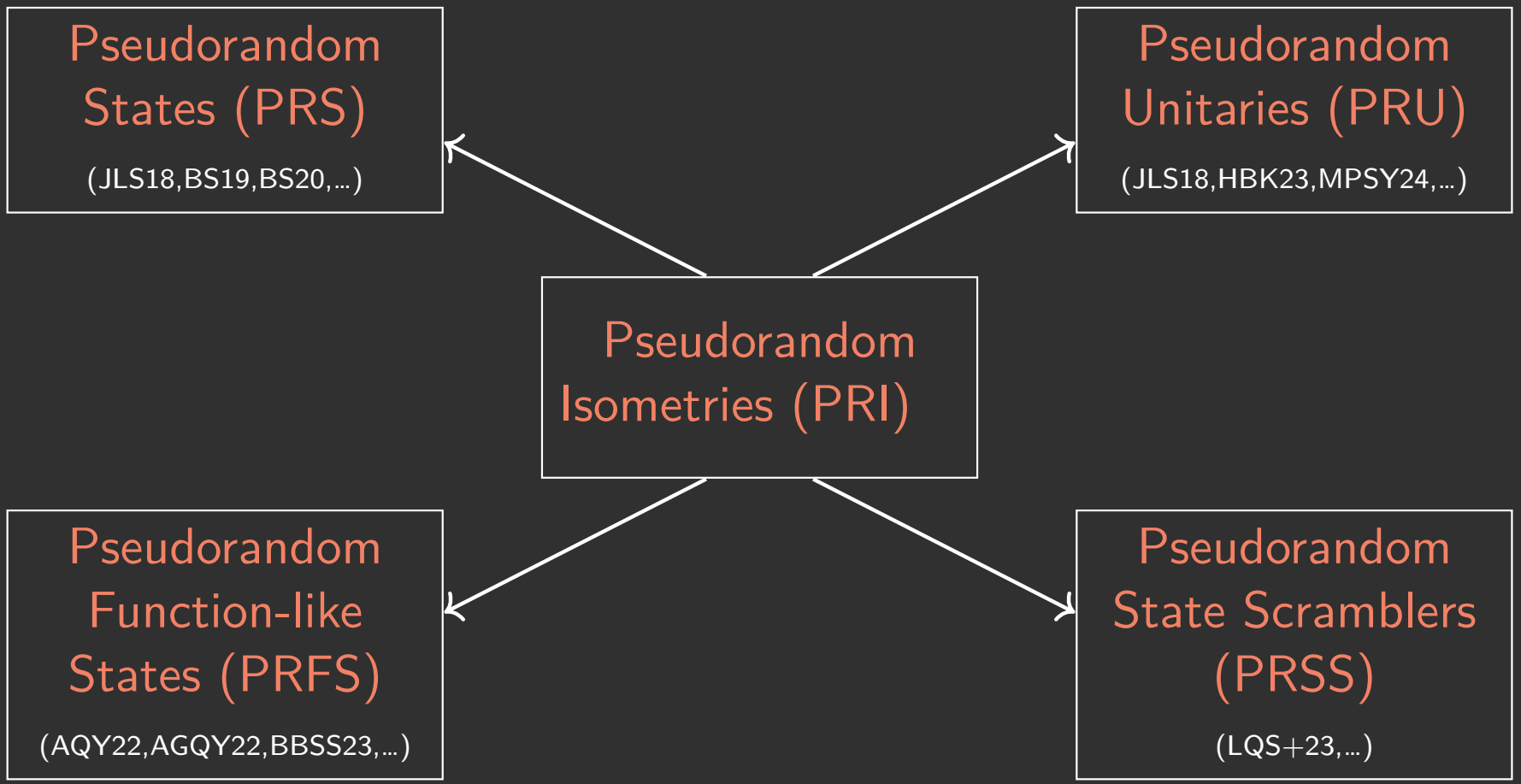

# Results

- **Pseudorandom Isometries (PRI):** Definition and framework to study other primitives.
- **Almost Invariance:** A technique to prove psuedorandomness.
- **Construction:** Using OWF to achieve PRI.
- **Applications:**
  - Length Extension Theorem
  - Quantum Message Authentication Codes (MACs)
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# Pseudorandom Isometries

## What are Pseudorandom Isometries?

# Haar Random Isometries

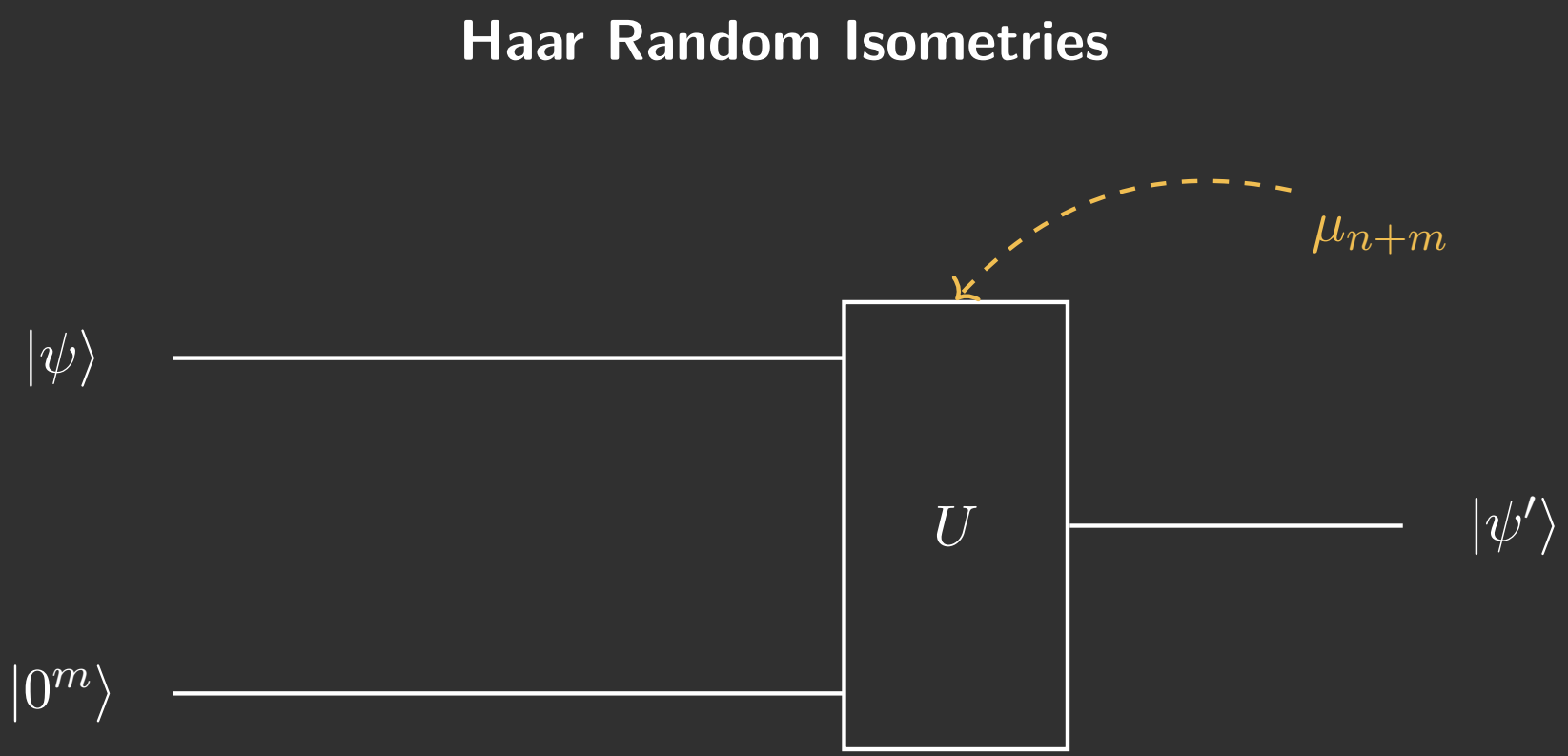


Figure: Quantum circuit representation of Haar isometries

# Pseudorandom Isometries (PRI)

Efficiently implementable circuits that "behave like" Haar random isometries.

# Pseudorandom Isometries (PRI)

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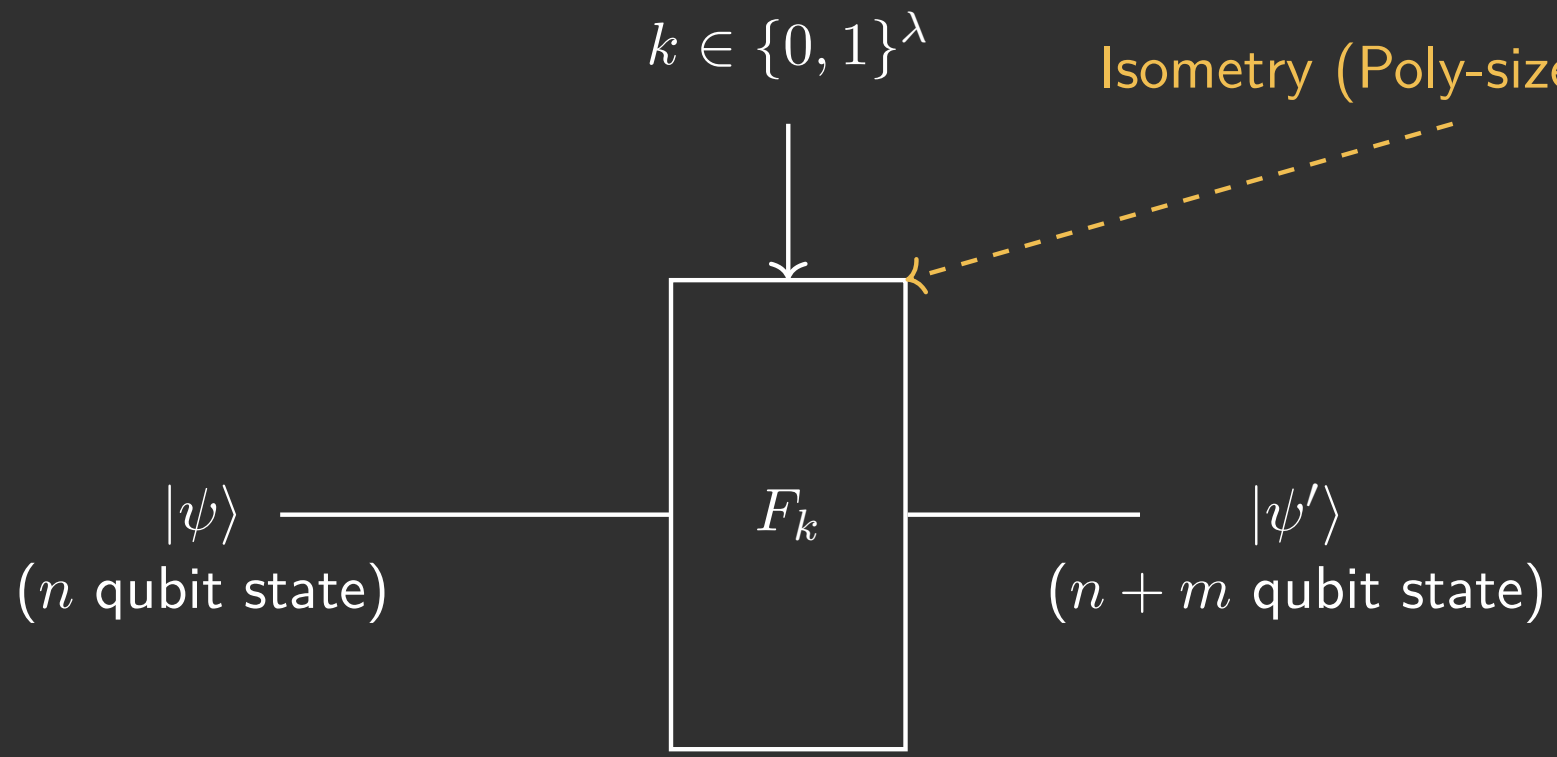


Notation: $(n, n+m)$-$PRI$

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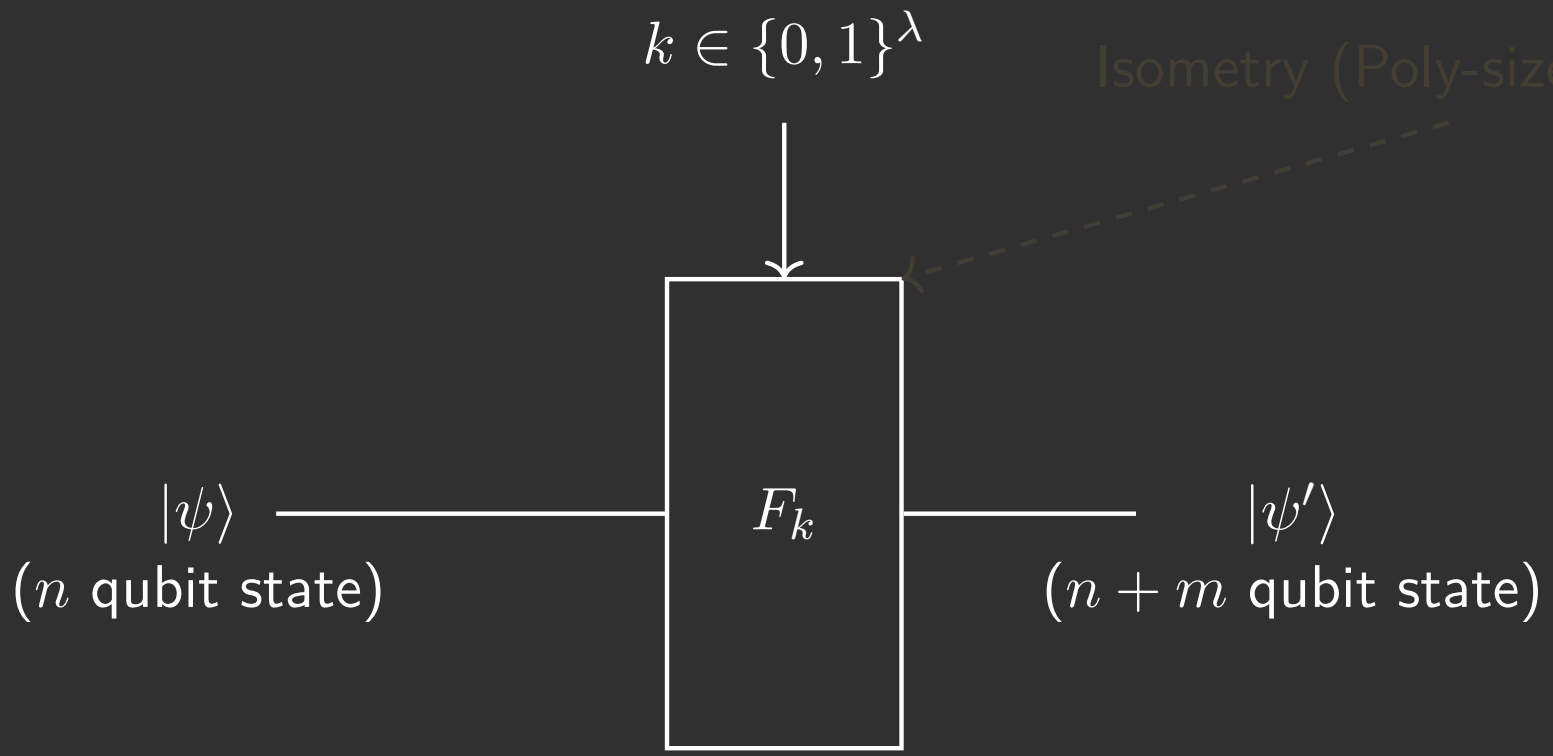


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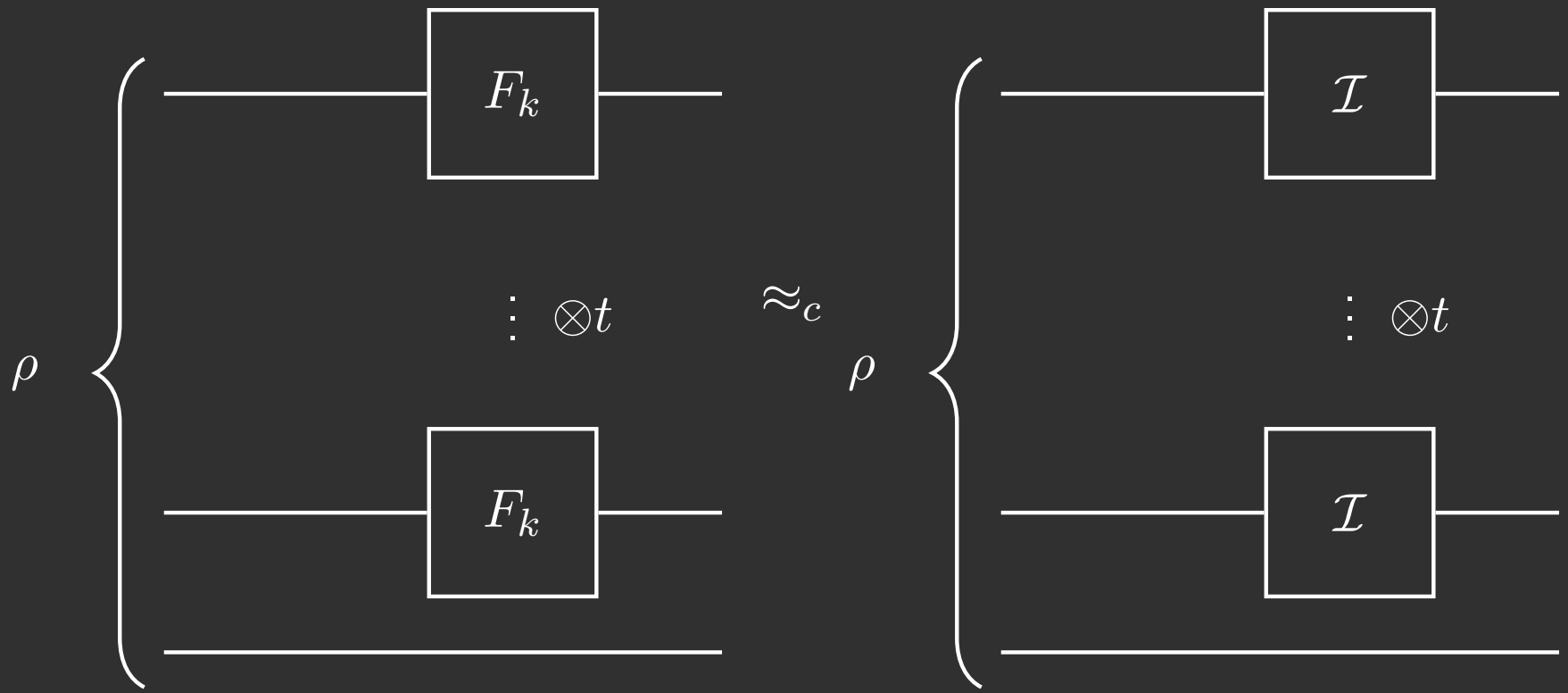
2. $\mathcal{Q}$-security: $\rho \in \mathcal{Q}$

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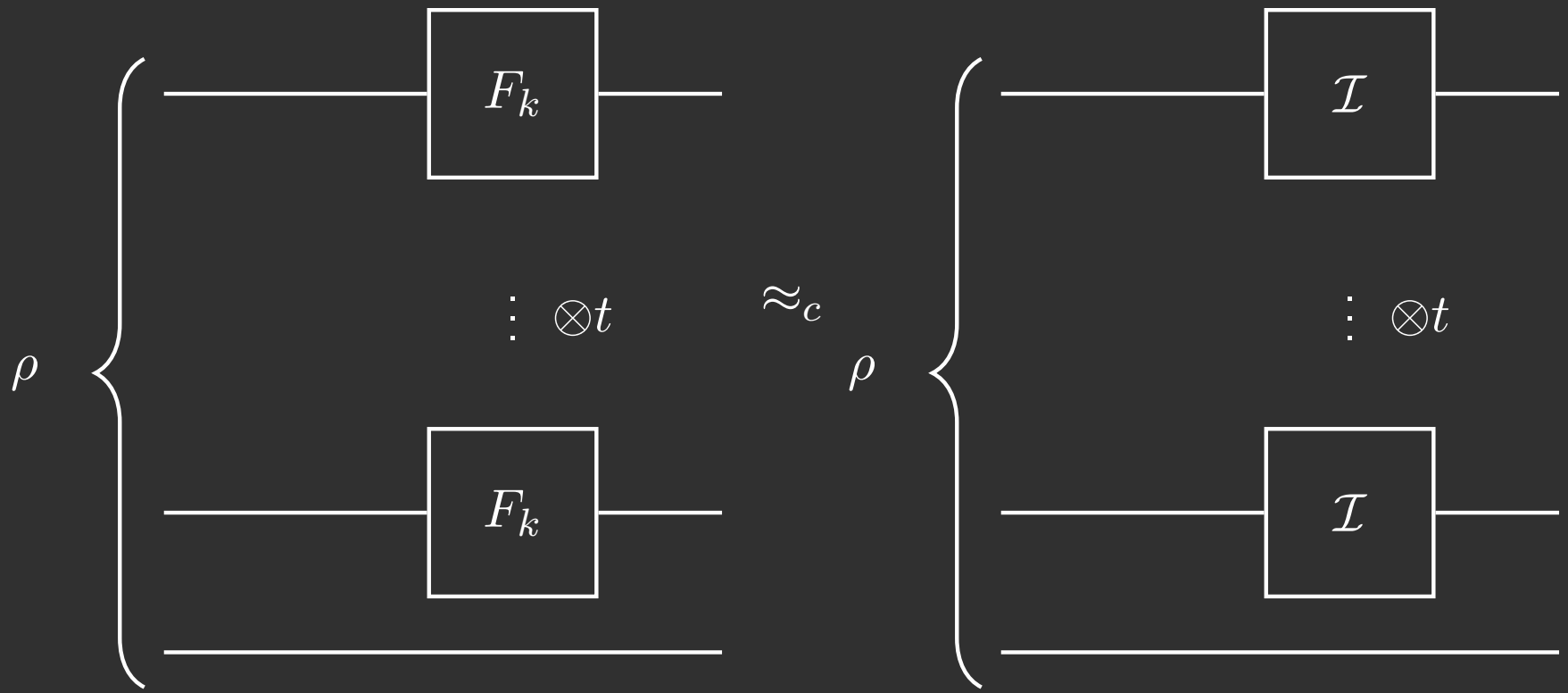


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# PRI as a Framework to Analyze Other Primitives

How are Pseudorandom Isometries related to other primitives?

# Relationship with Other Primitives

## Relationship with Other Primitives

- **Pseudorandom States:**
  PRI secure on $|0^n\rangle$ is a PRS.

$$\mathcal{Q} = \{|0^n\rangle^{\otimes t}\}$$

- **Pseudorandom State Scrambler:**
  PRI on $t$-tensor same state is a PRSS.

$$\mathcal{Q} = \{|\psi\rangle^{\otimes t} : |\psi\rangle \text{ is an } n\text{-qubit state}\}$$

- **Pseudorandom Function-like States:**
  PRI secure on computation basis states is a PRFS.

$$\mathcal{Q} = \{\otimes_{i=1}^{t} |x_i\rangle : x_i \in \{0,1\}^n\}.$$

- **PRU:**
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  - PRU implies PRI.

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# Almost Invariance Implies Security

How do we analyse Pseudorandomness of PRIs (and other primitives)?

# Almost Invariance Implies Security

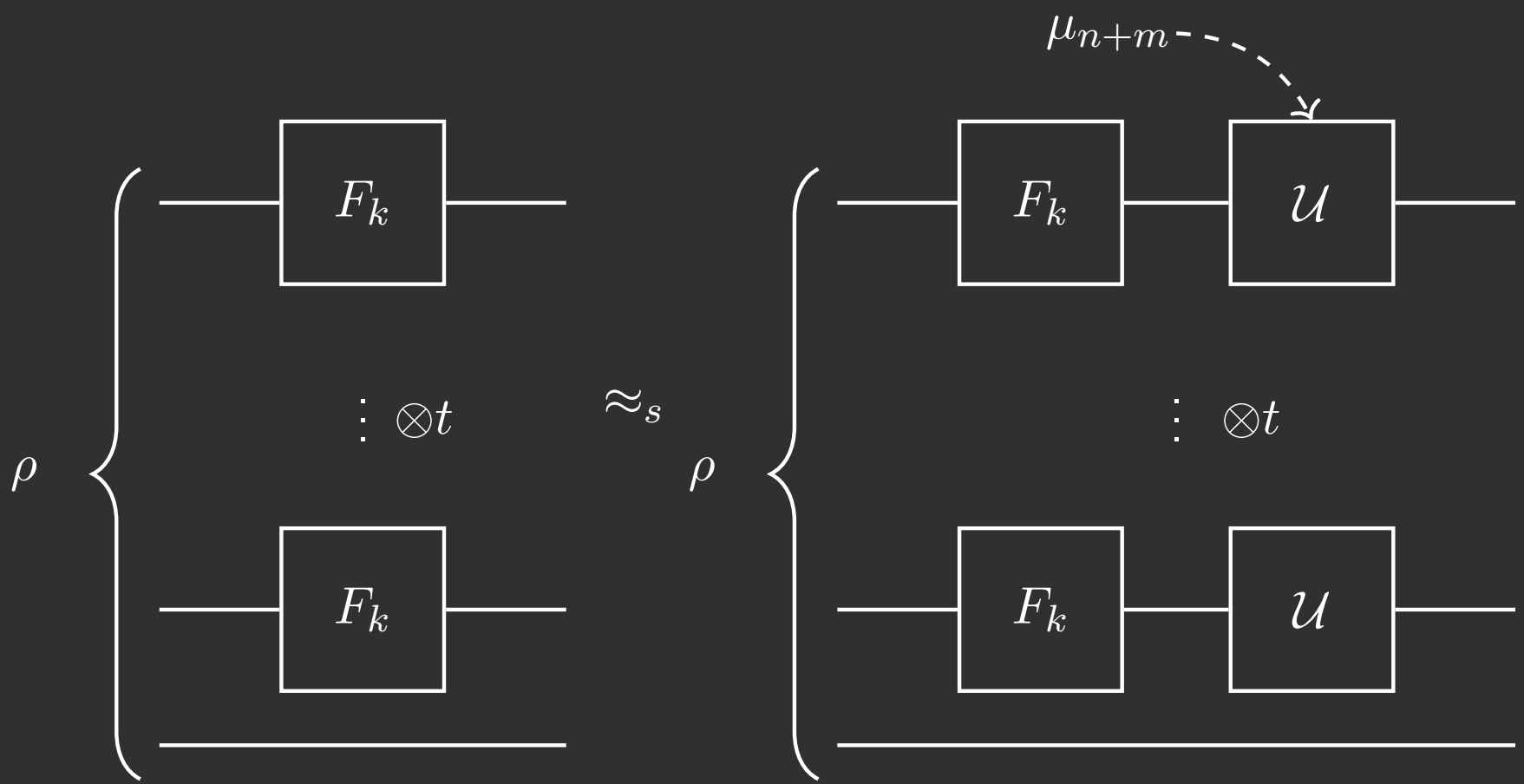
**Intuition: The output of Pseudorandom primitives are "maximally scrambled", hence applying $t$-fold Haar shouldn't change it much.**

# Almost Invariance Implies Security

**Almost Invariance implies security: If the output of a Pseudorandom primitive is invariant under $t$-fold Haar Unitary, then the Pseudorandom primitive is secure.**

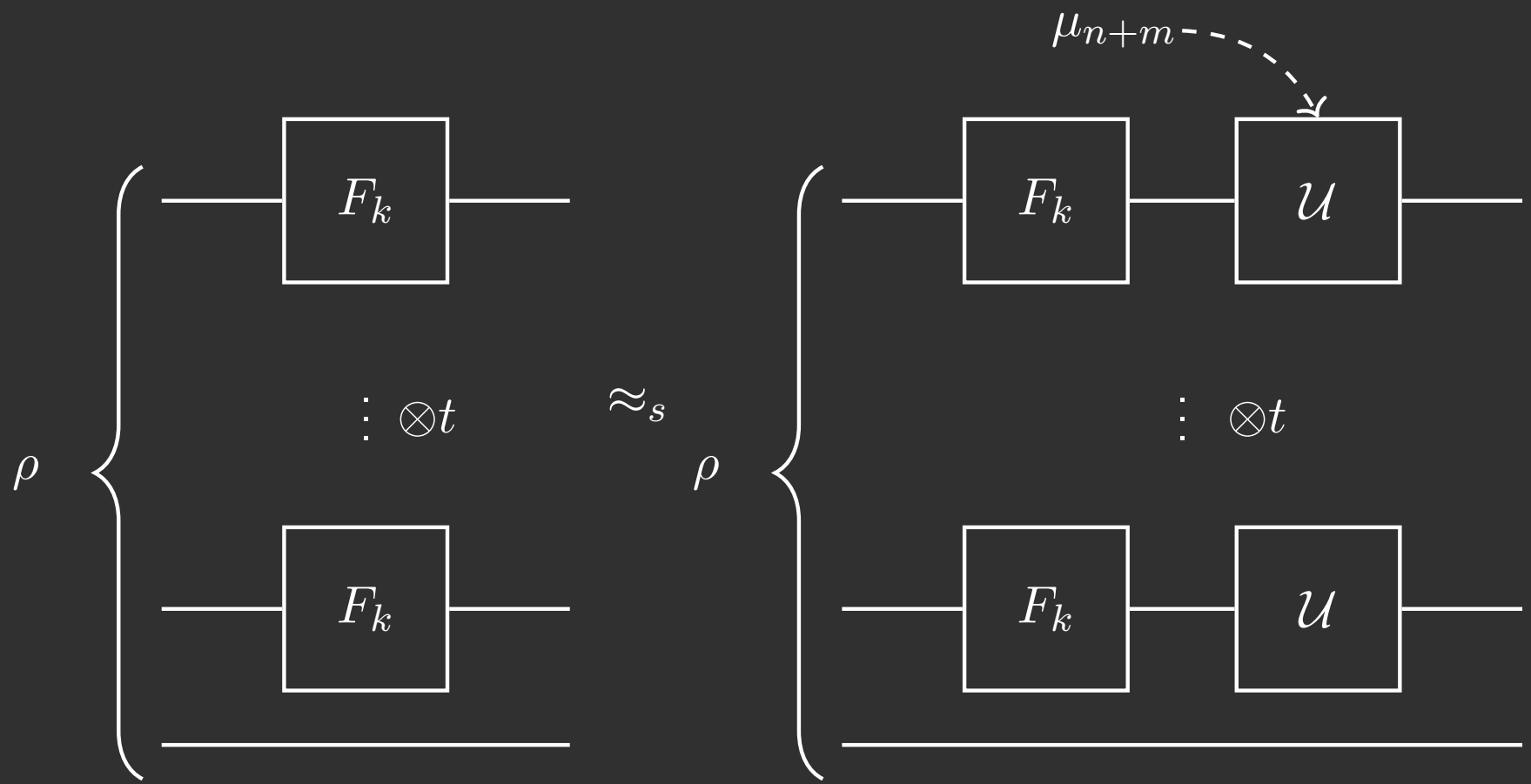
# Almost Invariance Implies Security

PRI is $\mathcal{Q}$-secure iff $\rho \in \mathcal{Q}$



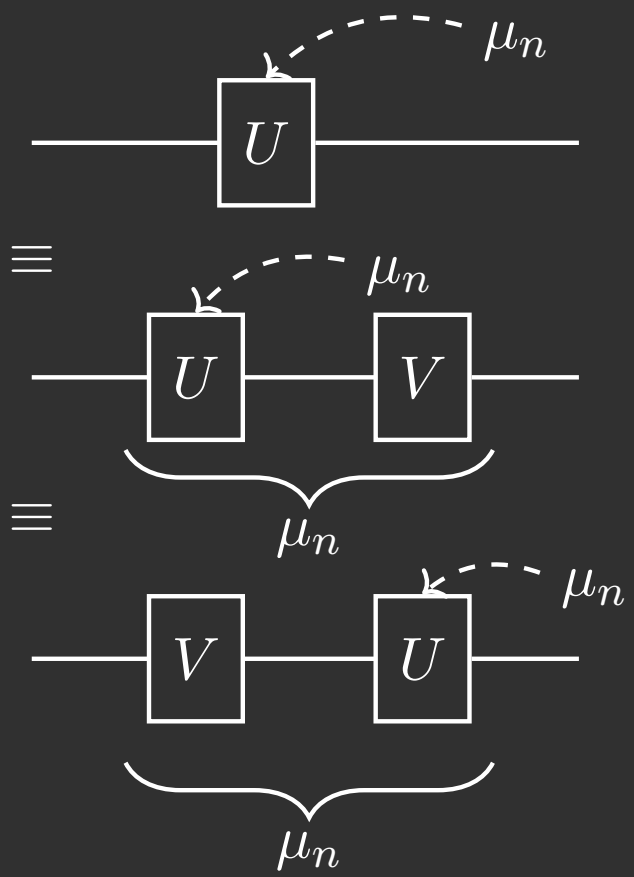
# Almost Invariance Implies Security

PRI is $\mathcal{Q}$-secure iff $\rho \in \mathcal{Q}$ (Used by BM'24 (subsequent work) to prove pseudorandomness of PRU's.)

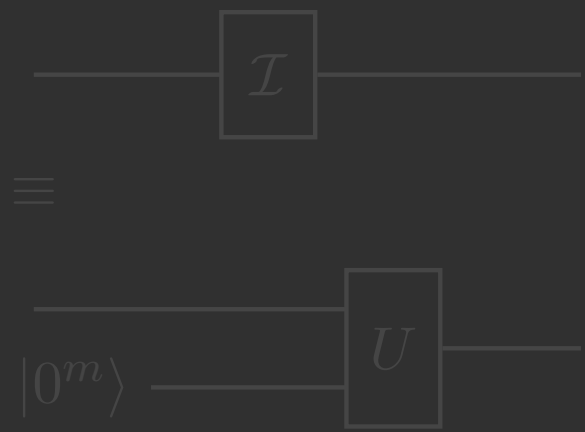


# Proof of Almost Invariance Implies Security

## Recall: Haar Invariance




## Isometry definition



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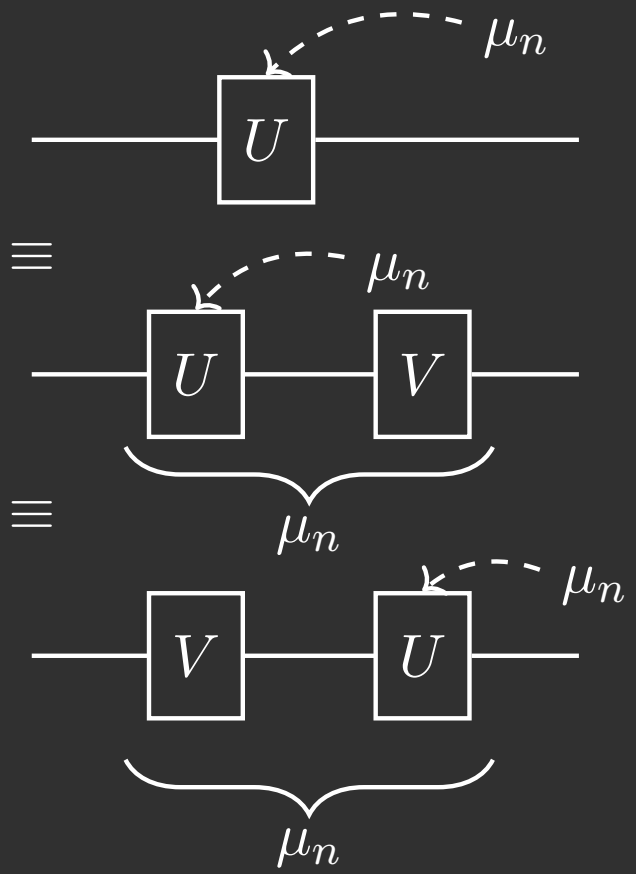


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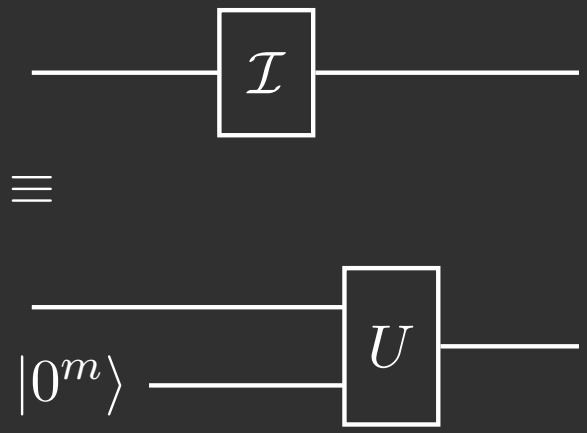

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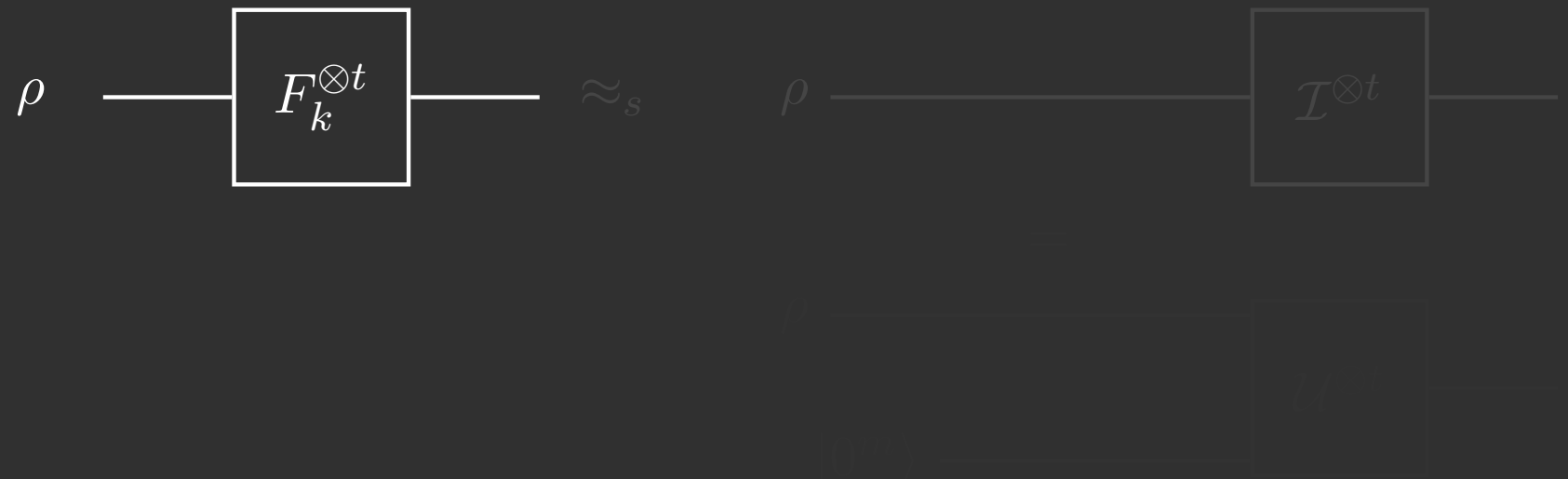

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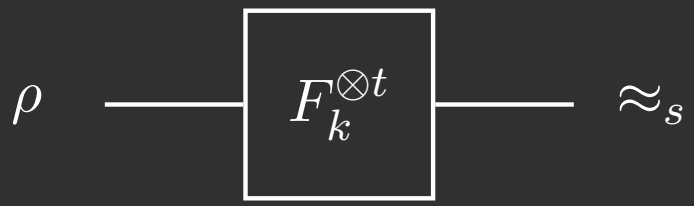


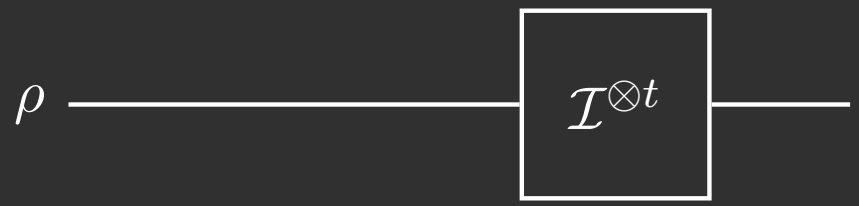


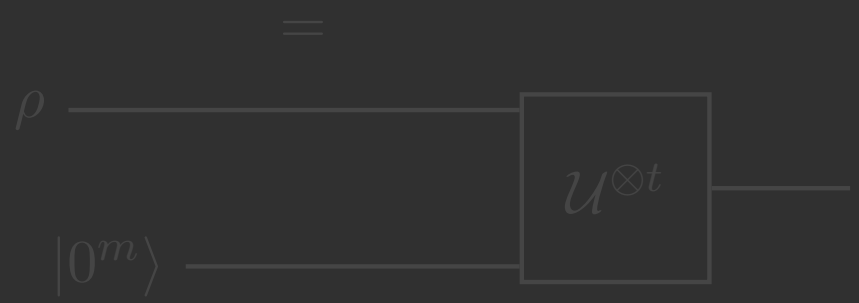

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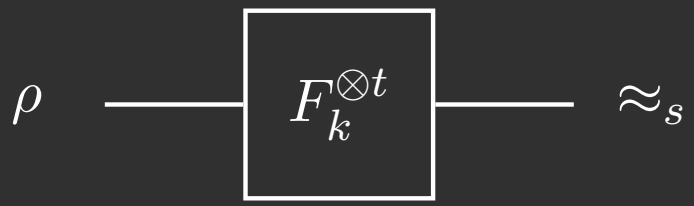


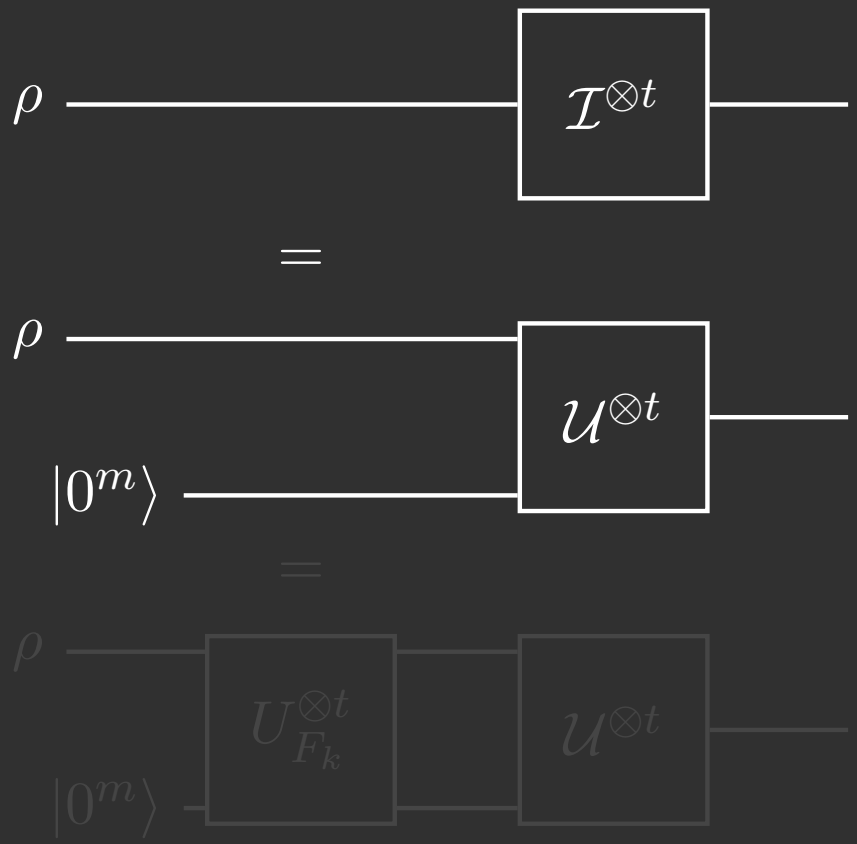

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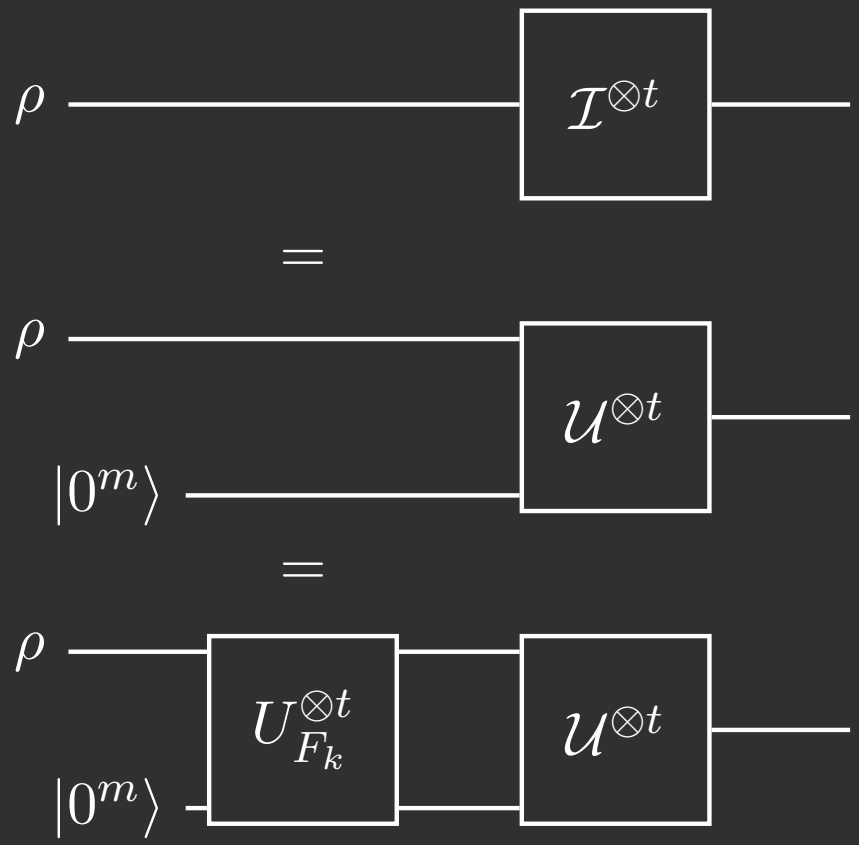


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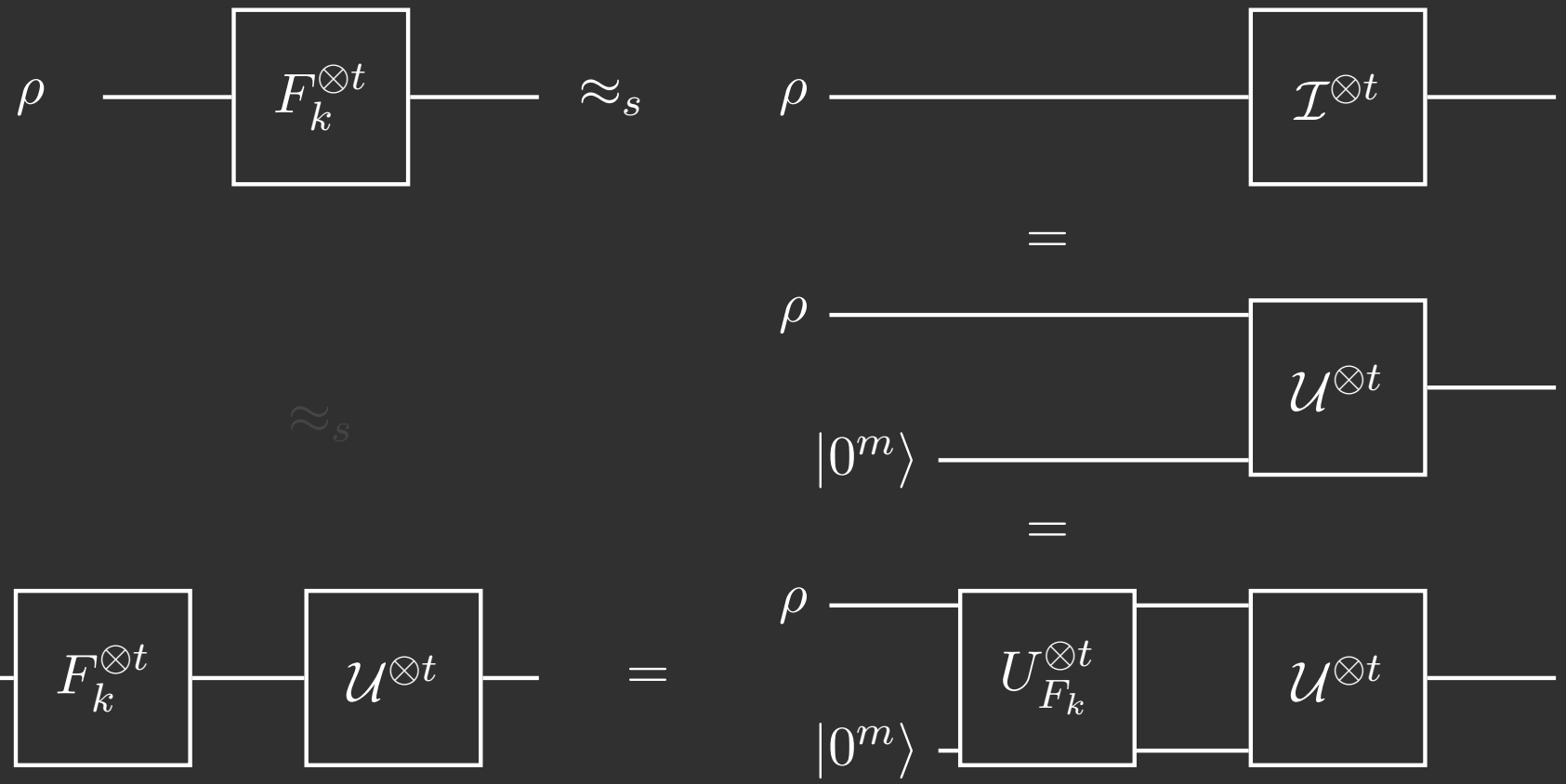

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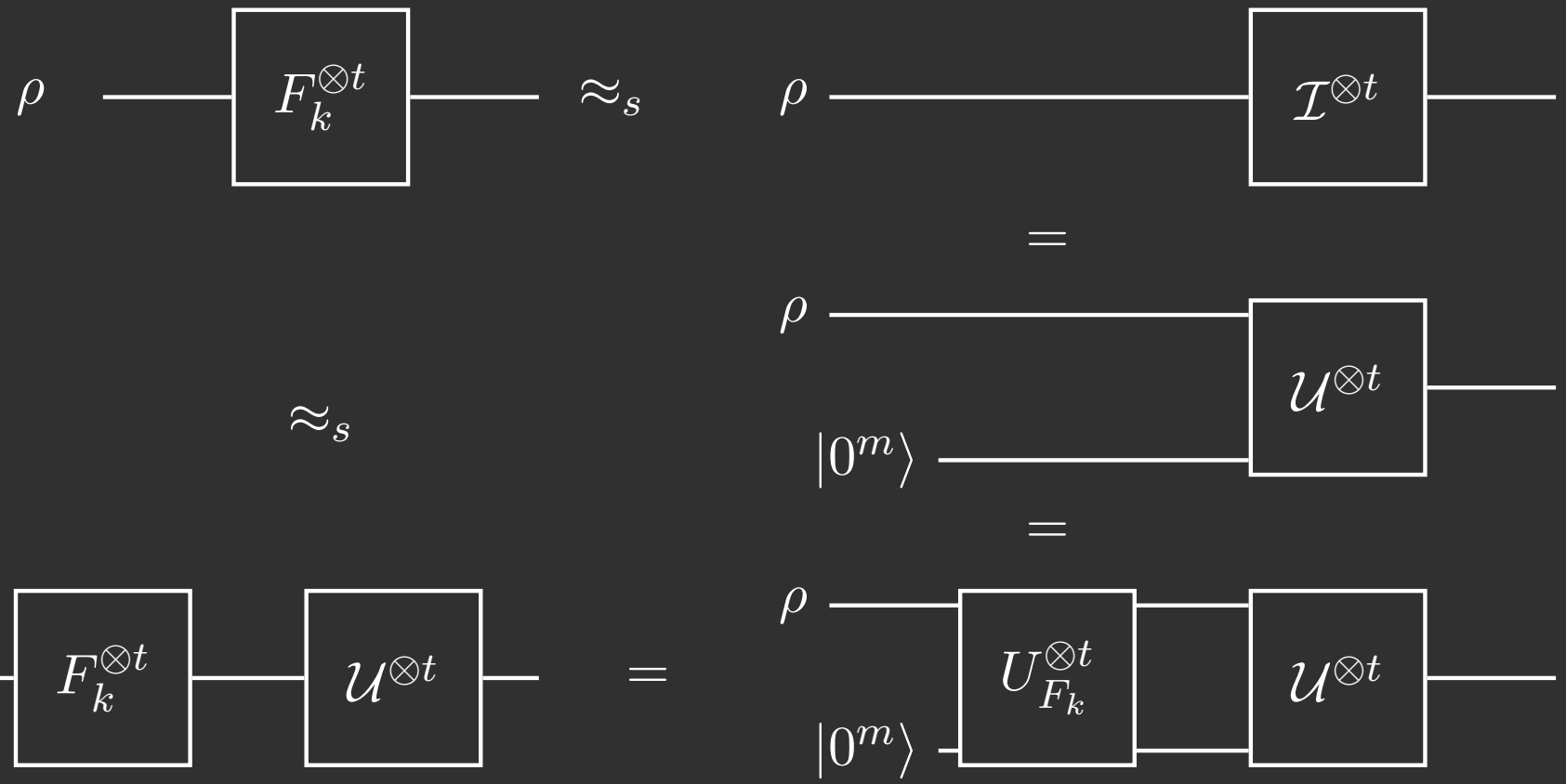
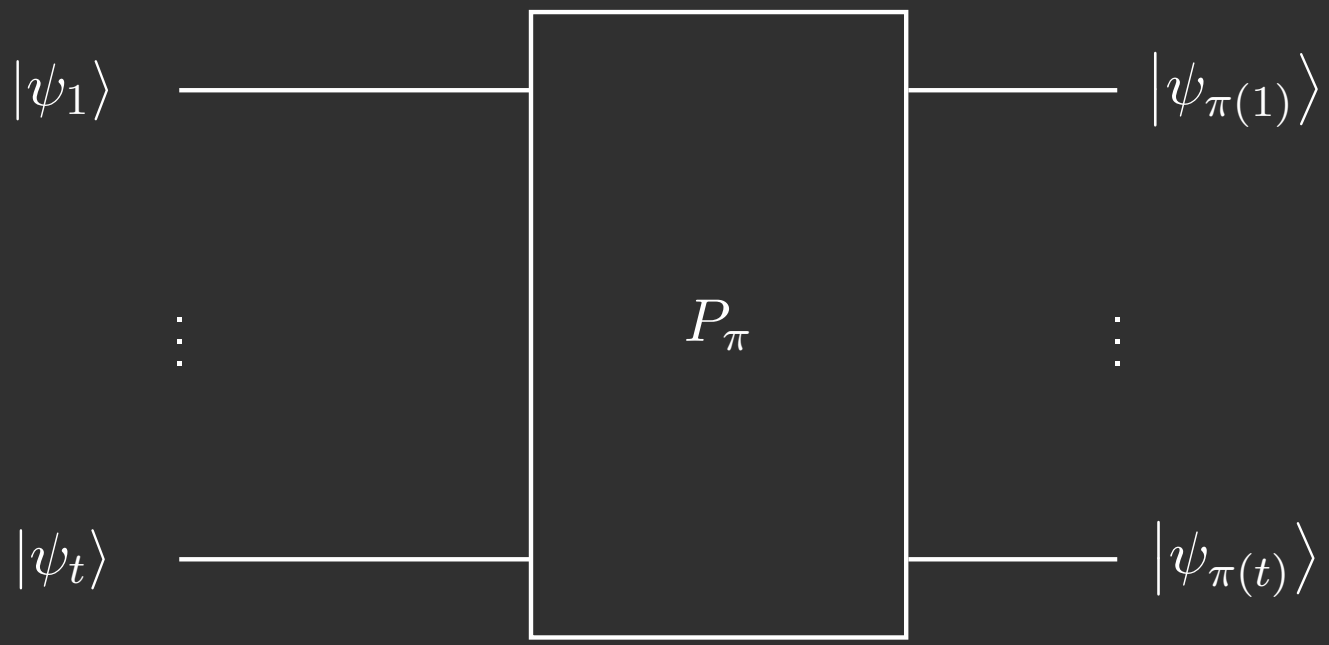


# Characterisation of Almost invariance

Can we characterise these "almost invariant" states?

# Characterisation of Almost Invariant states

Define $P_\pi$:



# Characterisation of Almost Invariant states

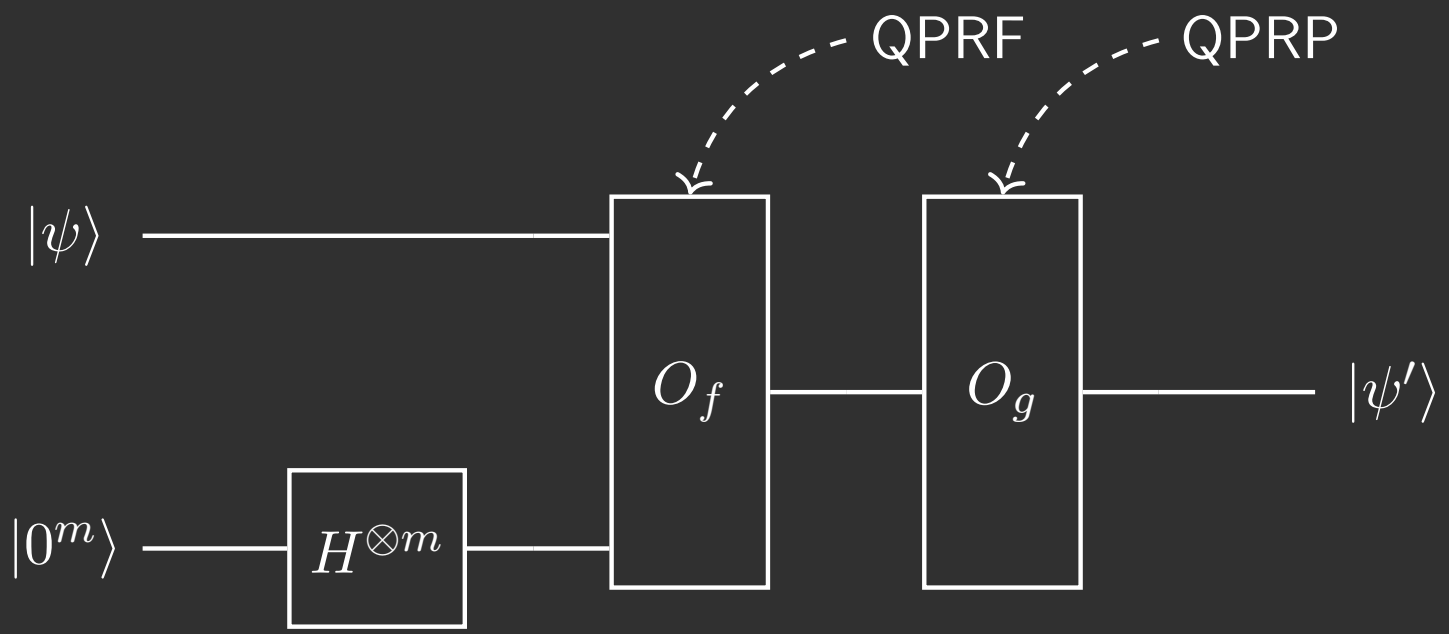
**Theorem**

*A state $\rho$ is almost invariant under $t$-fold Haar if and only if $\rho \approx_s \sum_\pi \alpha_\pi P_\pi$ for some $\pi \in S_t$.*

# Construction

Do we have a PRI from OWF?

# Implementation



# Security of the Construction

**Theorem**

*Assuming the existence of post-quantum one-way functions, the given construction is $\mathcal{Q}$-secure for $\mathcal{Q} \in \{\mathcal{Q}_{PRFS}, \mathcal{Q}_{PRSS}, \mathcal{Q}_{Haar}\}$.*

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**Theorem (MPSY'24 (subsequent work))**

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# Applications

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# Length Extension

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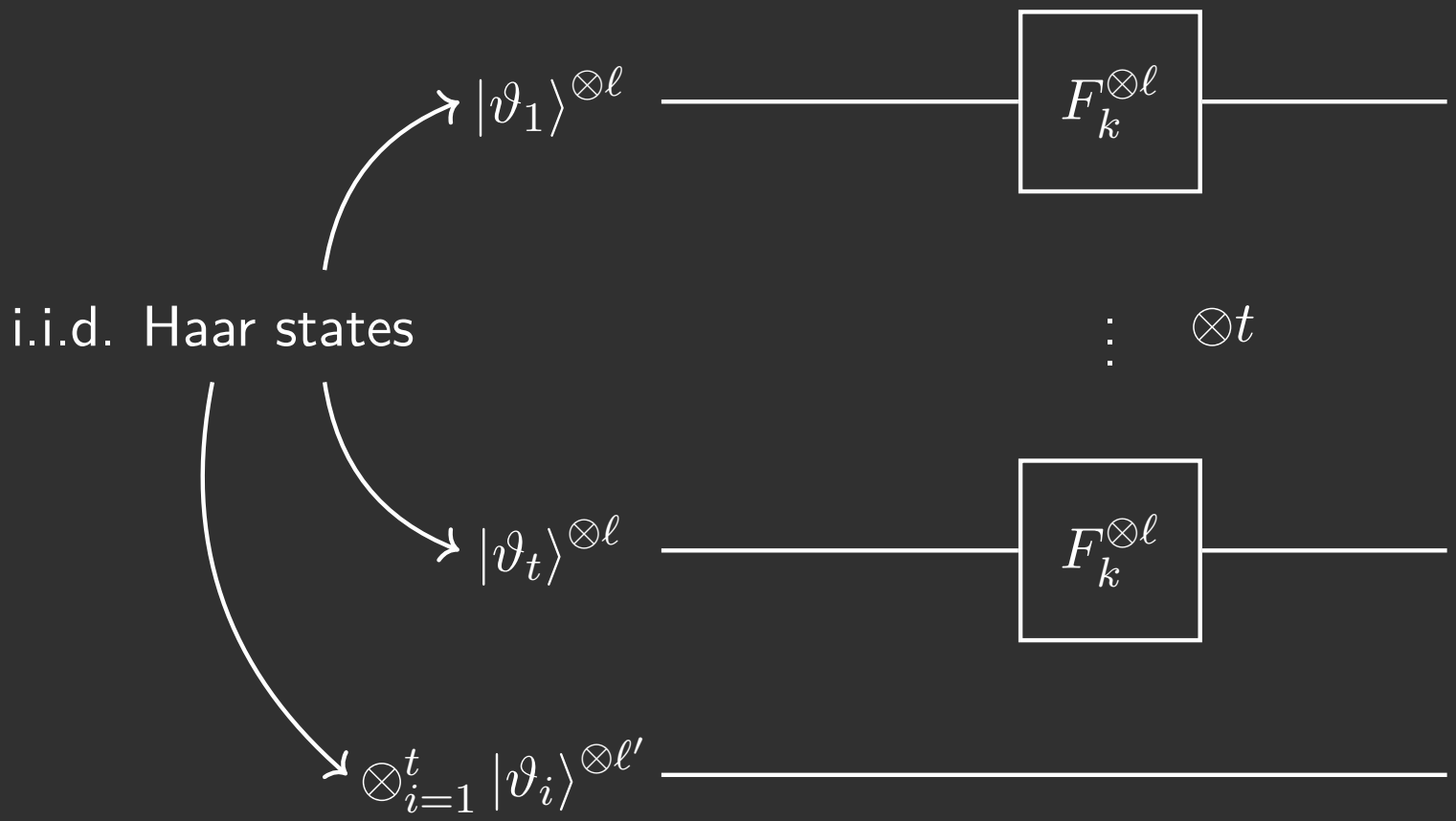
# Security against Haar-queries

$\mathcal{Q}_{\mathsf{Haar}}$-security:



# Length Extension Theorem: Formal Statement

- Assuming the existence a $\mathcal{Q}_{\mathsf{Haar}}$-secure $(n, n+m)$-pseudorandom isometry, then given an $n$-qubit PRSG, there exists an $n+m$-qubit PRSG.
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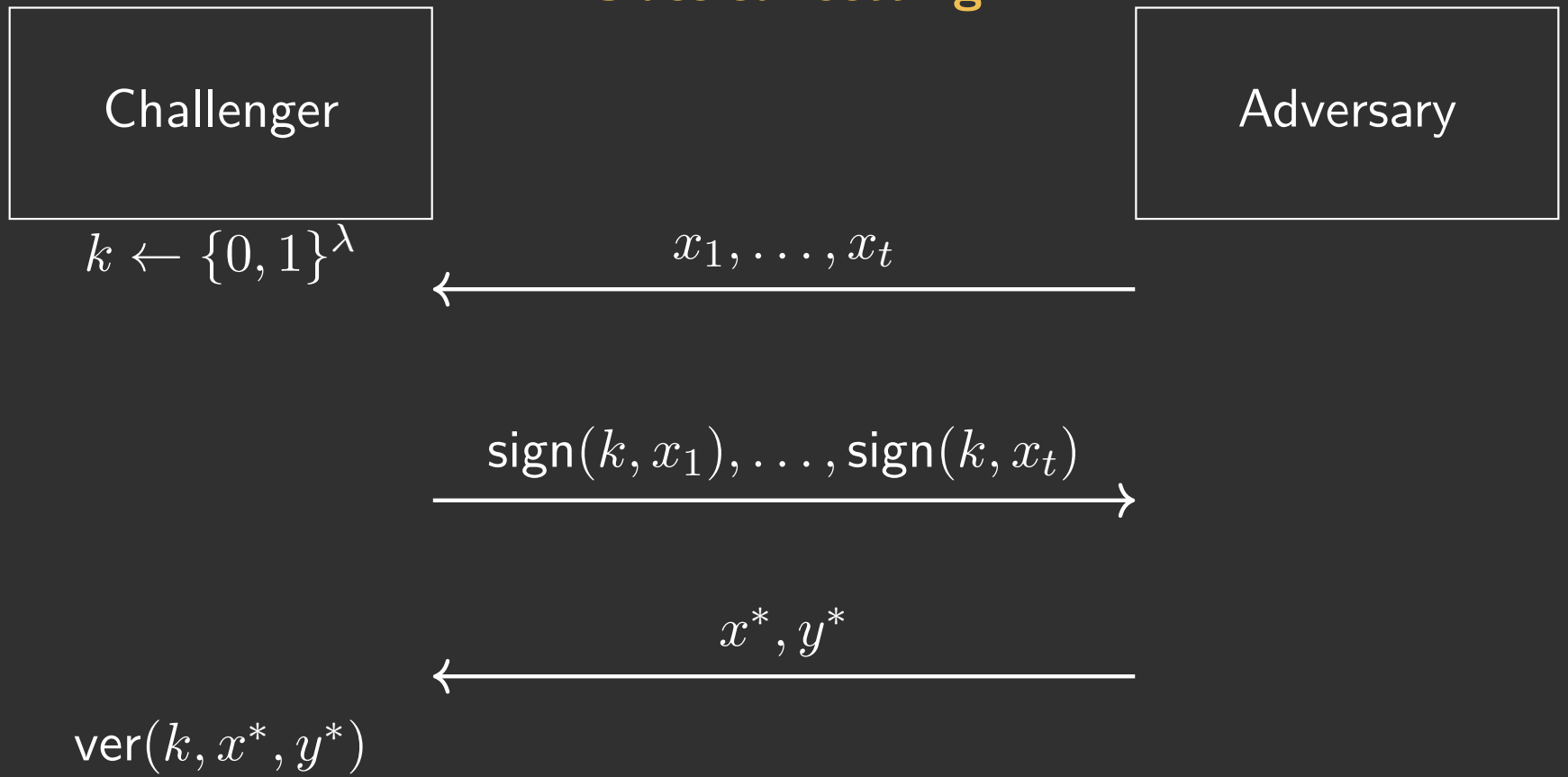
# Other Applications of PRI

- Quantum Message Authentication Codes (MACs).
- Multi-copy secure encryption schemes.
- Succinct quantum commitments.

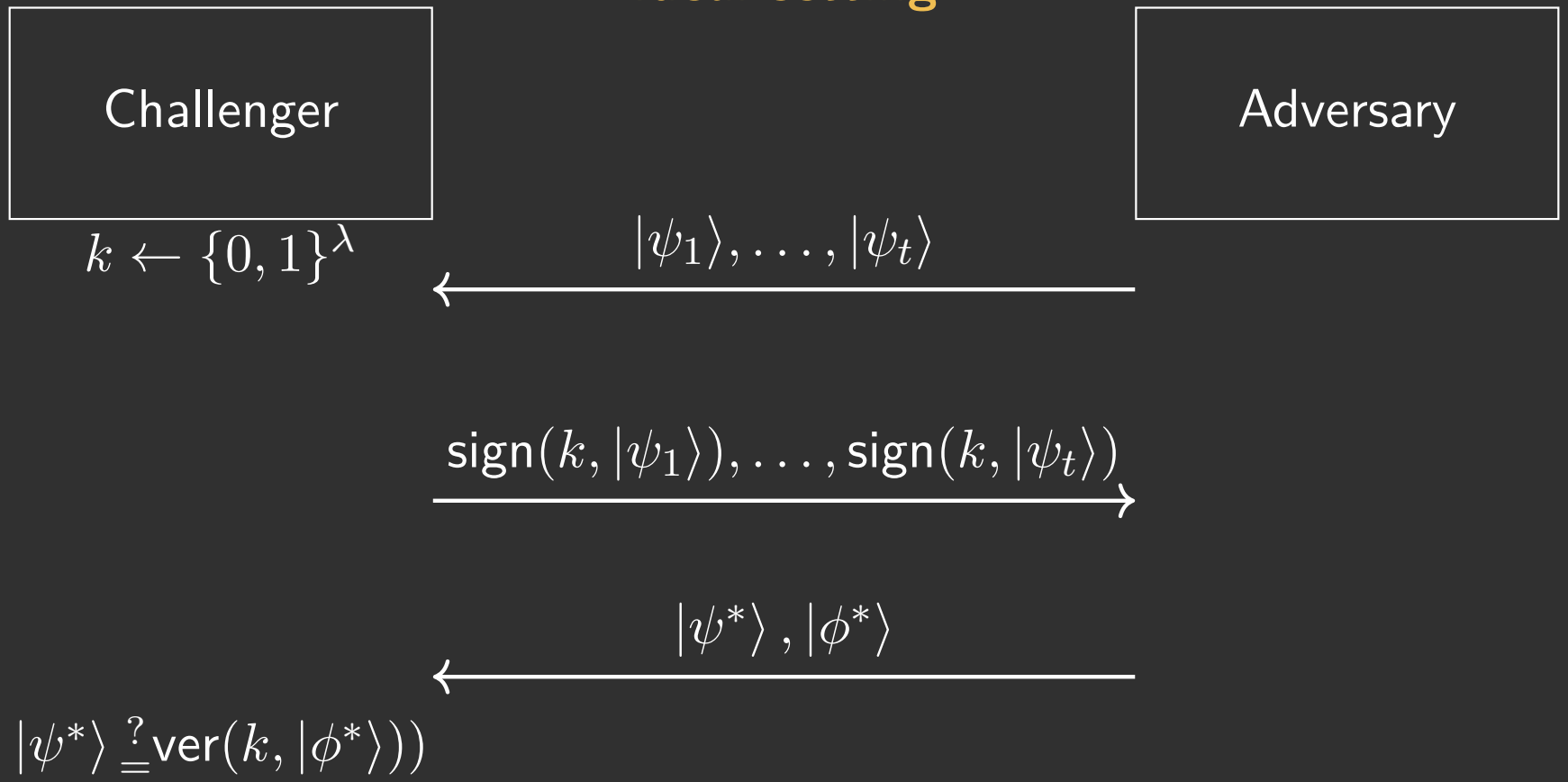
# Quantum Message Authentication Code (MAC)

**Classical setting**



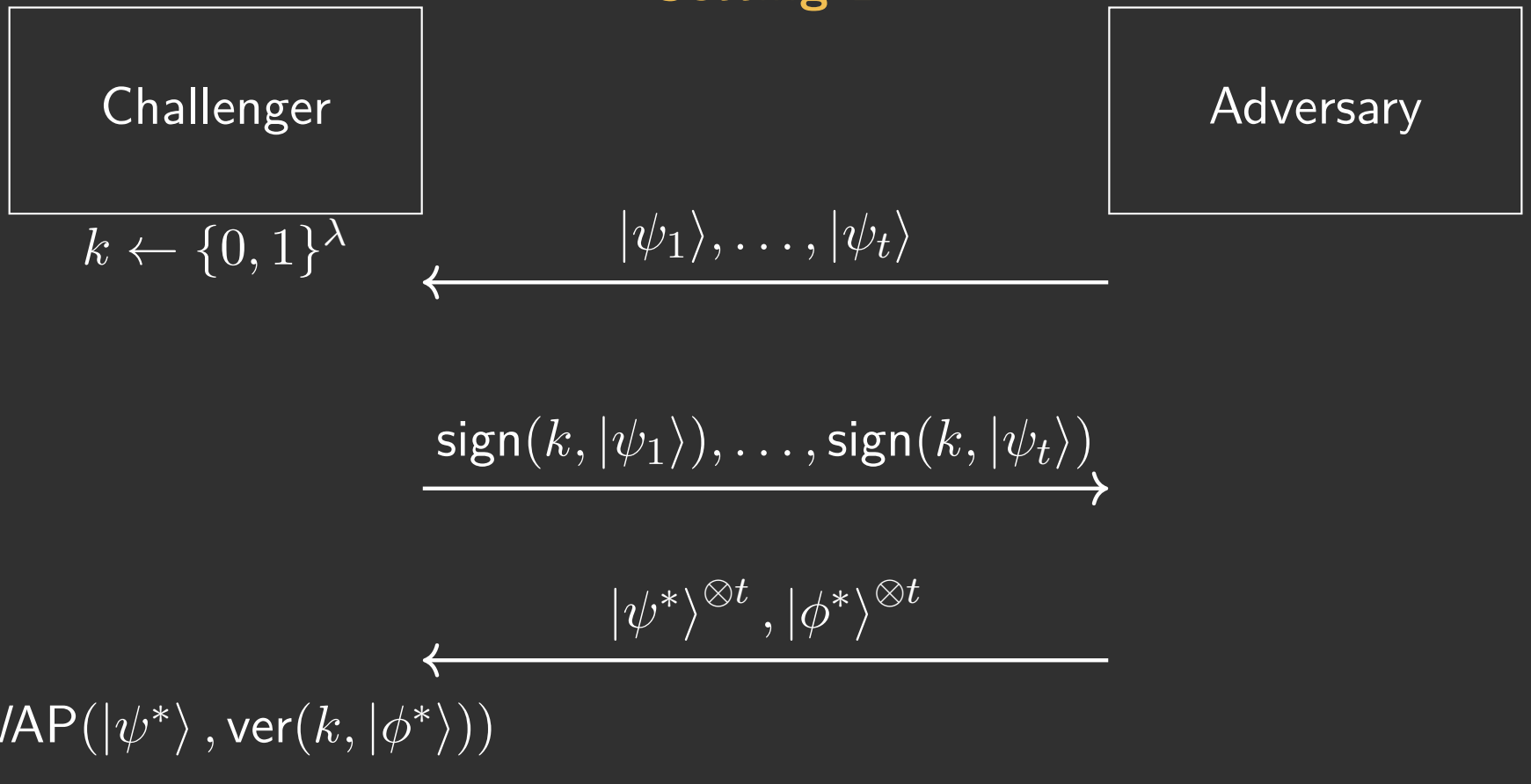
# Quantum Message Authentication Code (MAC)

**Ideal setting**



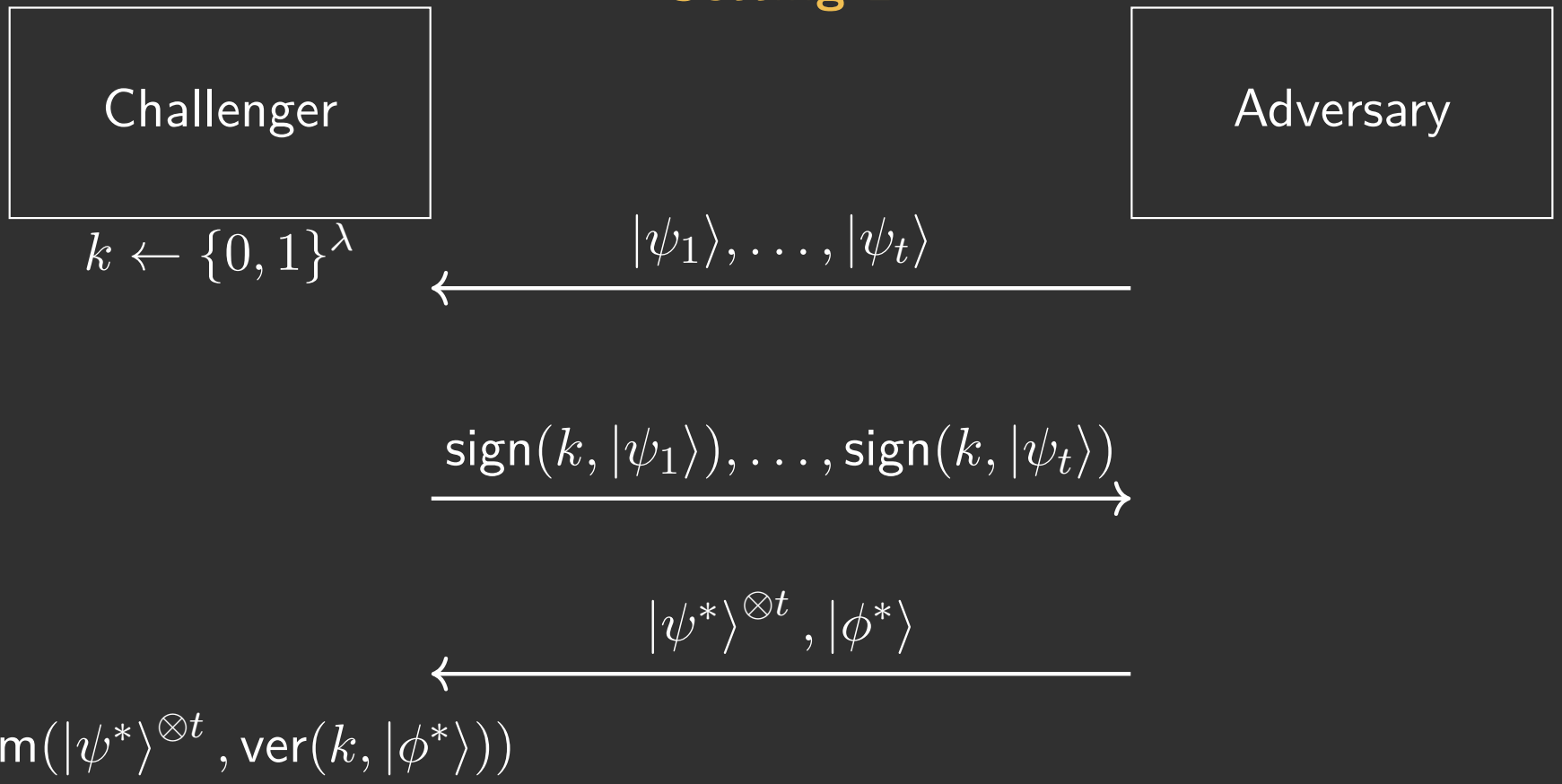
# Quantum Message Authentication Code (MAC)

**Setting 1**



# Quantum Message Authentication Code (MAC)

**Setting 2**



# Multi-Copy Secure Encryption and Succinct Quantum Commitments

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- The security should hold even if the adversary receives polynomially many copies of the ciphertext state.
- $\mathcal{Q}_{\mathsf{single}}$-PRI provides the necessary security for both public-key and private-key encryption schemes. We use techniques similar to those of [LQS + 23].

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# Thank You