Constructing Leakage-resilient Shamir's Secret Sharing: Over Composite Order Fields

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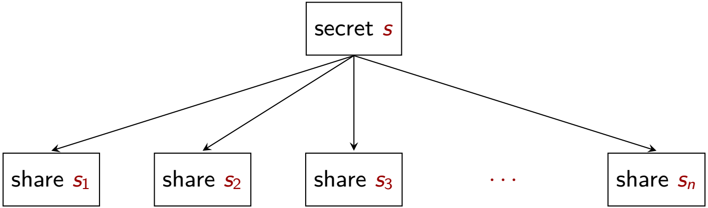
Threshold Secret Sharing [Shamir, Blakley]

Sharing:



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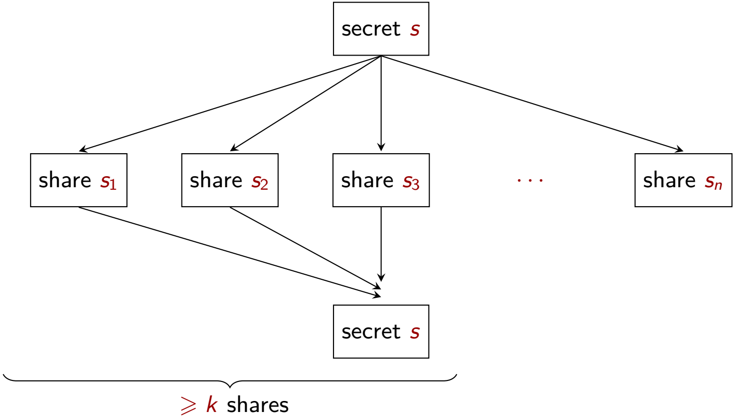
Sharing:

Reconstruction:



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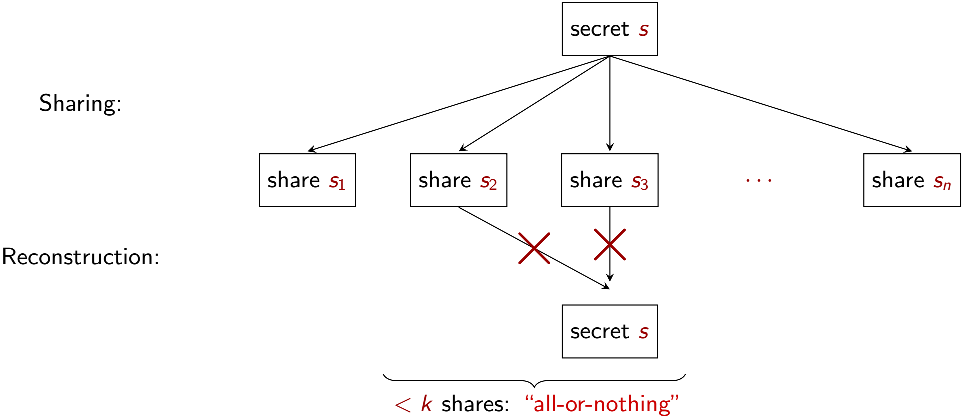

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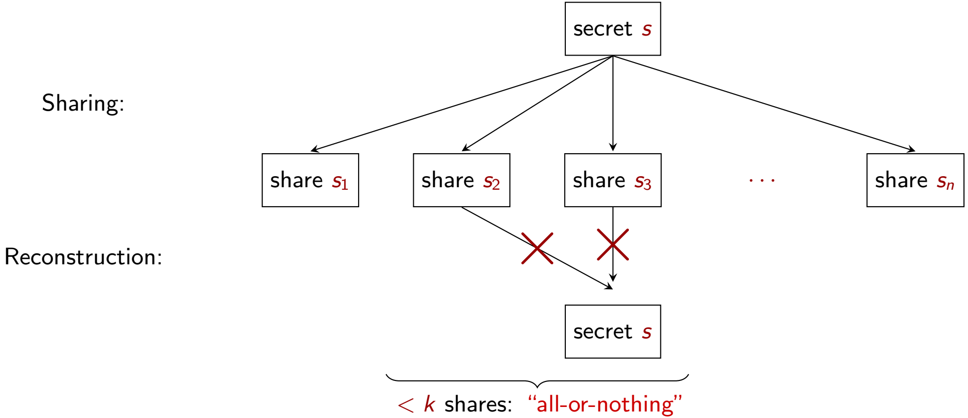
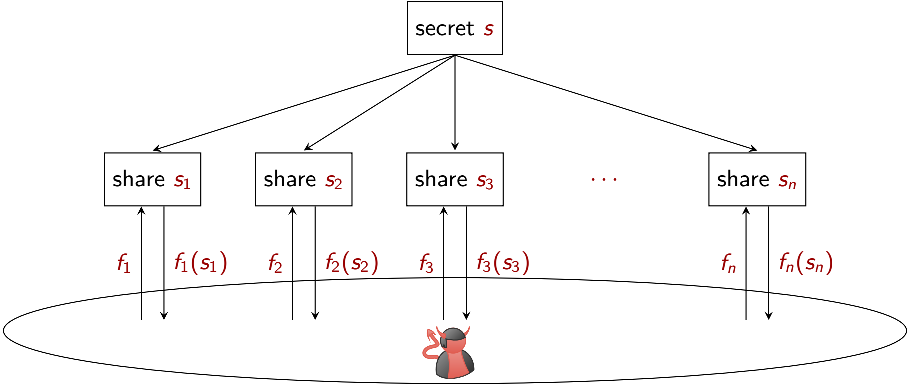


Concern: Side-channel attacks

- "All-or-nothing" no longer true
- Revealing partial or full information from every share

Local Leakage-resilient Secret Sharing

[Benhamouda-Degwekar-Ishai-Rabin-18, Goyal-Kumar-18]



Leakage resilience: Adversary view is essentially uncorrelated with the secret s.

Our Research Problem

Research Question

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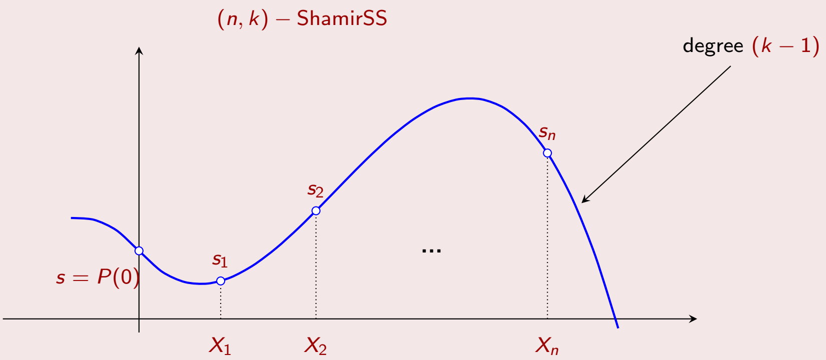
Question

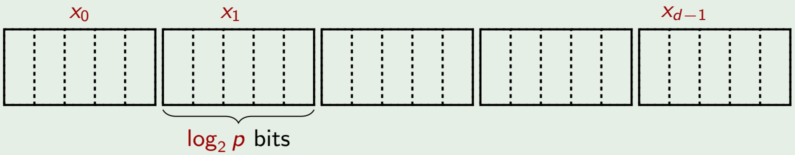
How about composite order fields?

Model: Shamir's Secret Sharing



Leakage Model: Physical Bit Probing [Ishai-Sahai-Wagner-03]

Representation of every field element $x \in F_{p^d}$



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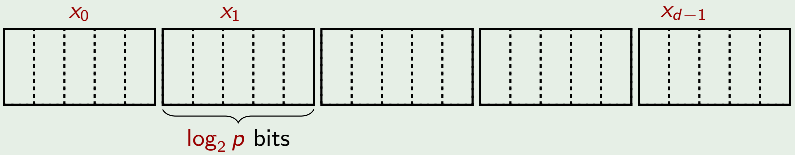
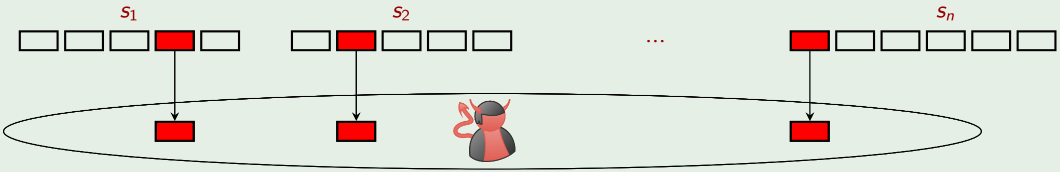


Leakage model

The adversary gets physical bits leakage from every share.

Example: single block leakage (a $log_2 p$ physical bits leakage)



Main Result I

Theorem (Randomized construction for composite order fields)

Let $\lambda = d\lceil \log_2 p \rceil$ *be the security parameter.*

If the total leakage $\leqslant \rho(k-1)\lambda$, *where* $\rho = \begin{cases} 1 - 1/p & \text{if } 2 \leqslant p \leqslant k-1, \\ 1 & \text{otherwise,} \end{cases}$

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Our result holds for any $k \geqslant 2$ and large characteristic-2 fields.

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Comparison with the result over prime fields [MNPSW-21]

- $\rho = 1$ for prime fields.
- The permissible leakage tolerance may be slightly smaller for composite order fields.

Main Result II

Theorem (Classifying evaluation places: a dichotomy)

Against single block leakage, $(n, 2)$-Shamir's scheme is either perfectly secure or completely insecure.

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Classifying Secure Evaluation Places Algorithm for Single Block Leakages

- **Input:** Distinct evaluation places $X_1, X_2, \ldots, X_n \in F_{p^d}$.
- **Output:** Whether $(n, 2)$-Shamir's secret sharing with evaluation places $X_1, X_2, \ldots, X_n$ are secure.
- **Algorithm:**
  1. Compute the set of shift factors S (of size d).
  2. If exist $\eta_1, \eta_2, \ldots, \eta_n \in S$ such that $X_1\eta_1, X_2\eta_2, \ldots, X_n\eta_n$ are F_p-linearly dependence, then return "insecure".
  3. Otherwise, return "secure".

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Comparison with [Hwang-Maji-Nguyen-Ye-24]

- Consider similar problems over Mersenne/Fermat prime fields, one-bit leakage per share.
- Derandomize the construction over prime fields.

Prior Works

Relevant work	Finite Field F	Evaluation Places	Leakage family	Bounds on k
BDIR'18&21	prime	any	local	$k \geqslant 0.85n$
MNPSW'21	prime	random	physical bit	$k \geqslant 2$
MNPW'22	prime	any	local	$k \geqslant 0.78n$
MNPSWYY'22	prime	random	bounded joint	$k > 0.5n$
KK'23	prime	any	local	$k \geqslant 0.69n$
This work	composite	random	physical bit	$k \geqslant 2$

Table 1: Summary of prior works and ours for 1-bit leakage, where $\lambda = \log_2 |F|$.

Technical Overview

Extend the analysis of [MNPSW'21] to composite order fields: Fourier analysis & probabilistic method.

Reductions

1. For any leakage function, for any two secrets, the distinguishing advantage is small over randomly chosen evaluation places.

$$\mathrm{E}_{\vec{X}}\ \mathrm{SD}\left(f(s)\ ,\ f(s')\right) \leqslant \mathrm{E}_{\vec{X}} \sum_{\vec{t}\in\{0,1\}^n} \sum_{\vec{\alpha}\in F^n\setminus\{0\}} \left(\prod_{i=1}^{n} \left|\widehat{\mathbb{1}_{t_i}}(\alpha_i)\right|\right) \cdot \Pr_{\vec{X}}\left[\vec{\alpha} \in C_{\vec{X}}^{\perp}\right] \leqslant \exp(-\Theta(\lambda))$$

2. Applying standard probabilistic techniques (union bound and Markov inequality) yields most evaluation places are secure.

Bound on the Number of Solutions of a System of Equations

System of equations

Fix $\vec{\alpha} \in (F_{p^d}^*)^k$, consider the following system of equations

$$\begin{pmatrix} X_1 & X_2 & \cdots & X_k \\ X_1^2 & X_2^2 & \cdots & X_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ X_1^k & X_2^k & \cdots & X_k^k \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \iff \begin{cases} f_1(\vec{X}) = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_k X_k & = 0 \\ f_2(\vec{X}) = \alpha_1 X_1^2 + \alpha_2 X_2^2 + \cdots + \alpha_k X_k^2 & = 0 \\ \vdots \\ f_k(\vec{X}) = \alpha_1 X_1^k + \alpha_2 X_2^k + \cdots + \alpha_k X_k^k & = 0 \end{cases}$$

How many solutions $\vec{X} \in (F_{p^d}^*)^k$ satisfying X_i's are distinct?

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Bound on the number of solutions

- Employ a contemporary Bézout-like theorem over composite order fields [Bafna-Sudan-Velusamy-Xiang-21].
  - Maji et al. used [Wooley-96] result for prime fields.
- Subtlety arises for composite order fields
  - A naive analysis would not work.

Bézout-like Theorems

Definition

Consider $f_i \in F_{p^d}[X_1, X_2, \ldots, X_k]$ of degree d_i for $1 \leqslant i \leqslant k$.
An $\vec{a} \in F_{p^d}^k$ is an *isolated zero* of the square system $\vec{f} = \vec{0}$, if $\vec{f}(\vec{a}) = \vec{0}$ but $J(\vec{f}; \vec{a}) \neq 0$.

Jacobian

$$J(\vec{f}) = \det \left(\frac{\partial f_j}{\partial X_i} \right)_{i,j \in \{1,2,\ldots,k\}} \in F_{p^d}[X_1, X_2, \ldots, X_k].$$

Theorem ([Wooley'96] for prime fields, [Zhao'12,BZXV'21] for composite order fields)

The number of isolated zeroes of the system of equations $\vec{f} = \vec{0}$ *is at most* $d_1 \cdot d_2 \cdots d_k$.

How the proof in [MNPSY'21] works?

Consider $k = 3$, $\vec{\alpha} = \vec{1}$, and a prime field F_p with large p.

$$J(\vec{f}) = \det \begin{pmatrix} 1 & 1 & 1 \\ 2X_1 & 2X_2 & 2X_3 \\ 3X_1^2 & 3X_2^2 & 3X_3^2 \end{pmatrix} = 6(X_1 - X_2)(X_2 - X_3)(X_3 - X_1)$$

$J(\vec{f}, \vec{X}) \neq 0$ iff X_i's are distinct. So #solutions = #isolated zeroes.

Illustrating Examples

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Over composite order fields

1. When $p > k = 3$, the same idea works since $J(\vec{f}, \vec{X}) \neq 0$ iff X_i's are distinct.
2. When $p = 2$, the same analysis does not work since $J(\vec{f}, \vec{a}) = 0$ for every $\vec{a}$.

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Our solution when $p = 2$

1. Remove equation with even power
2. Fix X_3 arbitrarily, consider a new system $g_1 = X_1 + X_2 + c_1$, and $g_2 = X_1^3 + X_2^3 + c_2$.

$$J(\vec{g}) = \det \begin{pmatrix} 1 & 1 \\ 3X_1^2 & 3X_2^2 \end{pmatrix} = 3(X_1 - X_2)(X_1 + X_2) = 3(X_1 - X_2)^2$$

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What if $p = 3$?

1. $J(\vec{g}, \vec{X}) = 0$ iff $X_1 = X_2$ or $X_1 + X_2 = 0$ – a new and unexpected way of making Jacobian zero.
2. $J(\vec{g})$ is a *generalized Vandermonde determinant*. We prove that the number of zeroes is small.
   - Identifying their zeroes is an open research problem in Mathematics.

Summary

Theorem (Randomized construction for composite order fields)

Random evaluation places yield leakage-resilient Shamir's scheme.

Theorem (Classifying evaluation places: a dichotomy)

1. *Against single block leakage, $(n, 2)$-Shamir is either perfectly secure or completely insecure.*
2. *Given evaluation places $(X_1, X_2, \ldots, X_n)$, our algorithm classifies them as secure or not.*

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