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The supersingular Endomorphism Ring and One Endomorphism problems are equivalent

(speaker)

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Elliptic curves

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a curve of the form

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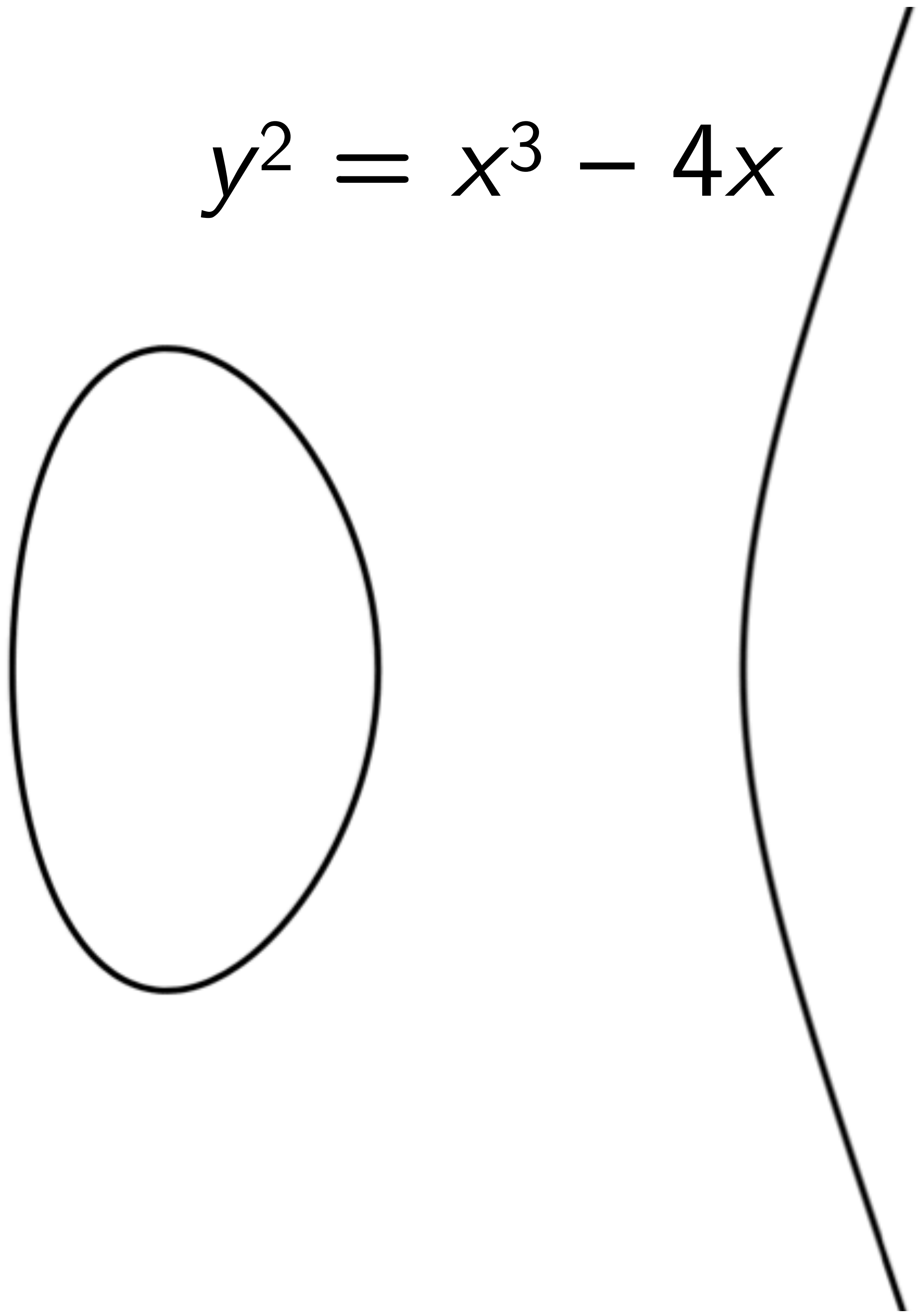

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a map between two curves

$\varphi : E_1 \rightarrow E_2$

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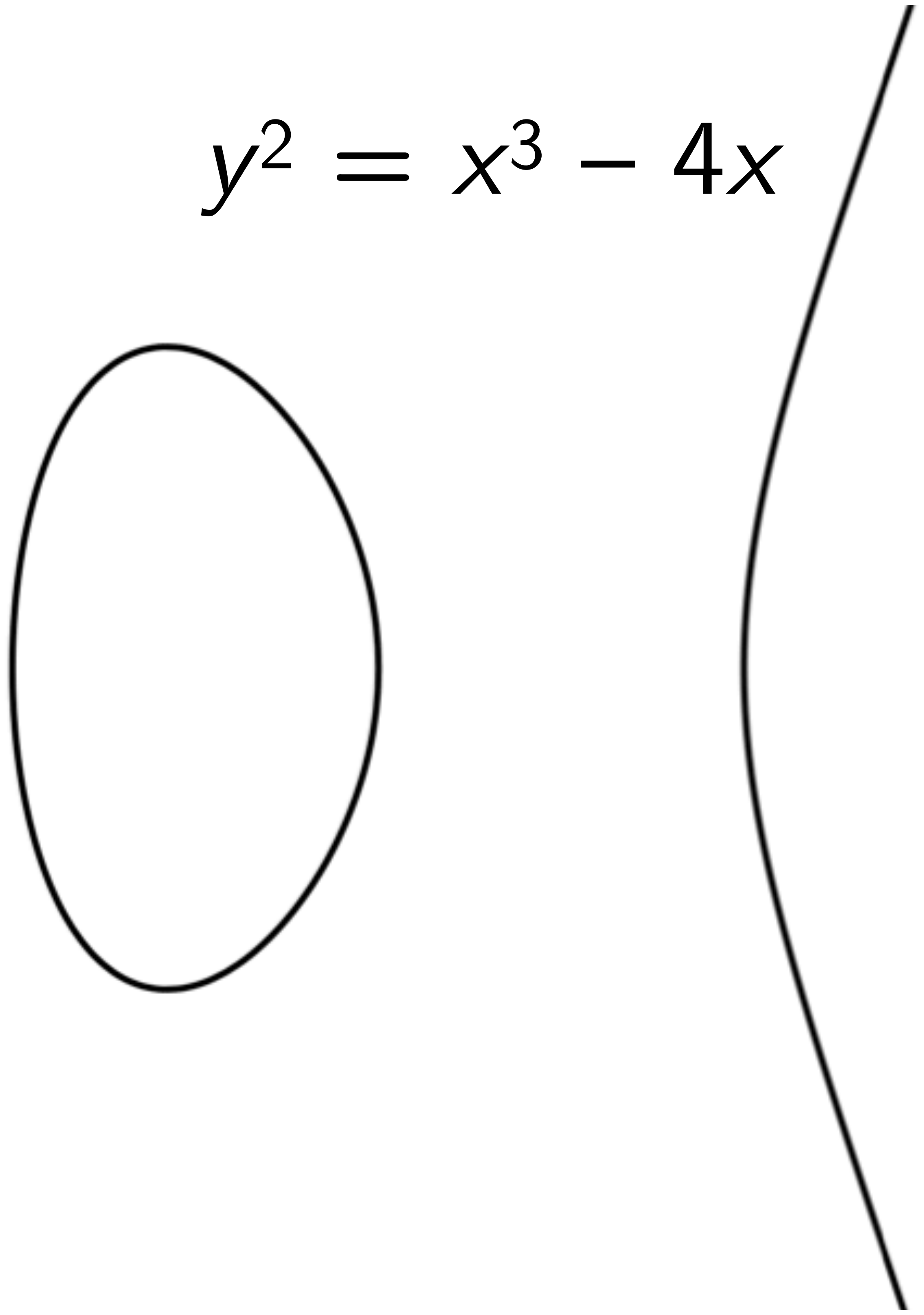

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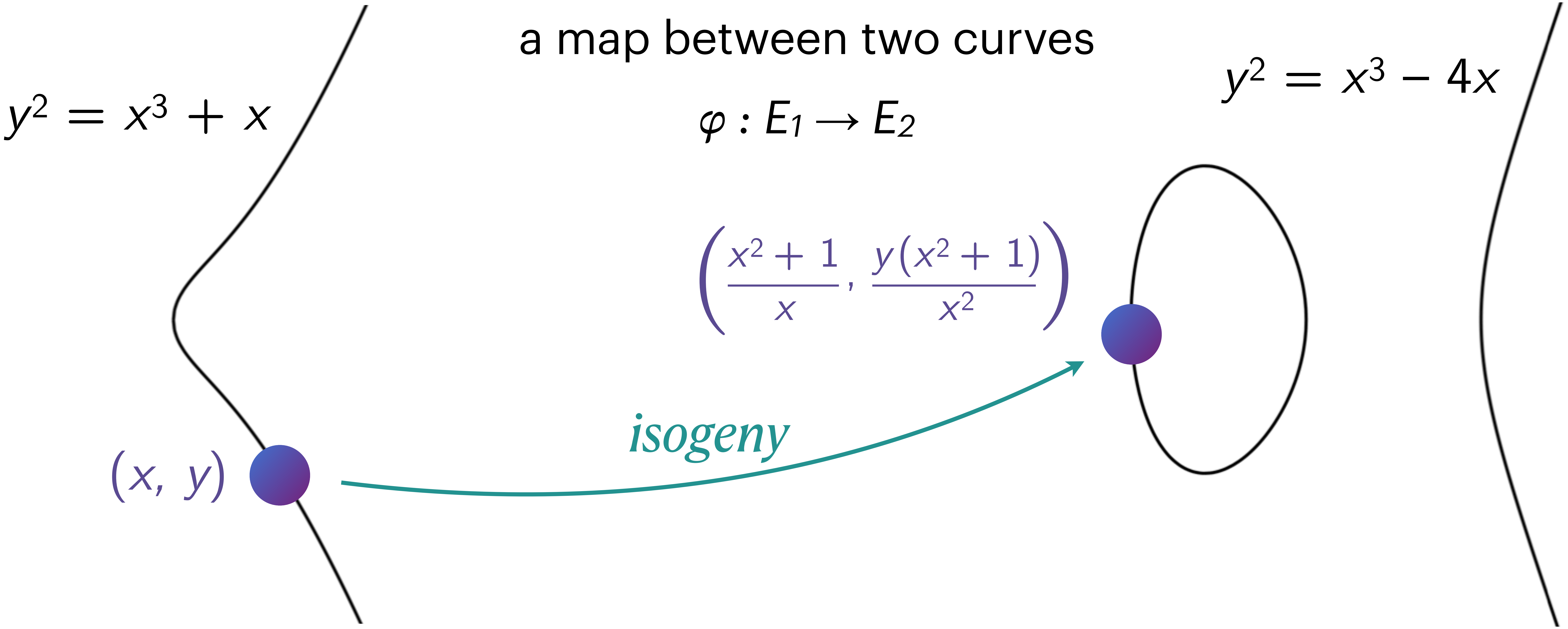

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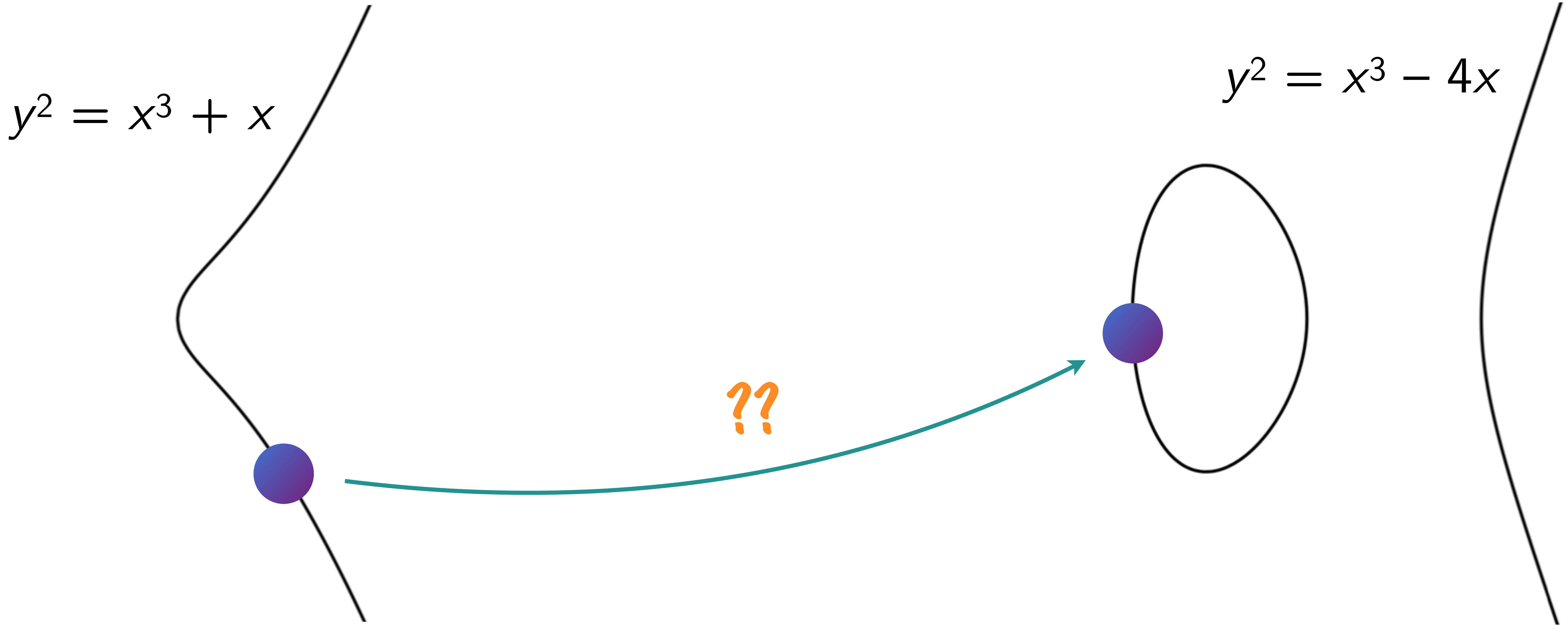
Isogeny problem: Given two elliptic curves E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$



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ℓ-IsogenyPath: Given two elliptic curves E_1 and E_2, find an $\boldsymbol{\ell}$-isogeny paths $E_1 \to E_2$

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Special kind of easy-to-work-with isogenies

Isogeny-based cryptography

Expectations: cryptosystems as secure as **ℓ-IsogenyPath** is hard

ℓ-IsogenyPath = **Security of cryptosystems**

Hard even for quantum algorithms

Post-quantum cryptography

Isogeny-based cryptography

Reality: upper and lower bounds

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Endomorphisms

OneEnd and EndRing

Endomorphism ring

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Ordinary elliptic curve

Supersingular elliptic curve

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Theorem [W. – FOCS 2021]: **EndRing** is equivalent to ℓ-**IsogenyPath**
(assuming the Generalised Riemann Hypothesis)

Earlier **heuristic** reductions in:

[Petit, Lauter – preprint 2017] *Hard and Easy Problems for Supersingular Isogeny Graphs.*

[Eisenträger, Hallgren, Lauter, Morrison, Petit – Eurocrypt 2018] *Supersingular isogeny graphs and endomorphism rings: Reductions and solutions.*

The [one] endomorphism [ring] problem

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Theorem (main result of this work): OneEnd is equivalent to **EndRing**, under probabilistic polynomial time reductions

Applications

of OneEnd = EndRing

New security reductions

Reality: upper and lower bounds

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ℓ-IsogenyPath	$=$	CGL hash function (preimage)
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Theorem (Application 1): CGL is **collision-resistant** if and only if **EndRing** is hard

Theorem (Application 2): SQIsign is **sound** if and only if **EndRing** is hard

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Previous work:

- **Isogeny** ≤ **EndRing**: already known (assuming GRH **[W. – FOCS 2021]**)
- **EndRing** ≤ **Isogeny**: only known for special case **ℓ-IsogenyPath** (assuming GRH)

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Idea of the proof: Suffices to show that **OneEnd** ≤ **Isogeny**

- Given E (an instance of **OneEnd**), sample random isogeny $\varphi : E \to F$, solve **Isogeny** to find $\psi : F \to E$, and return $\psi \circ \varphi$ (a solution of **OneEnd**)
- No need to assume GRH!

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Idea of the proof:

- By the previous application, **EndRing** ≤ **Isogeny** (unconditionally!)
- Meet-in-the-middle solves **Isogeny** with complexity $\tilde{O}(p^{1/2})$

Sketch of the proof

Main ideas and obstacles

Reducing EndRing to OneEnd

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Let E be an instance of **EndRing**: we wish to find generators of $\mathrm{End}(E)$

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Randomize the oracle…

Enriching the oracle

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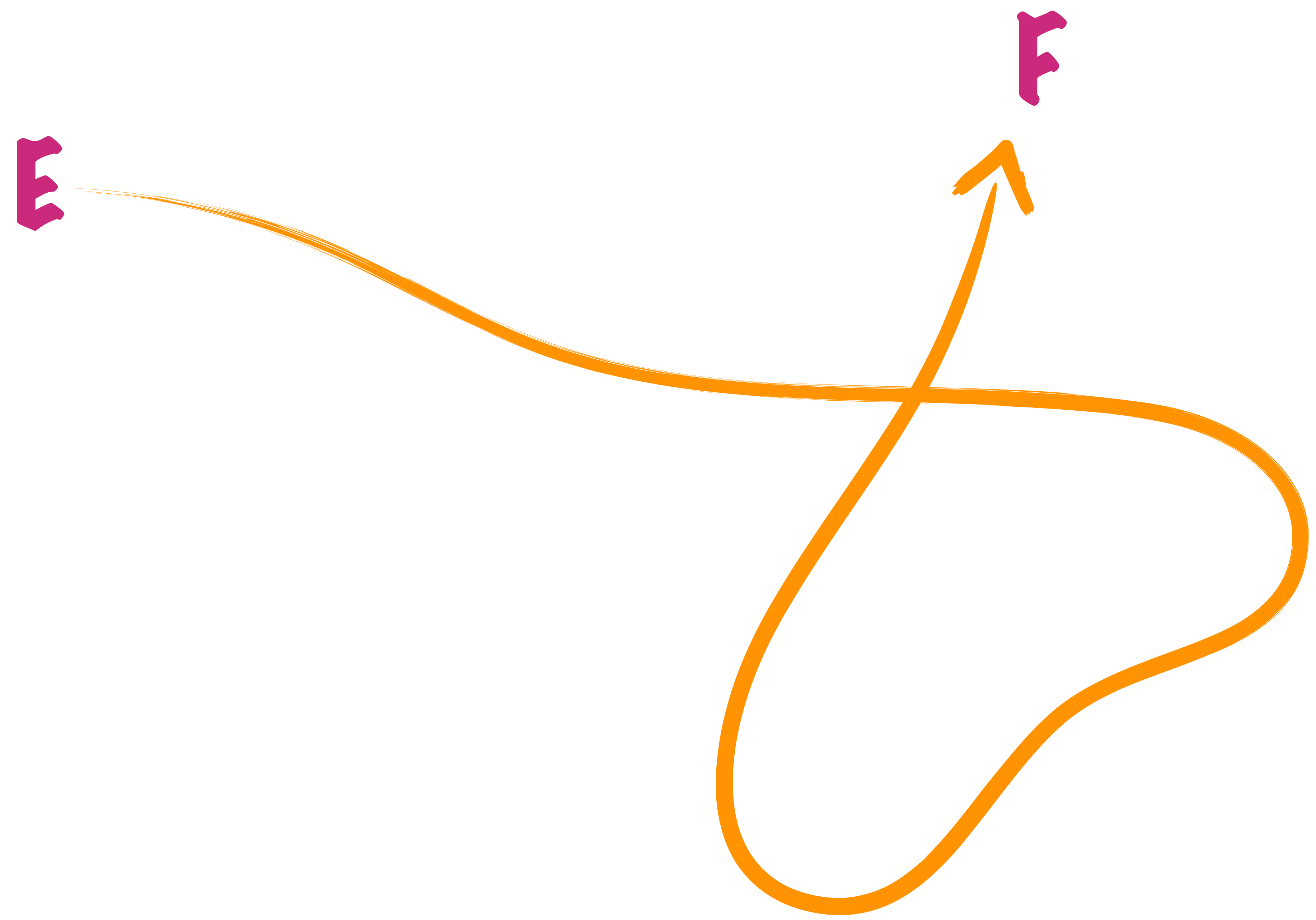

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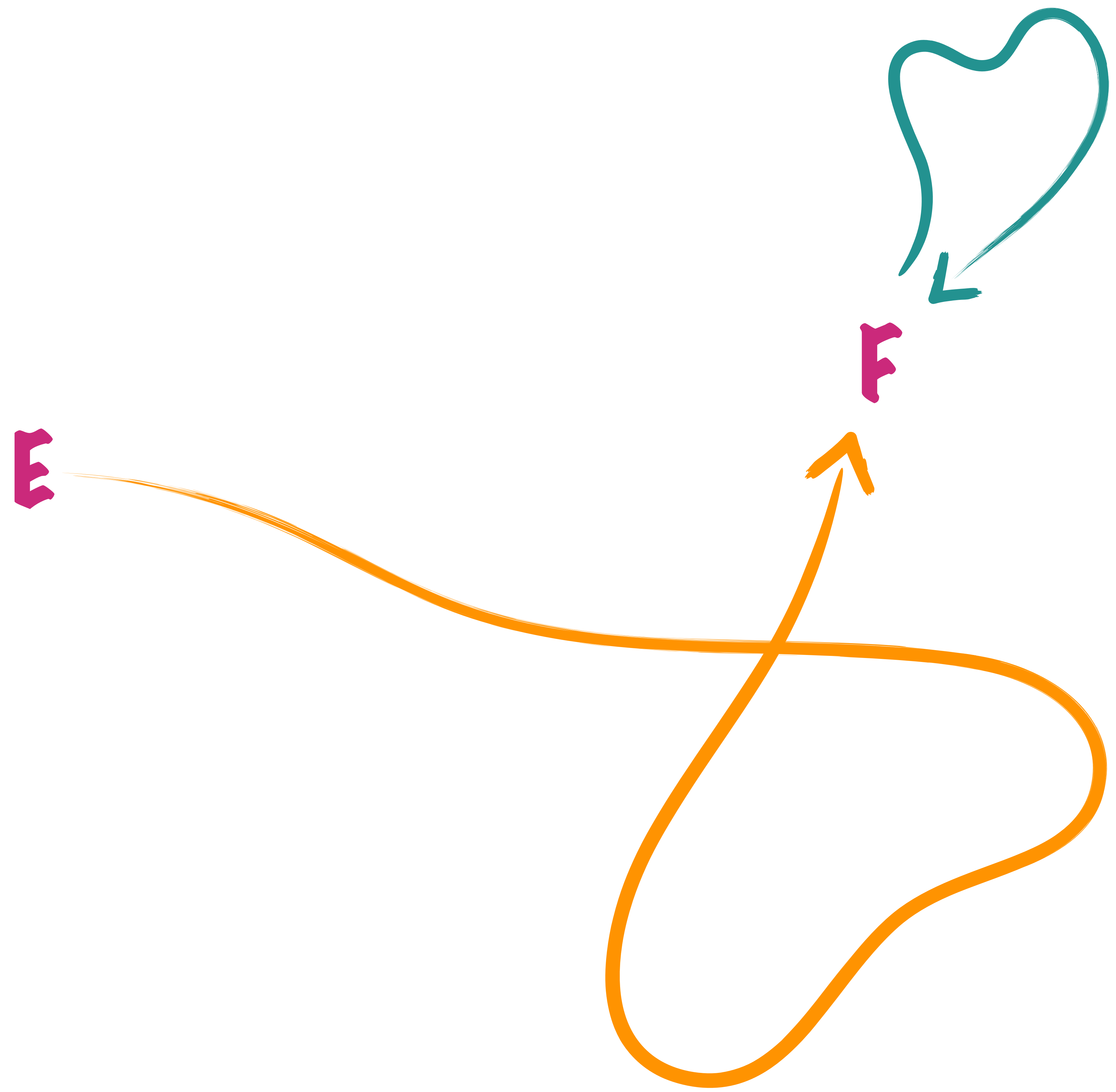

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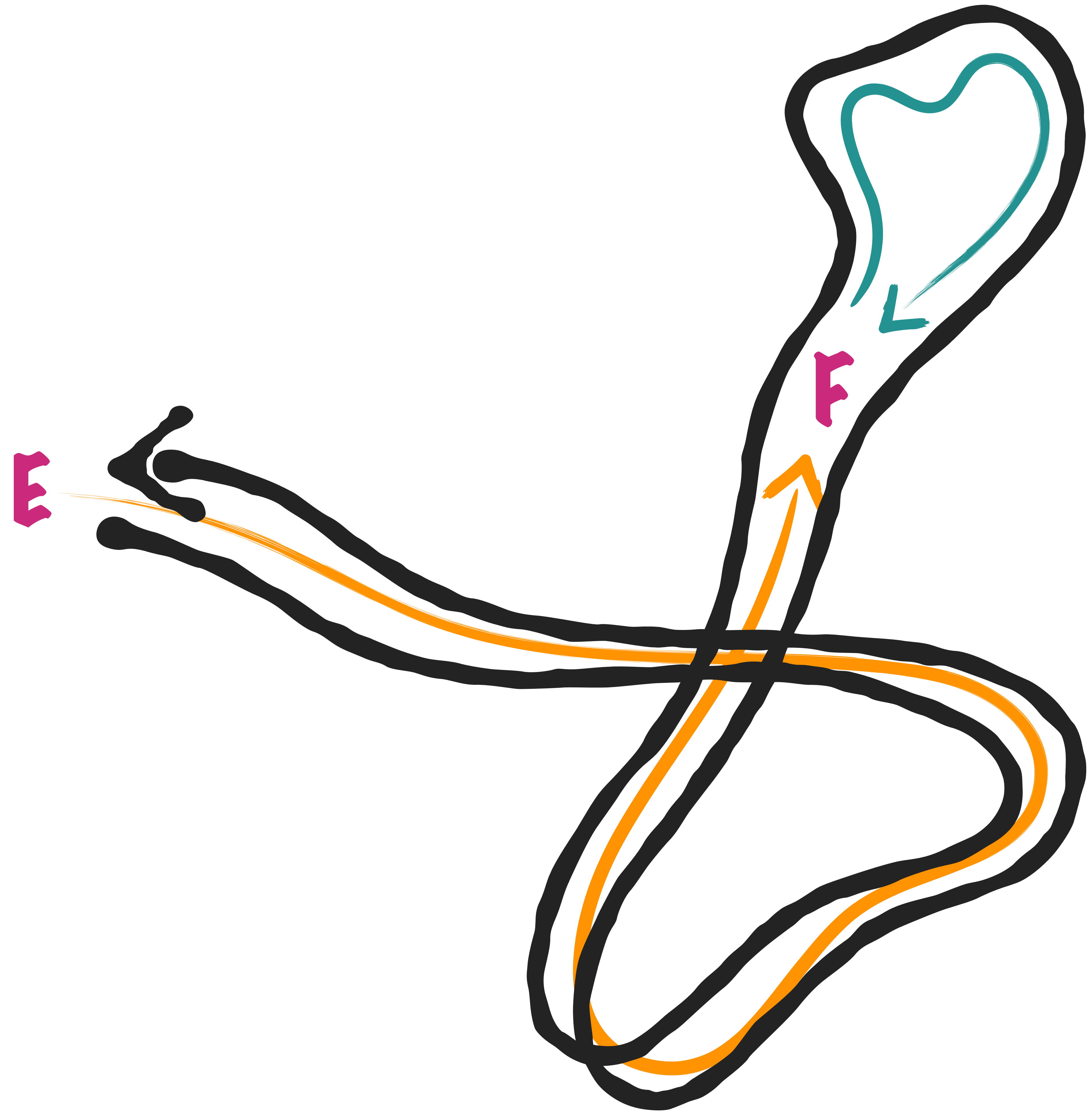

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Problem: It **fails**. There exist oracles $\mathcal{O}$ for which the algorithm does not terminate

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Deligne's bound on coefficients of modular forms

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"Eventually" = exponential time 👎

Reducing EndRing to OneEnd

Outline of the reduction:

1. Initialize $S = \{ 1 \}$
2. While S does not generate a ring of the form $\mathbb{Z} + M \cdot \mathrm{End}(E)$, do:
    3. Sample $\alpha \leftarrow$ **`Rich`**$^{\mathcal{O}}(E)$
    4. $\alpha \leftarrow$ **`LazyReduce`**(α) **(Idea 3)**
    5. Add α to S
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Polynomial time!

OneEnd to find them all