Improved Cryptanalysis of HFERP

PKC 2024

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In this talk, we provide two attacks on the multivariate cryptosystem HFERP

Hidden Field Equations (HFE)

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$\mathcal{P} = T \circ F \circ S$ where $F = \phi^{-1} \circ f \circ \phi$ and

$$f(X) = \sum_{i \leq j}^{q^i + q^j < D} \alpha_{ij} X^{q^i + q^j} + \sum_{q^i < D} \beta_i X^{q^i} + \gamma$$

$$\begin{tikzcd}
 & \mathbb{F}_{q^n} \arrow[r, "f"] & \mathbb{F}_{q^n} \arrow[d, "\phi^{-1}"] & \\
\mathbb{F}_q^n \arrow[r, "U"] & \mathbb{F}_q^n \arrow[u, "\phi"] & \mathbb{F}_q^n \arrow[r, "T"] & \mathbb{F}_q^n
\end{tikzcd}$$

Jaques Patarin, "Hidden Fields Equations (HFE) and Isomorphisms of Polynomials (IP): Two New Families of Asymmetric Algorithms." Eurocrypt (1996)

Unbalanced Oil and Vinegar

$$\mathbf{x} \in \mathbb{F}_q^n \longrightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_v \\ x_{v+1} \\ \vdots \\ x_n \end{pmatrix}$$

Patarin, "The Oil and Vinegar Signature Scheme." (1997)
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$$f^{(k)}(\mathbf{x}) = \sum_{i=1}^{v} \sum_{j=i}^{v} \alpha_{ijk} x_i x_j + \sum_{i=1}^{v} \sum_{j=v+1}^{n} \beta_{ijk} x_i x_j + \sum_{i=1}^{n} \gamma_{ik} x_i$$

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Rainbow

$$P : \mathbb{F}_q^n \to \mathbb{F}_q^{(v+1)(n-v_\ell)}, \quad P = U \circ F \circ T, \quad F = \left(F^{(1)}, F^{(2)}, \ldots, F^{(v+1)}\right)$$

$$f_\ell^{(k)}(\mathbf{x}) = \sum_{i=1}^{v_\ell} \sum_{j=1}^{v_\ell} \alpha_{ij\ell} x_i x_j + \sum_{i=1}^{v_\ell} \sum_{j=v_\ell+1}^{n} \beta_{ij\ell} x_i x_j + \sum_{i=1}^{n} \gamma_{i\ell} x_i + \delta_\ell$$

$$0 < v_1 < v_2 < \ldots < v_L < n$$

Ding, Schmidt, "Rainbow, a New Multivariable Polynomial Signature Scheme." Applied Cryptography and Network Security (2005)

Plus Modifier

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}$$

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$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \longrightarrow P^+ = \begin{pmatrix} p_1 \\ \vdots \\ p_m \\ p_{m+1} \\ \vdots \\ p_{m+a} \end{pmatrix}$$

HFERP

- Multivariate Encryption Scheme composed of:
  - An instance of HFE
  - A single layer Rainbow map
  - A plus modifier

Ikematsu, Perlner, Smith-Tone, Takagi, Vates, "HFERP - A New Multivariate

Structure of HFERP

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$$n = o + d$$
$$m = o + d + s + r$$

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$$\mathbb{F}_q^n \xrightarrow{U} \mathbb{F}_q^n$$

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$$\begin{tikzcd}
& & \mathbb{F}_{q^d} \arrow[r, "\mathcal{F}_{HFE}"] & \mathbb{F}_{q^d} \arrow[d, "\phi^{-1}"] \\
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\end{tikzcd}$$

$$P : \mathbb{F}_q^n \to \mathbb{F}_q^m \quad P = T \circ (\mathcal{F}_{HFE} || \mathcal{F}_R || \mathcal{F}_P) \circ U$$

MinRank Attacks

MinRank Problem: Given $A_1, \ldots, A_k \in \mathbb{F}_q^{M \times N}$ and positive integer R, find $y_1, \ldots, y_k \in \mathbb{F}_q$ such that

$$\text{rank}\left(\sum_{i=1}^{k} y_i A_i\right) \leq R.$$

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A Few MinRank Algorithms:

- Kipnis, Shamir, "Cryptanalysis of the HFE Public Key Cryptosystem by Relinearization." Crypto, (1999)
- Faugére, Levy-dit-Vehel, Perret, "Cryptanalysis of Minrank." Crypto (2008)
- Bardet, Bros, Cabarcas, Gaborit, Perlner, Smith-Tone, Tillich, Verbel, "Improvements of Algebraic Attacks for Solving the Rank Decoding and Minrank Problems." Asiacrypt (2020)

HFE Central Maps

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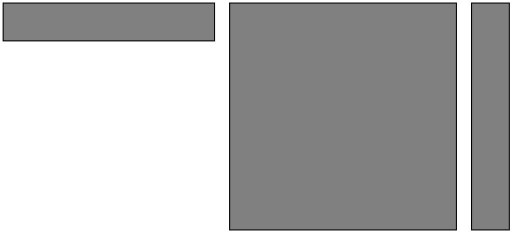

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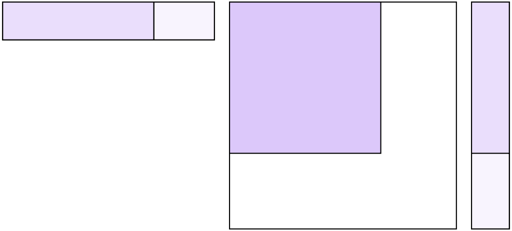

UOV Central Maps

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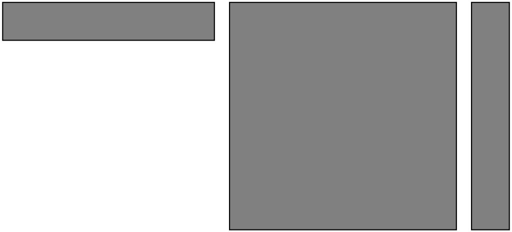

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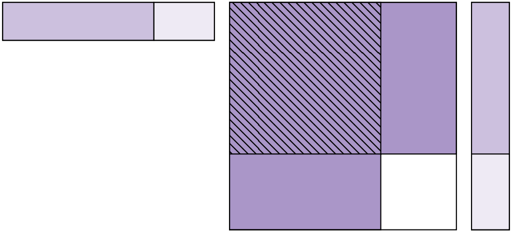

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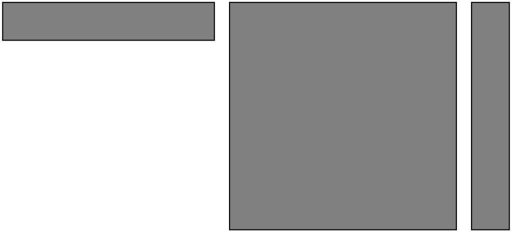

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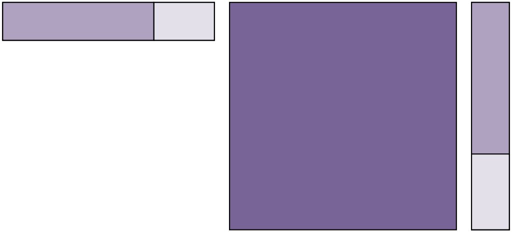


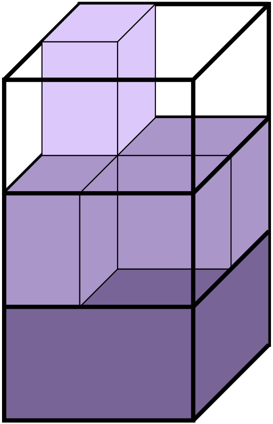
Simple Attack of Rainbow

- Find a layer two oil vector y by solving:

$$\begin{cases} P(y) = 0 \\ D_x(y) = 0 \end{cases}$$

Beullens, "Breaking rainbow takes a weekend on a laptop." Crypto (2022)

HFERP Central Maps



Simple Attack on HFERP

Finding y

- Let P_i be the matrix representation of the i^{th} public key polynomial
- Choose a random $\mathbf{z} \in \mathbb{F}_q^n$ and create the $m \times n$ matrix

$$A_{\mathbf{z}} = \begin{bmatrix} \mathbf{z}P_1 \\ \mathbf{z}P_2 \\ \vdots \\ \mathbf{z}P_m \end{bmatrix}$$

- Goal: Find a vector $\mathbf{y} \in \mathbb{F}_q^m$ such that

$$\begin{cases} \mathbf{y} \in Ker_L(A_{\mathbf{z}}) \\ Rank(\sum_{i=1}^{m} y_i P_i) \leq d. \end{cases}$$

Simple Attack on HFERP

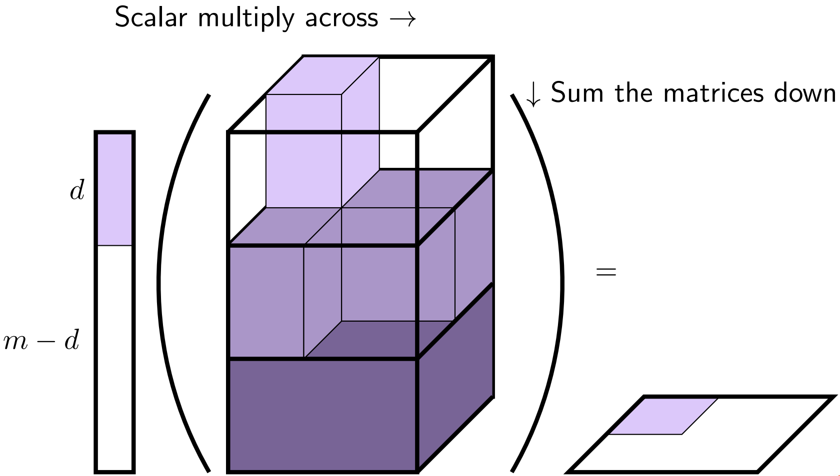
Finding y

If we can find a $\mathbf{y} \in \mathbb{F}_q^m$, then with high probability,

$$\mathbf{yT} = (\mathbf{a}||\mathbf{b}) \implies \mathbf{y} = (\mathbf{a}||\mathbf{b})\mathbf{T}^{-1},$$

where

- $\mathbf{a}$ is a length d vector in the kernel of the upper left $d \times d$ block of $A_{\mathbf{z}}$
- $\mathbf{b}$ is the length $m - d = o + r + s$ zero vector.


Scalar multiply across $\rightarrow$
$\downarrow$ Sum the matrices down
d
$m - d$
$=$

Finding the Oil and Vinegar Spaces

Simple Attack on HFERP: Inverting U

- Once we have a valid $\mathbf{y}$, compute the $n \times n$ matrix

$$P_{\mathbf{y}} = \sum_{i=1}^{m} y_i P_i.$$

- Note:

$$\text{rank}(P_{\mathbf{y}}) \leq d$$
$$\text{nullity}(P_{\mathbf{y}}) \geq o$$

- Recall,

$$\text{rank}(F_{HFE}^{(i)}) = d$$
$$\text{nullity}(F_{HFE}^{(i)}) = o$$

- Use basis vectors of $\text{Ker}(P_y)$, extend to a basis of $\mathbb{F}_q^n$.

Finding the Oil and Vinegar Spaces

Simple Attack on HFERP: Inverting U

We obtain the block matrix

$$\widehat{\mathbf{B}}\mathbf{P}_{\mathbf{y}}\widehat{\mathbf{B}}^{\top} = \begin{bmatrix} \alpha_i \mathbf{P}_{\mathbf{y}} \alpha_j^{\top} & 0 \\ 0 & 0 \end{bmatrix}$$

where $i, j \leq d$.

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Observe:

- Only the upper left $d \times d$ block has nonzero entries

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Conclusion: We found equivalent U^{-1}

Inverting T

Simple Attack on HFERP

Consider the matrix representation of public key as composition of the secret keys.

$$\mathcal{P} = \mathcal{T} \circ \begin{bmatrix} F_{HFE} \\ F_R \\ F_p \end{bmatrix} \circ \mathcal{U}$$

$$\begin{bmatrix} p_1(\mathbf{x}) \\ \vdots \\ p_m(\mathbf{x}) \end{bmatrix} = \mathcal{T} \circ \begin{bmatrix} \mathbf{x} U F_1 U^\top \mathbf{x}^\top \\ \vdots \\ \mathbf{x} U F_m U^\top \mathbf{x}^\top \end{bmatrix}$$

$$\implies P_i = \sum_{k=1}^{m} t_{ik} \mathbf{U} \mathbf{F}_k \mathbf{U}^\top = \mathbf{U} \left(\sum_{k=1}^{m} t_{ik} \mathbf{F}_k \right) \mathbf{U}^\top$$

Inverting T

$$\widehat{\mathbf{B}} P_i \widehat{\mathbf{B}}^\top = \widehat{\mathbf{B}}\mathbf{U}\left(\sum_{k=1}^{m} t_{ik}\mathbf{F}_k\right)\mathbf{U}^\top\widehat{\mathbf{B}}^\top \cong \sum_{k=1}^{m} t_{ik}\mathbf{F}_k$$

Inverting T

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Each $\widehat{\mathbf{B}}P_i\widehat{\mathbf{B}}^\top$ is now seen to be a linear combination of the m central maps.

Structure of each F_k

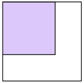
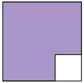
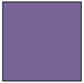
	$1 \leq k \leq d$	$d+1 \leq k \leq o+r$	$o+r+1 \leq k \leq m$
Structure of $\mathbf{F}_k$			
Rank$(\mathbf{F}_k) \leq$	d	n	n
Guaranteed zeros	Rows $d \leq i \leq m$ Columns $d \leq j \leq m$	Lower right $o \times o$ submatrix	None

Table: The table summarizes notable properties of the symmetric matrices corresponding to the F_{HFE}, F_R, and F_P polynomials.

Divide and Conquer

- We now have equivalent keys for every part of the secret map.
- Invert the HFE maps using Berlekamp's algorithm.
- We know O and V, so invert the UOV maps as they are linear in the oil variables.

Updated Security

Table: Complexity of the Simple Attack on proposed parameters of HFERP, where $D = 3^7 + 1$ is the degree bound of the HFE central map polynomial.

(q, d, o, r, s)	Claimed Sec	Simple Attack	MinRank Type
$(3, 42, 21, 15, 17)$	80 bit	$3 \cdot 3^{32} \cdot 63^{\omega} \approx 2^{69}$	Search
$(3, 63, 21, 11, 10)$	80 bit	$3 \cdot 3^{21} \cdot 84^{\omega} \approx 2^{52}$	Search
$(3, 60, o_i = 40, r_i = 23, 40)$	128 bit	$3 \cdot 3^{59} (86^{\omega} + 140^{\omega}) \approx 2^{115}$	Lin Alg

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(q, d, o, r, s)	Degree Bound	Claimed Sec	Update Comp	Algth Type
$(3, 42, 21, 15, 17)$	$D = 3^7 + 1$	80 bit	2^{57}	Wiedemann
$(3, 63, 21, 11, 10)$	$D = 3^7 + 1$	80 bit	2^{59}	Strassen
$(3, 85, o_i = 70, r_i = 89, 61)$	$D = 3^9 + 1$	128 bit	2^{63}	Strasssen
$(3, 60, o_i = 40, r_i = 23, 40)$	$D = 3^9 + 1$	128 bit	2^{69}	Wiedemann

Baena, Briaud, Cabarcas, Perlner, Smith-Tone, Verbel, "Improving Support-Minors Rank Attacks: Applications to GeMSS and Rainbow." (2021)

Thank you for your attention!
Any questions?

https://github.com/maxcartor/HFERP-Cryptanalysis

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