Reducing the Share Size of Weighted Threshold Secret Sharing Schemes via Chow Parameters Approximation

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Universitat Rovira i Virgili

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GOBIERNO DE ESPAÑA — MINISTERIO PARA LA TRANSFORMACIÓN DIGITAL Y DE LA FUNCIÓN PÚBLICA — SECRETARÍA DE ESTADO DE DIGITALIZACIÓN E INTELIGENCIA ARTIFICIAL

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Introduction: Secret Sharing Schemes

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Cryptographic primitive that allows a dealer to **share** a secret among a set of parties, so that only **authorized** subsets of them can **recover** it.

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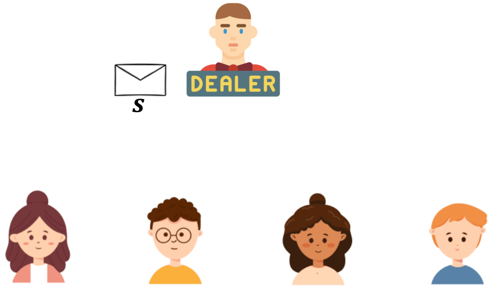

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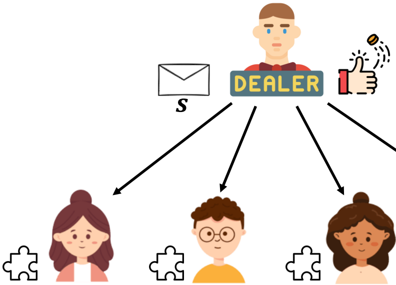

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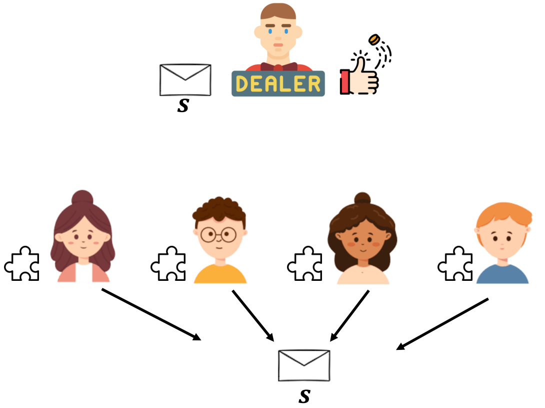

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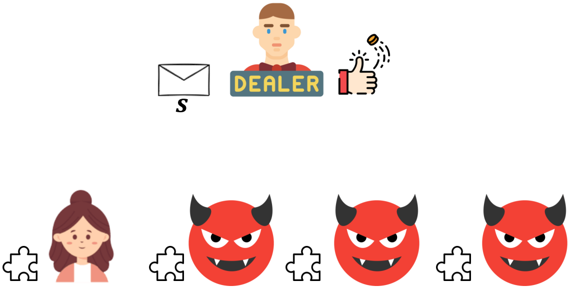

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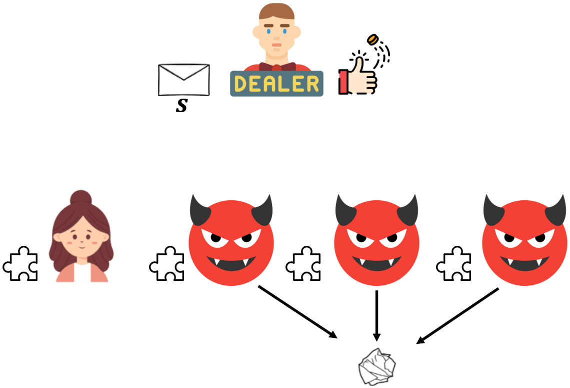

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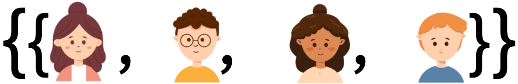

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$f($  ,  ,  ,  $) = 1$

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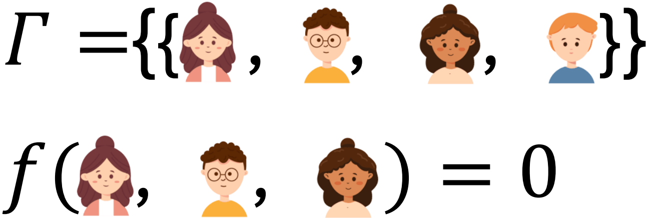

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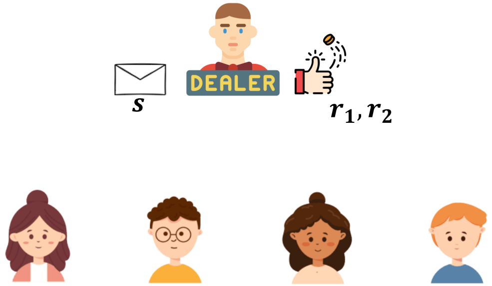

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$$p(x) = s + r_1 x + r_2 x^2$$

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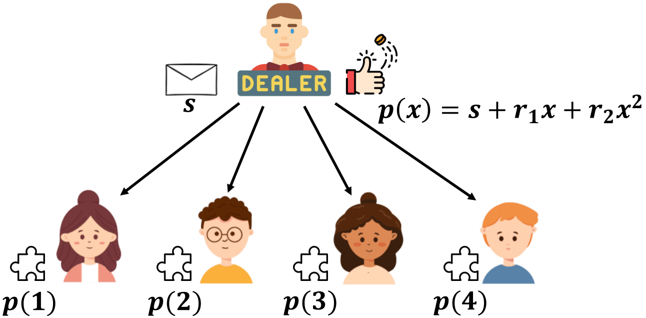

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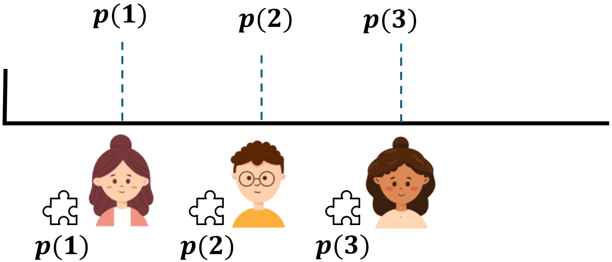

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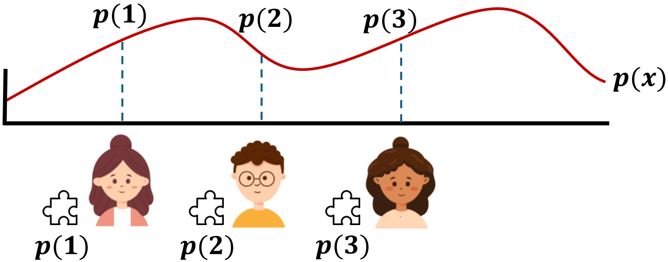

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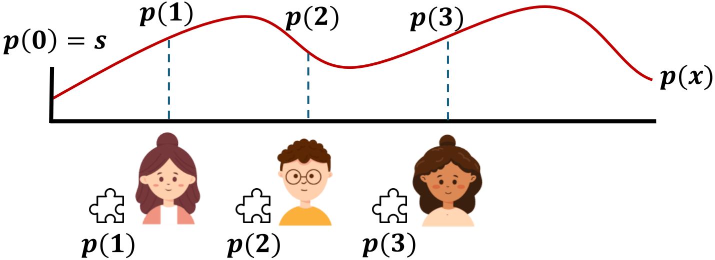

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Theorem [Beimel, Weinreb'06]

Weighted threshold access structures admit schemes with share size $\boldsymbol{n^{O(\log n)}}$.

Introduction: Weighted Threshold State-of-the-Art

Problem

Multiparty computation protocols and consensus mechanisms require efficient schemes for weighted threshold access structures.

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$(t, T)-$Ramp Weighted Threshold Access Structure

Generalization in which subsets **below** $\boldsymbol{t}$ are forbidden and subsets **above** $\boldsymbol{T}$ are authorized.

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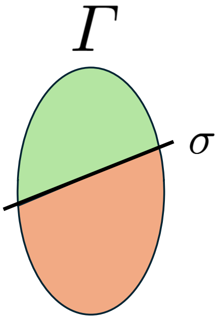
Generalization in which subsets **below t** are forbidden and subsets **above T** are authorized.

Work	Share Size	Access structure
GJMSWZ'23	$w_i = 2^{O(n \log n)}$	$(t, t + \Omega(\lambda))$-ramp WTAS
BHS'23 Rounding, TF'24	$O\left(\frac{n}{\beta-\alpha}\right)$	$(\alpha W, \beta W)$−ramp WTAS
BHS'23 BS Channels	$\max\left\{\lambda^2, \text{poly}\left(\frac{1}{\beta-\alpha}\right)\right\}$	$(\alpha W, \beta W)$−ramp WTAS

Introduction: Accuracy of the Approximation



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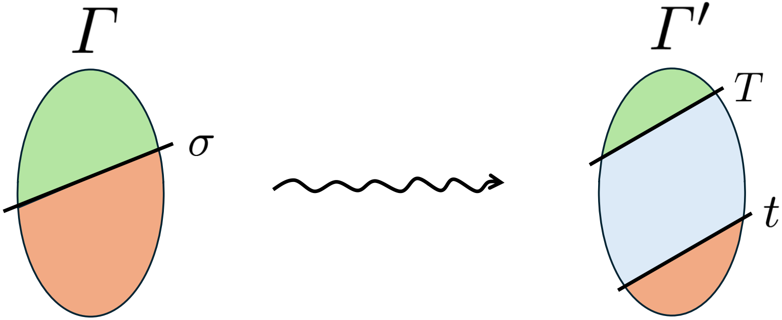

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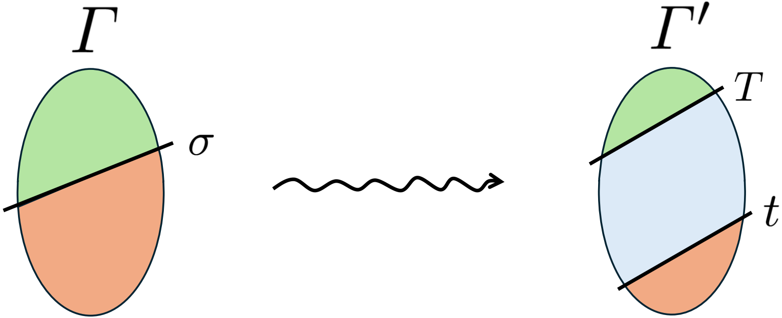
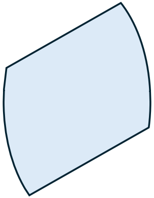


Question: How many subsets fall into  ?

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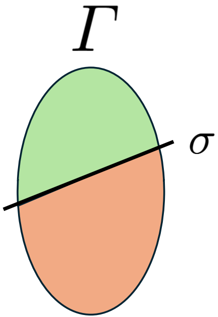

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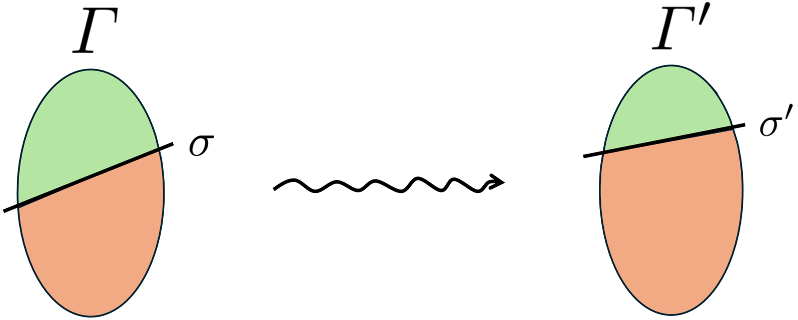

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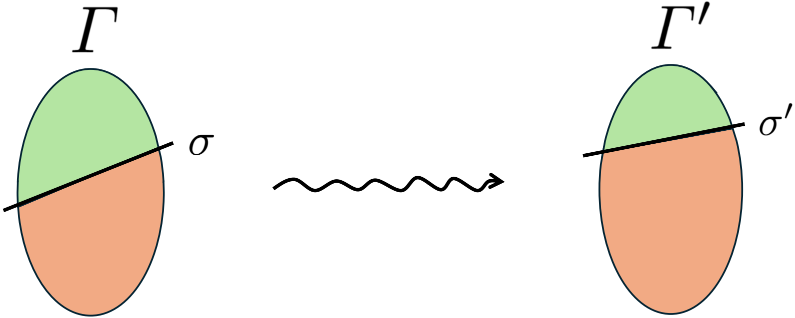

Question: How many subsets change their condition from Γ to Γ'?

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Goal: To find a good tradeoff between the efficiency and the accuracy of approximation.

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Theorem

For any weighted threshold access structure Γ, there exists a secret sharing scheme with share size $\boldsymbol{n^{1+o(1)}}$ whose weighted threshold access structure Γ' is $\boldsymbol{o(1)}$-close to Γ.

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$$\Gamma \xrightarrow{\varepsilon\text{-close}} \Gamma'$$

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f & & &
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Analysis of Boolean Functions: Chow Parameters & Distance

Chow Parameters

The $n+1$ values $\hat{f}(0) = \mathbf{E}[f(x)]$ and $\hat{f}(i) = \mathbf{E}[f(x)x_i]$ taking uniform distribution on its domain. The Chow vector is given by $\chi_f = (\hat{f}(0), \ldots, \hat{f}(n))$.

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Theorem [Chow'61]

Any WTF is **uniquely determined** in the space of Boolean functions by its Chow parameters.

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Theorem [O'Donnell, Servedio'11 + De, Diakonikolas, Feldman, Servedio'14]

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Idea

To use the Chow parameters as the **building block** for approximating WTFs.

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Theorem

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- The approximation preserves the influence of the coordinates and the weight hierarchy.

Approximation of Weighted Threshold Functions: The Algorithm

$$f(x) = \text{sign}(w_1x_1 + ... + w_nx_n - t)$$

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$$\downarrow$$

$$\chi_f$$

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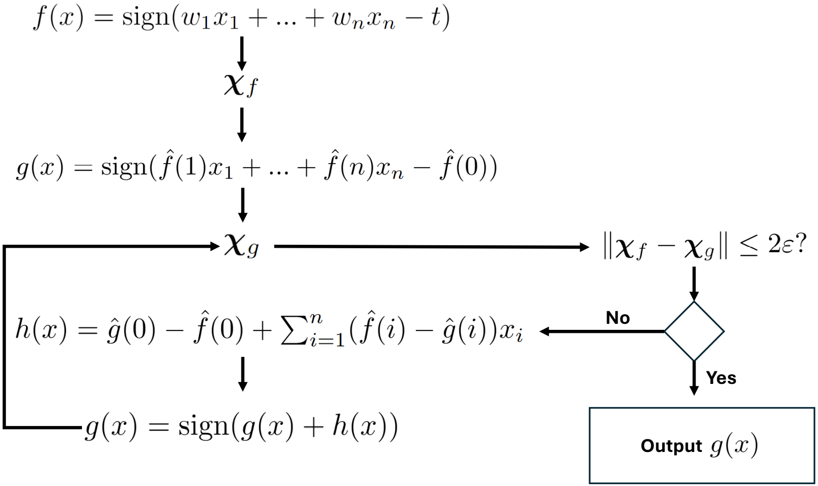

Secret Sharing Schemes Construction

Theorem

For any weighted threshold access structure Γ, there exists a secret sharing scheme with share size $\boldsymbol{n^{1+o(1)}}$ whose access structure Γ' is **$\boldsymbol{o(1)}$-close** to Γ.

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Side Note

Efficient schemes for **any** weighted threshold access structure can be build from **computational assumptions**.

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Thank You!

Check the Full Version of our Paper!