Quantum Pseudorandom Scramblers

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Pseudorandomness

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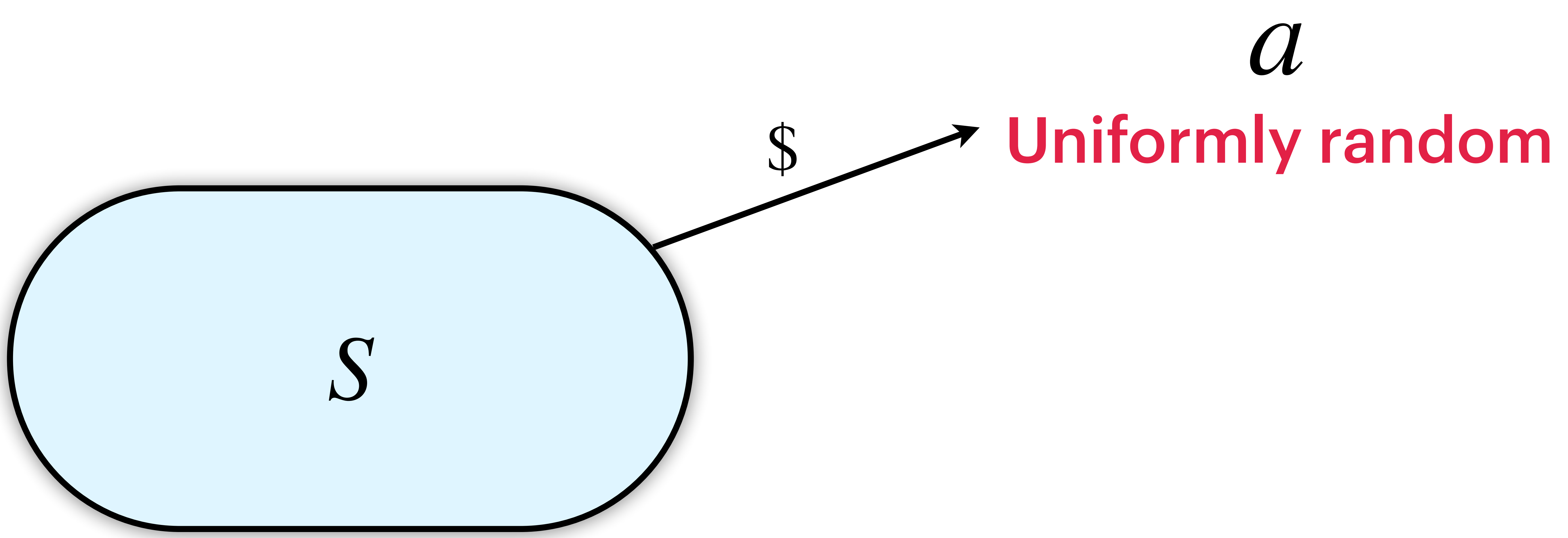

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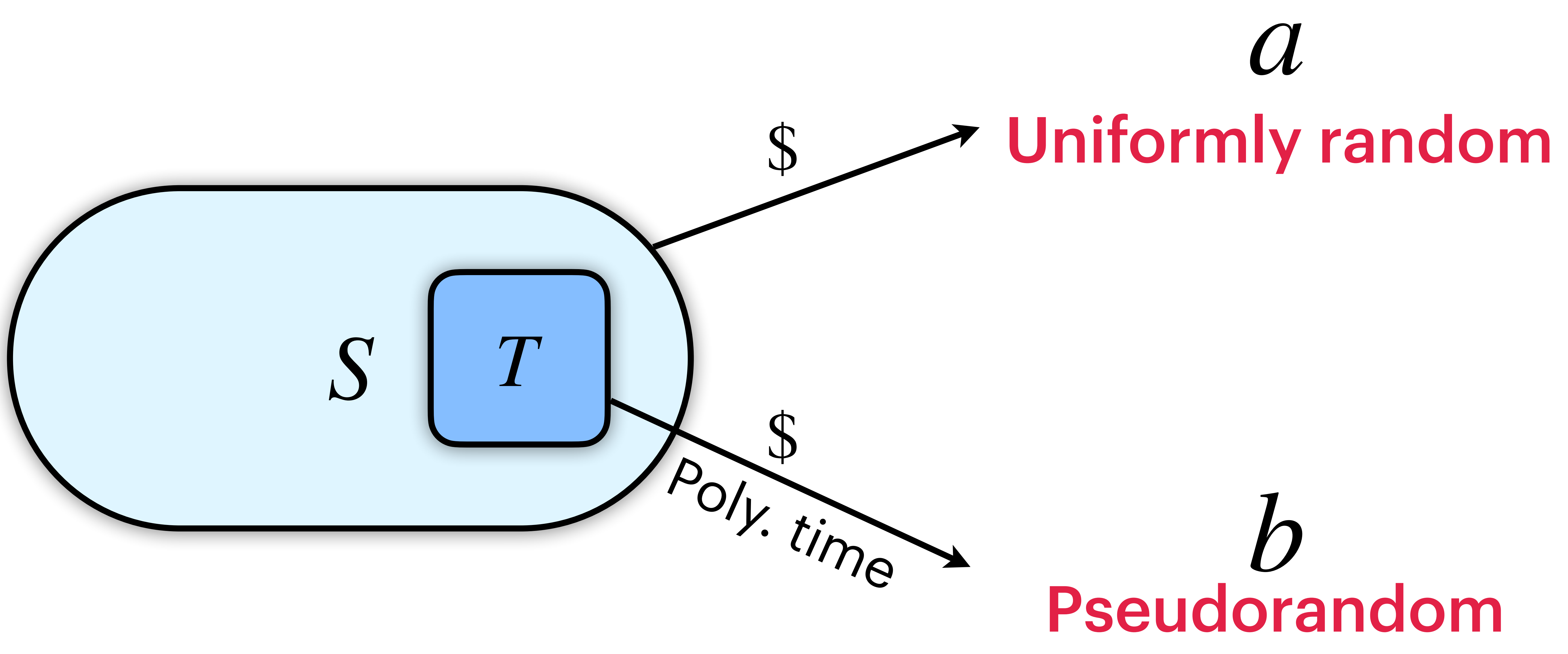

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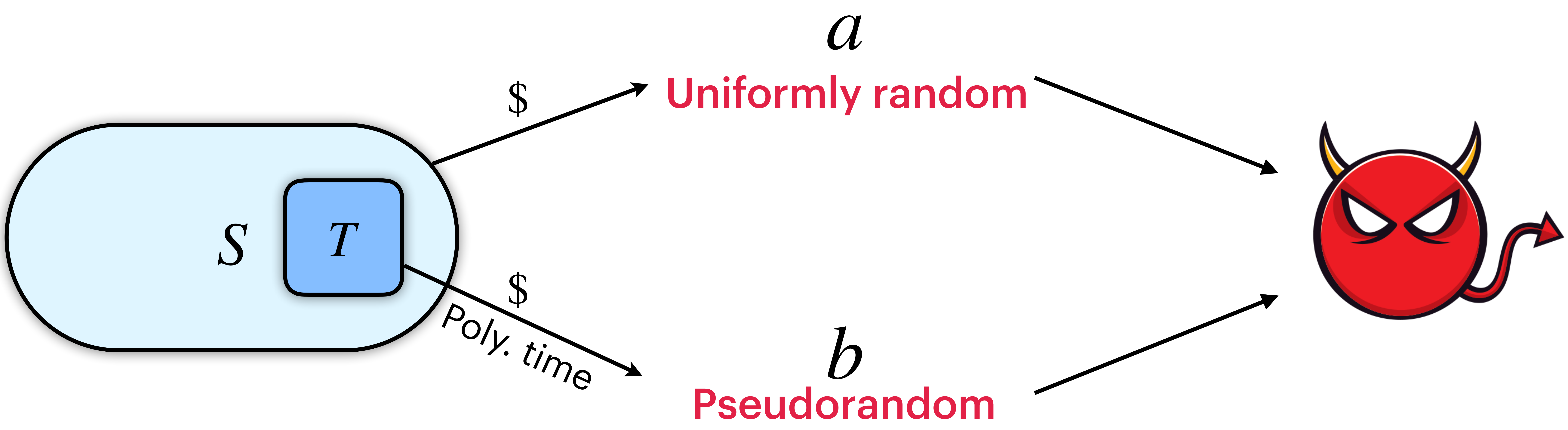

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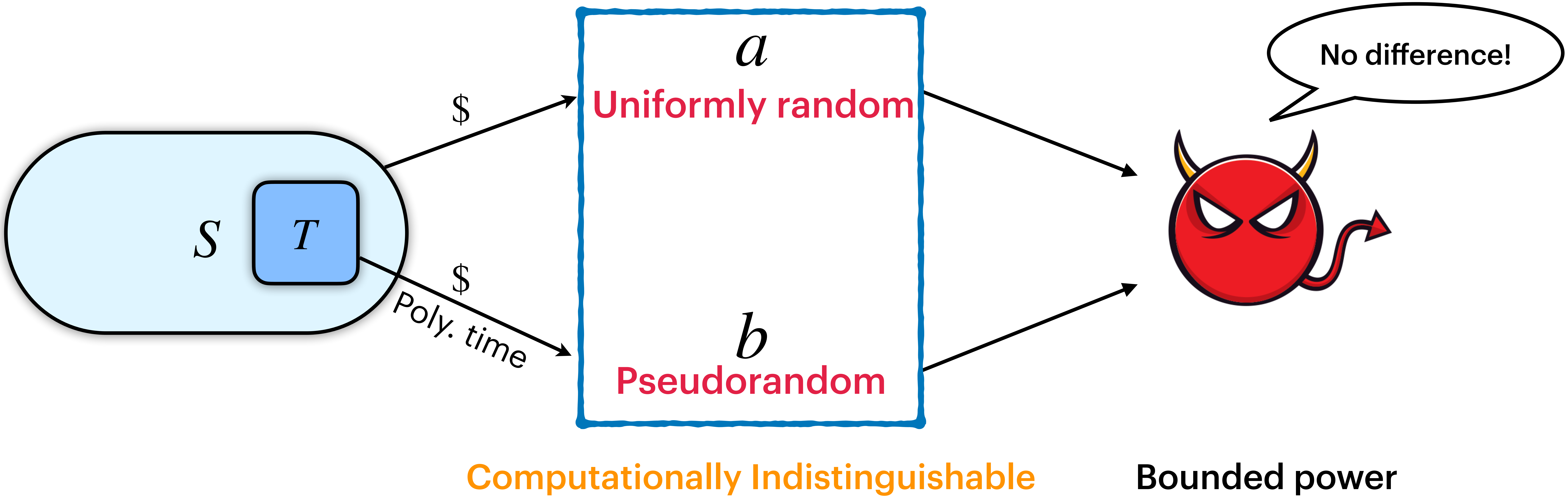

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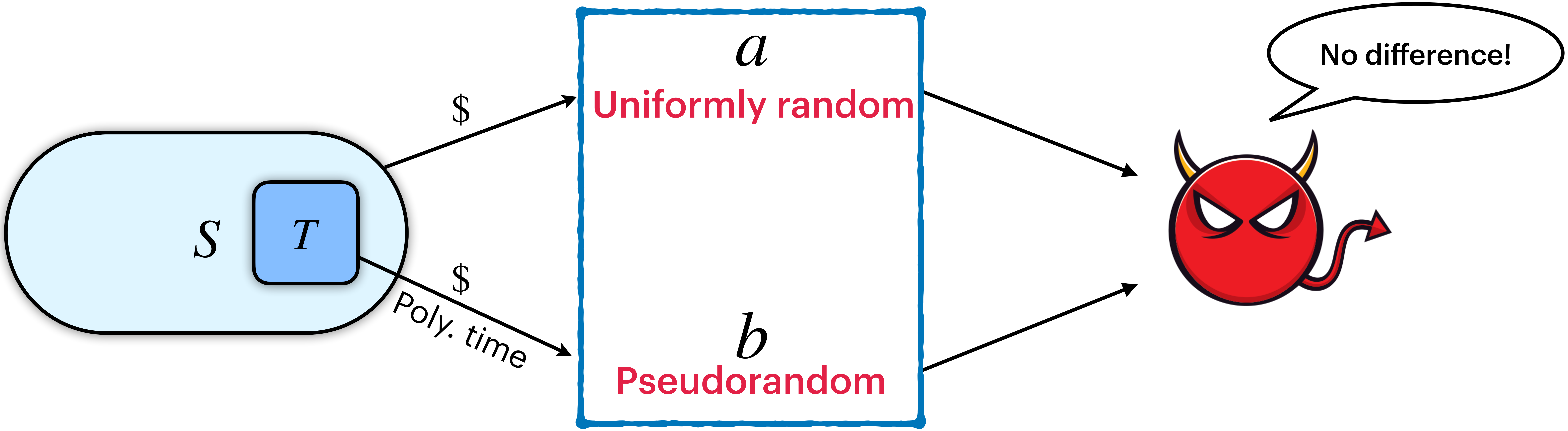

Computationally Indistinguishable — Bounded power

Statistically Indistinguishable — Unlimited power

Quantum Pseudorandomness

Pseudorandom State Generator (PRSG) [JLS18]

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Comp. indist. given any poly. copies

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PRSGs exist assuming the existence of QPRFs. [JLS18, BS19]

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Comp. indist. given any poly. copies

$$|\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} (-1)^{\mathrm{PRF}_k(x)} |x\rangle$$

$$|\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} \omega_N^{\mathrm{PRF}_k(x)} |x\rangle$$

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Applications:

- Quantum money [JLS18]
- Digital signature [MY22]
- Data encryption [AGQY23]
- Quantum bit commitment [AQY22, AGQY23]
- Quantum trapdoor function [Col23]
- Quantum gravity theory [BFV20]

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Haar random unitary

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- Quantum analogue of PRFs.
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Construction (from OWFs)?

An open problem until very recently.

Quantum Pseudorandomness

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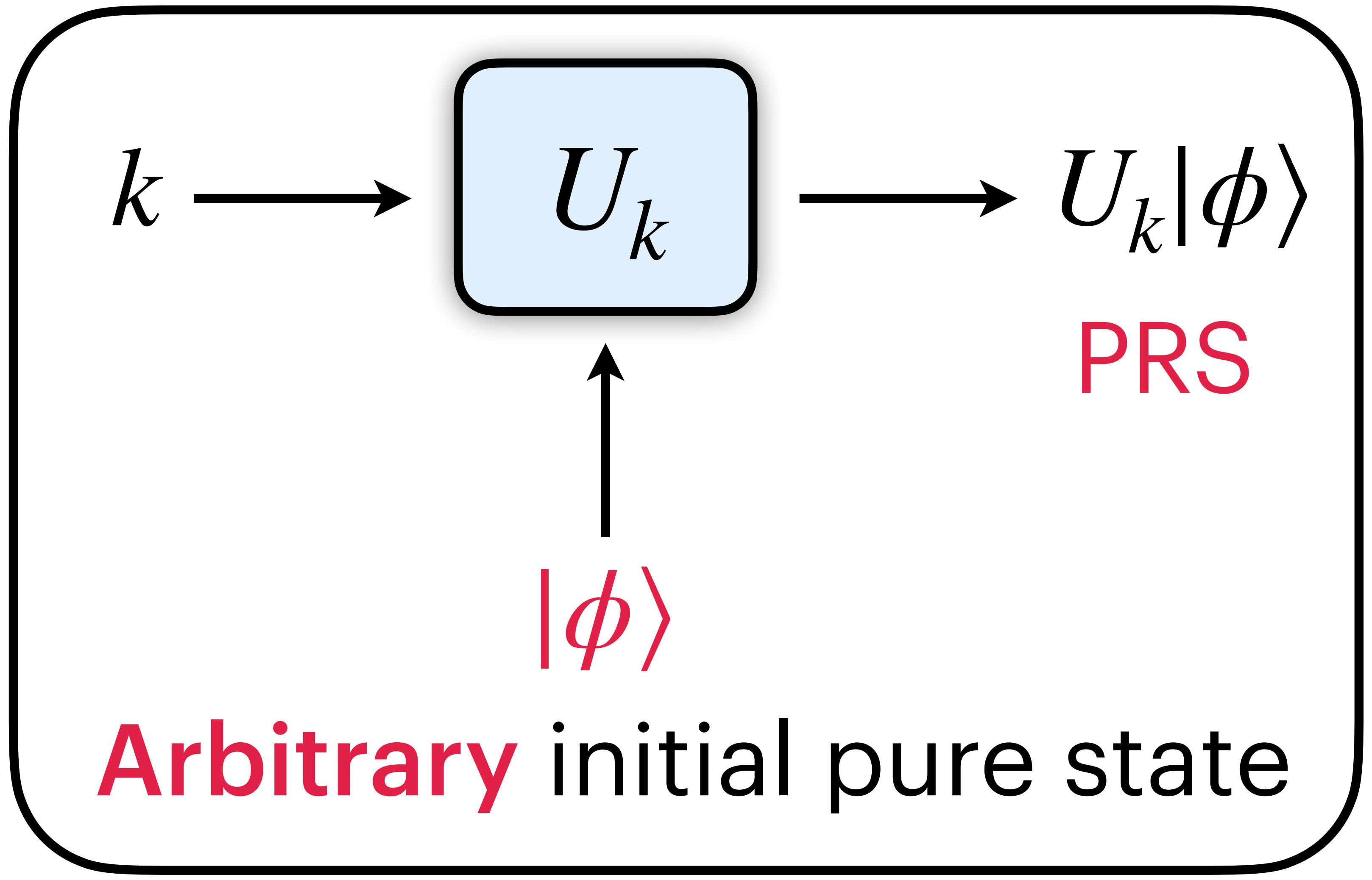
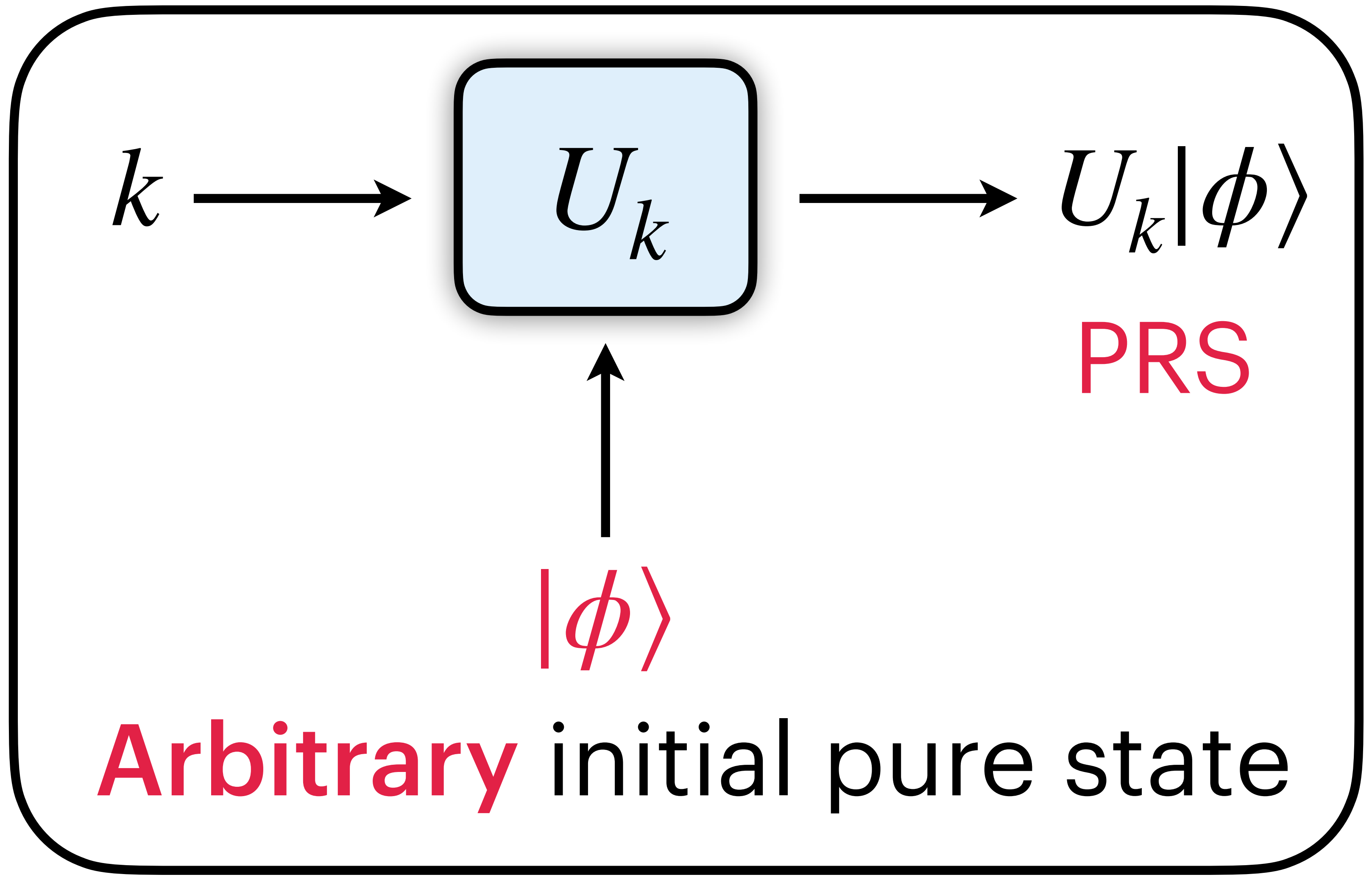


PRU

Quantum Pseudorandomness



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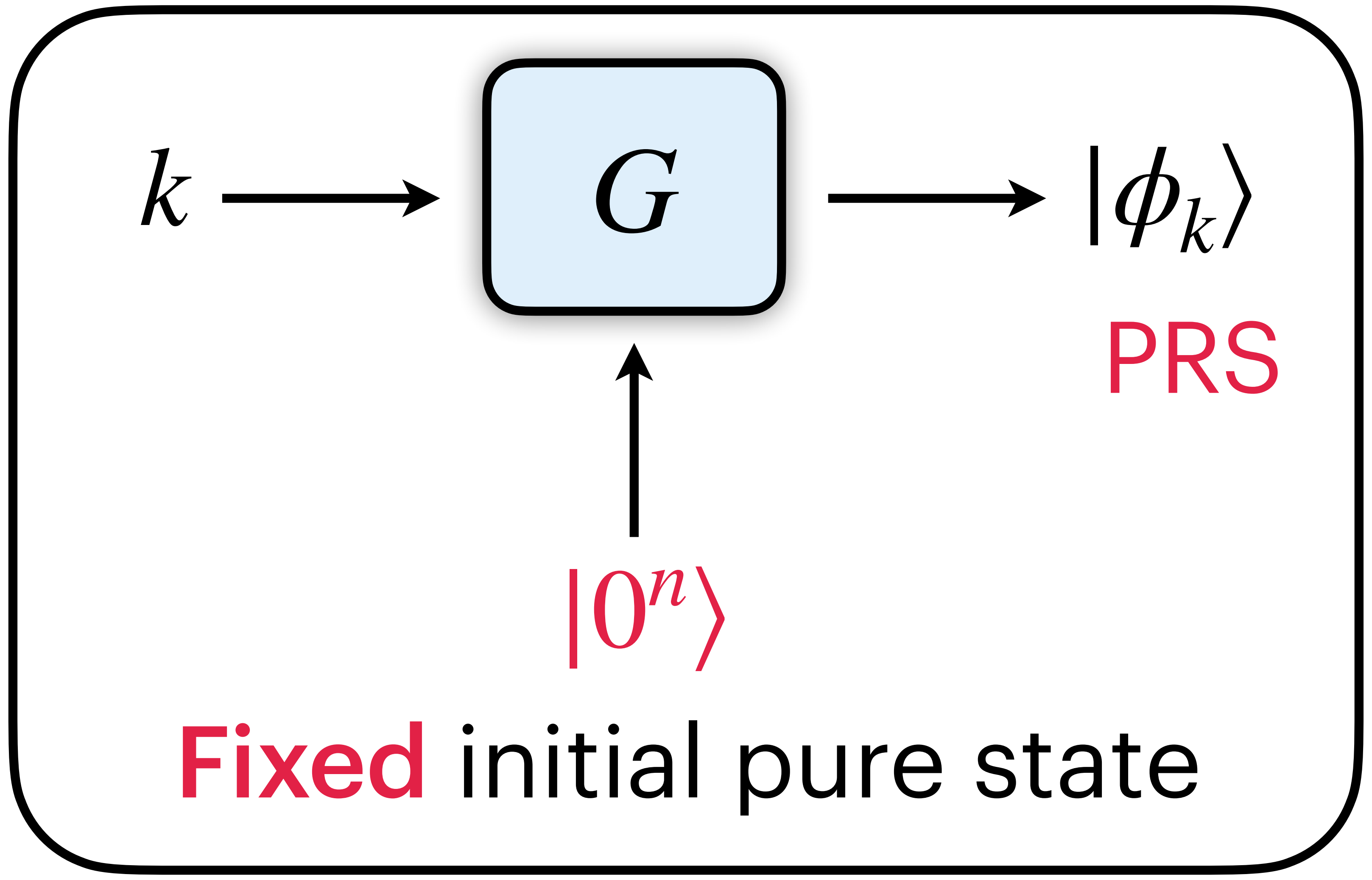

PRSG

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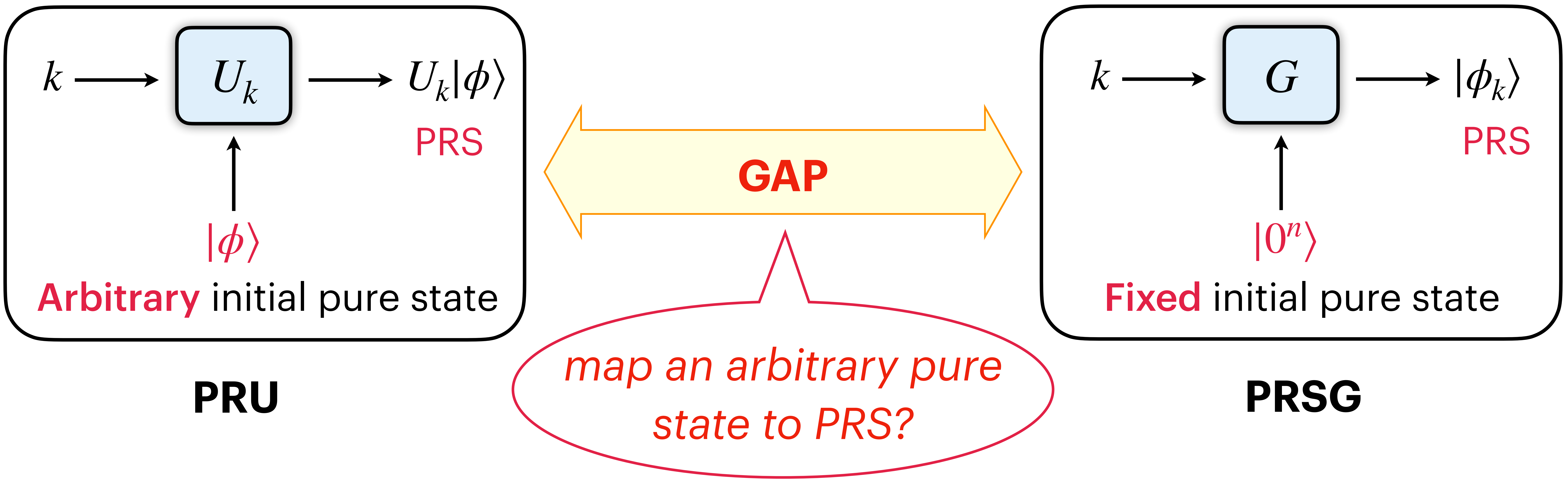

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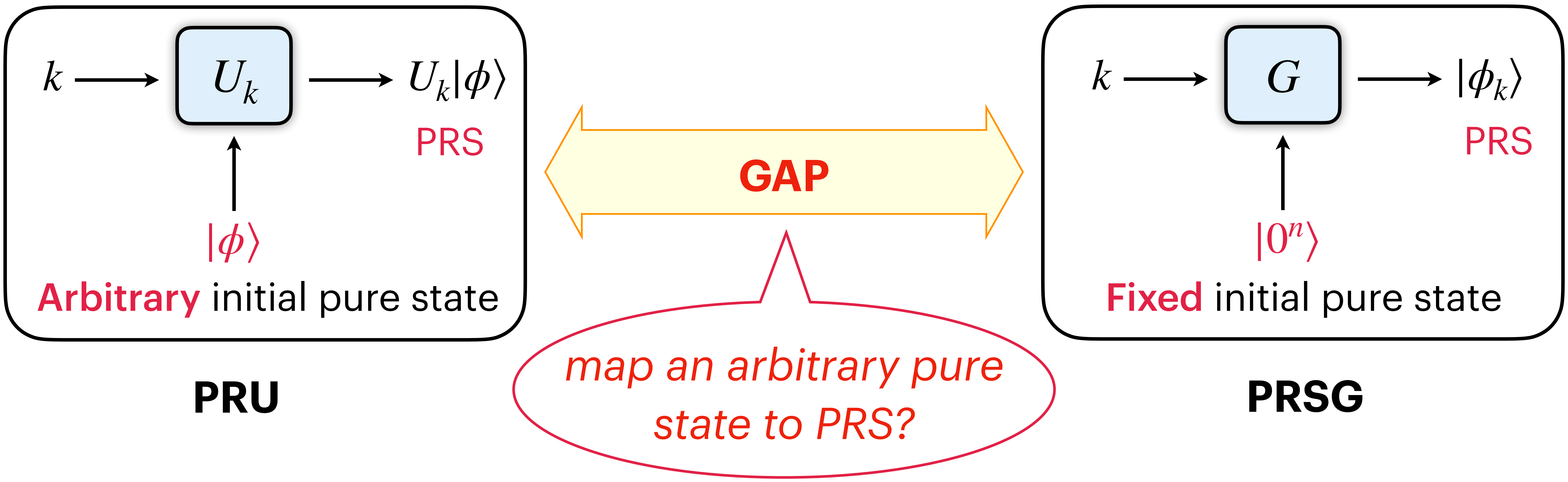

Pseudorandom State Scramblers

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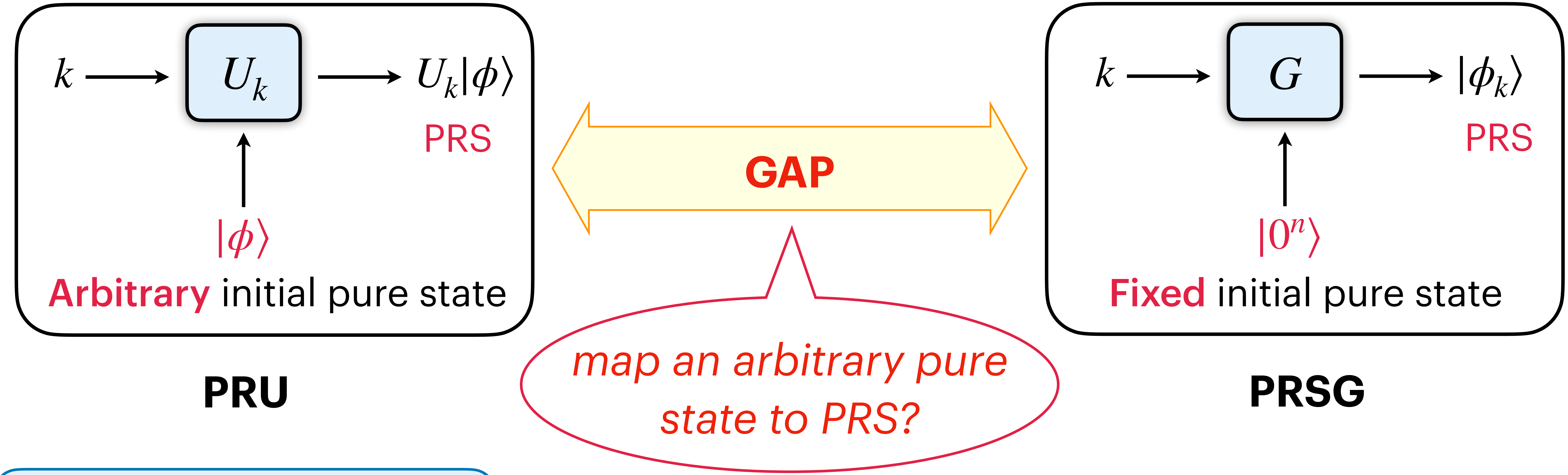

PRU

PRSG

PRUs exist assuming the existence of any quantum-secure OWF. [MH24]

Pseudorandom State Scramblers

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⇨ **Applications**

- compact quantum encryption
- succinct quantum state commitment

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Let $\mathscr{H}$ be a Hilbert spaces of dimension 2^n and λ be a security parameter. A family of unitary operators $\left\{R_k : \mathscr{H} \to \mathscr{H}\right\}_{k \in \mathscr{K}}$ is a PRSS, if

1. **Bounded key length**: $\log \left|\mathscr{K}\right| = \mathrm{poly}(n, \lambda)$
2. $\exists$ **efficient implementation** of R_k
3. **Comp. Indist.**: $\forall \, |\phi\rangle \in \mathcal{S}(\mathscr{H})$, $\forall$ poly-time quantum $\mathscr{A}$, $\forall \, t = \mathrm{poly}(\lambda)$

$$\left| \Pr_{k \leftarrow \mathscr{K}} \left[\mathscr{A}\left(\left(R_k|\phi\rangle\right)^{\otimes t}\right) = 1\right] - \Pr_{|\psi\rangle \leftarrow \mu} \left[\mathscr{A}\left(|\psi\rangle^{\otimes t}\right) = 1\right] \right| = \mathrm{negl}(\lambda)$$

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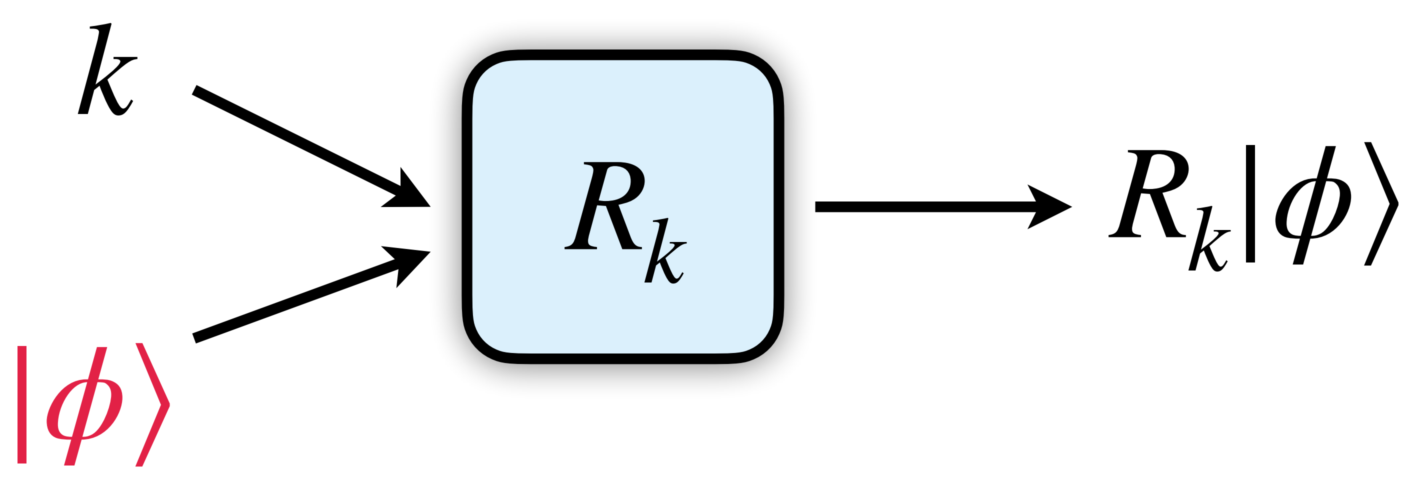
PRSS



Arbitrary pure state

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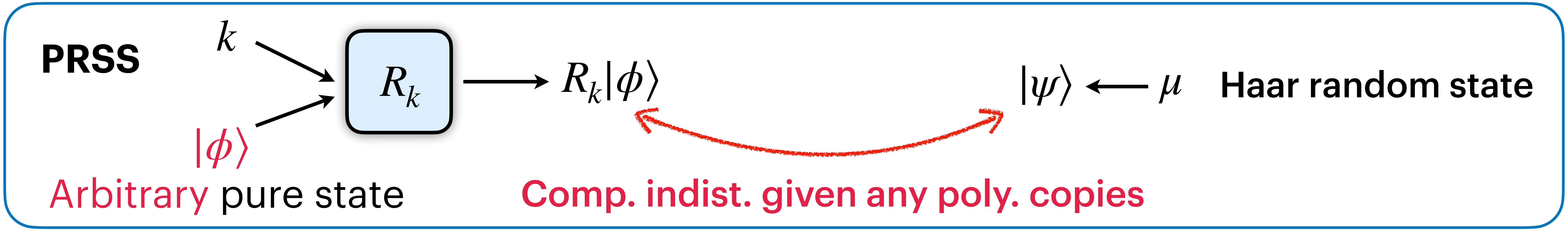

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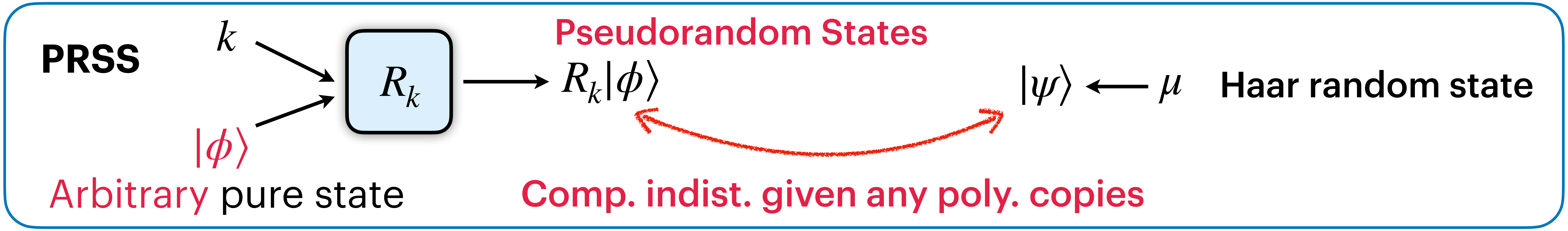

Pseudorandom State Scrambler (PRSS)

PRSS

k, $|\phi\rangle$ $\rightarrow$ $\boxed{R_k}$ $\longrightarrow$ $R_k|\phi\rangle$

Arbitrary pure state

Pseudorandom States

$|\psi\rangle \longleftarrow \mu$ Haar random state

Comp. indist. given any poly. copies

Pseudorandom State Scrambler (PRSS)

PRSS

k

$|\phi\rangle$

Arbitrary pure state

R_k

$$R_k|\phi\rangle \qquad\qquad |\psi\rangle \longleftarrow \mu$$

Pseudorandom States

Haar random state

Comp. indist. given any poly. copies

Random State Scrambler (RSS)

A family of unitary operators $\left\{R_k : \mathscr{H} \to \mathscr{H}\right\}_{k\in\mathscr{K}}$ is an RSS, if

1. **Stat. Indist.**: $\forall\ |\phi\rangle \in \mathcal{S}(\mathscr{H})$, $\forall\ t = \mathrm{poly}(\lambda)$

$$\left\| \mathop{\mathbb{E}}_{k\leftarrow\mathscr{K}}\left[\left(R_k|\phi\rangle\langle\phi|R_k^\dagger\right)^{\otimes t}\right] - \mathop{\mathbb{E}}_{|\psi\rangle\leftarrow\mu}\left[\left(|\psi\rangle\langle\psi|\right)^{\otimes t}\right] \right\|_1 = \mathrm{negl}(\lambda)$$

Constructing PRSS

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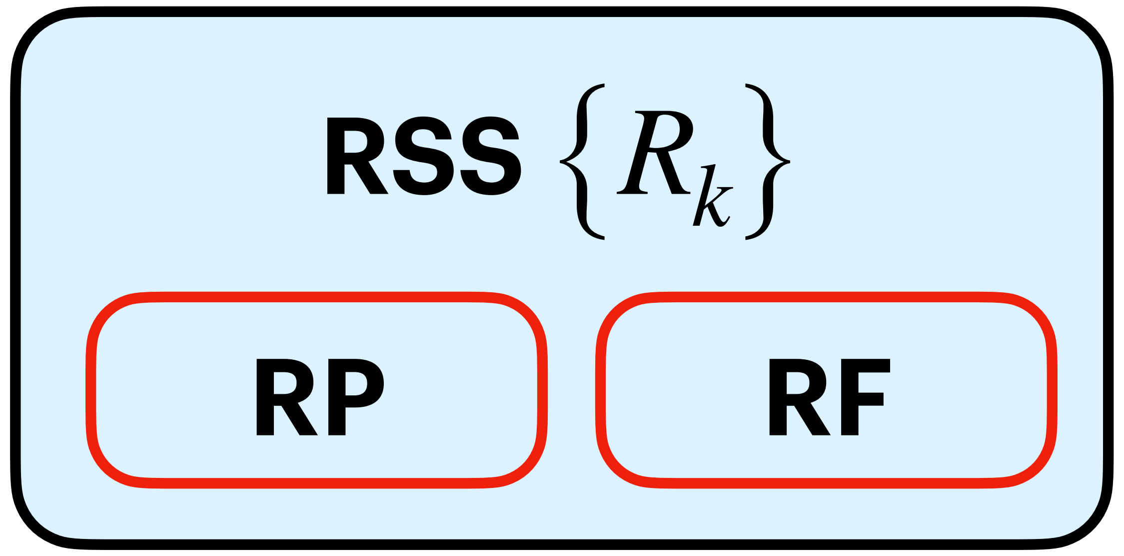

Constructing PRSS

RSS $\{R_k\}$

RP

RF

$$\left\{R_k|\phi\rangle\right\}$$

Stat. indist.

Haar random state

Constructing PRSS

RSS $\{R_k\}$

RP

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PRSS $\{R'_k\}$

QPRP

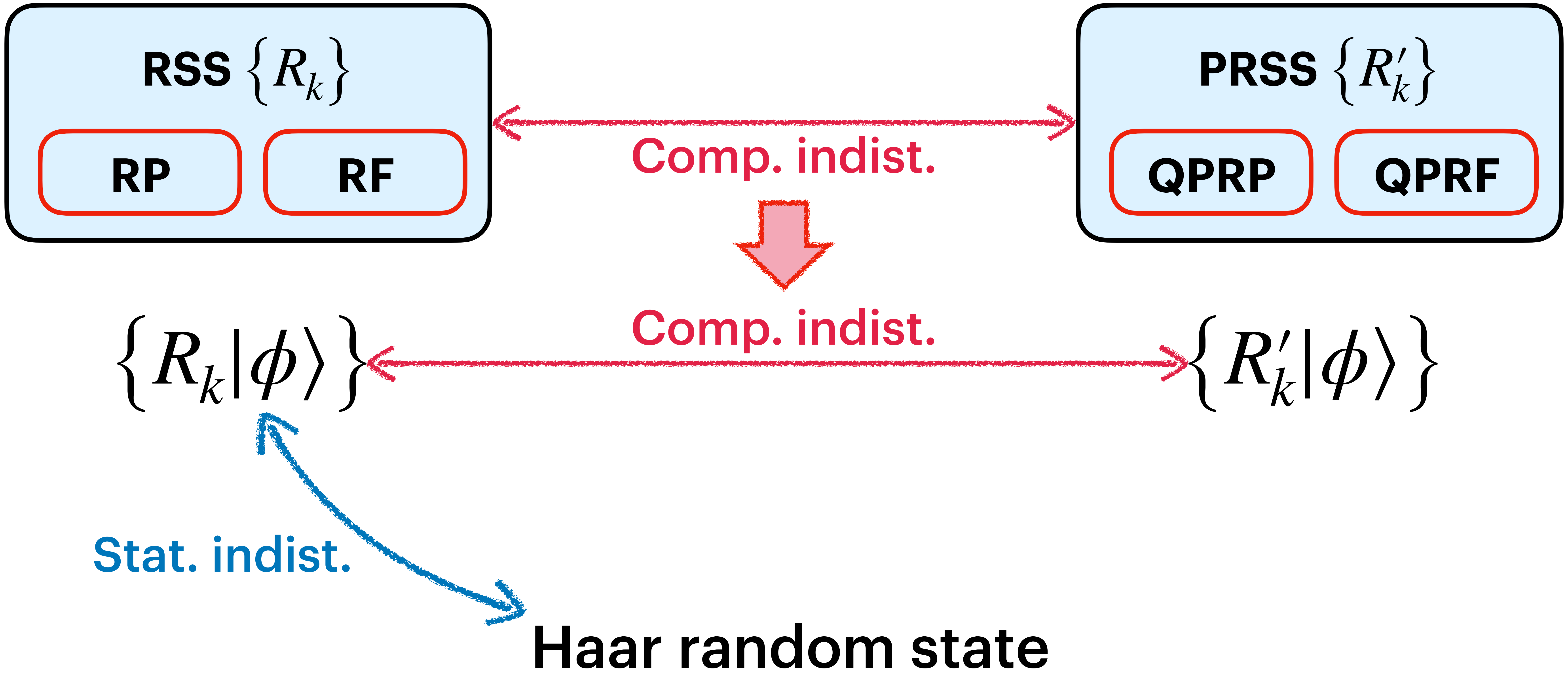
QPRF

$$\{R_k|\phi\rangle\}$$

Stat. indist.

Haar random state

Constructing PRSS



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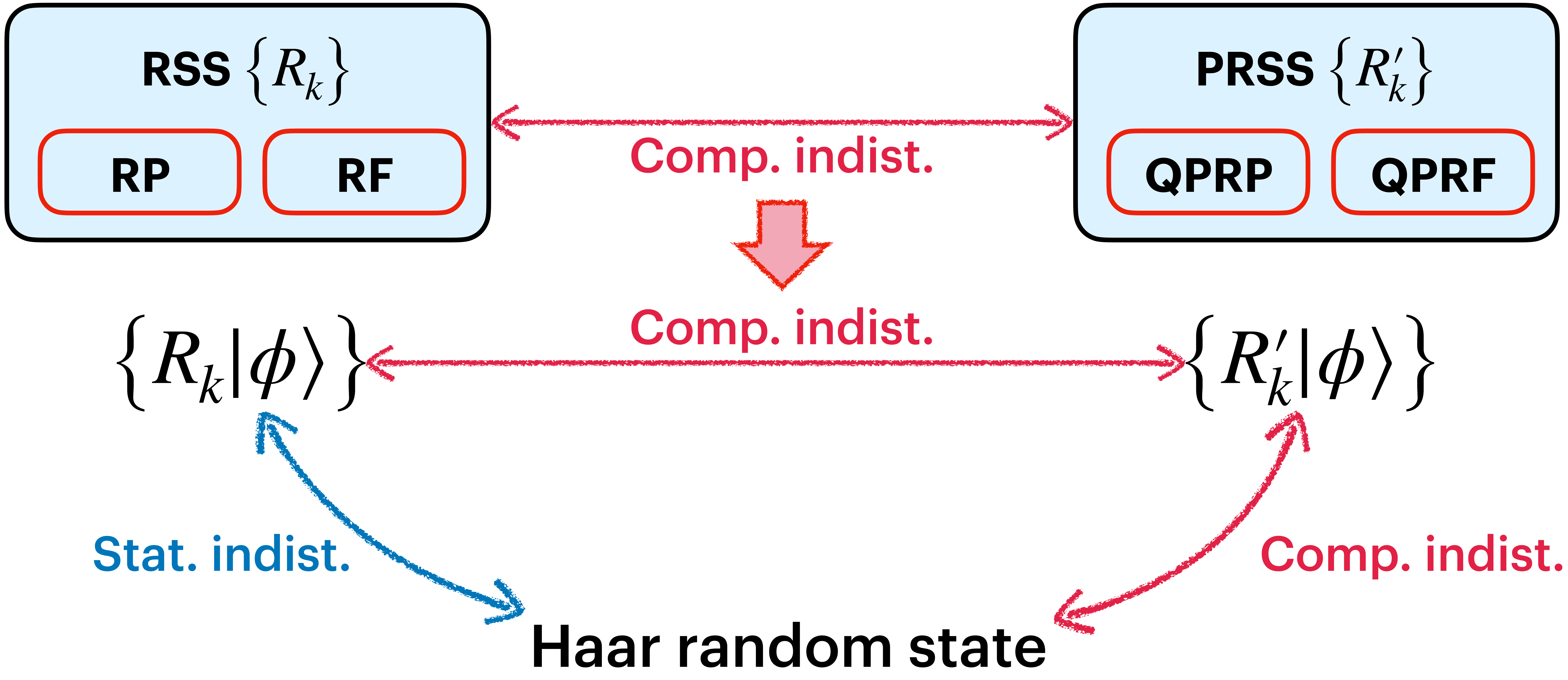

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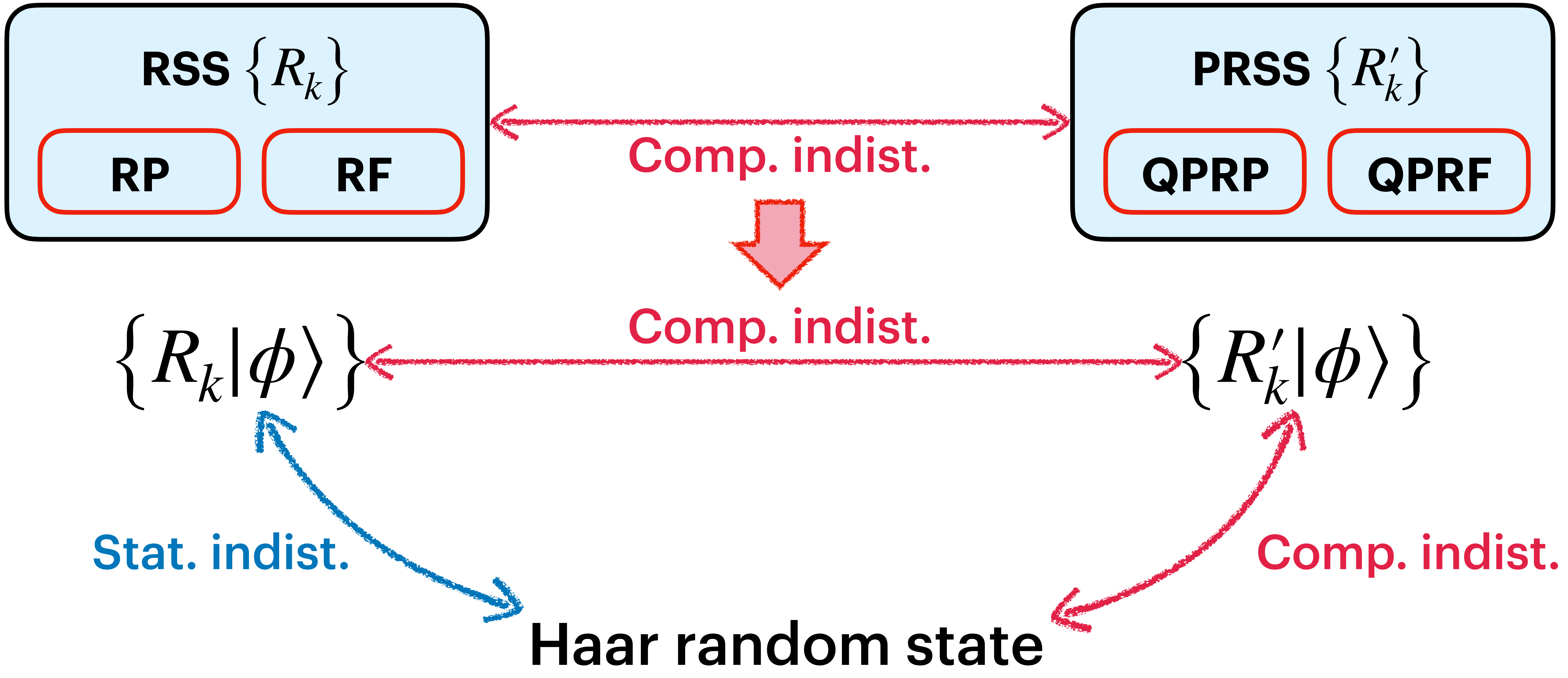

OUR APPROACH:

1. a novel **random walk** yields a state that is **stat. indist.** from a Haar random state.
2. an efficient quantum circuit simulates one step of the random walk.

Constructing RSS $(\mathscr{H} = \mathbb{R}^{2^n})$

Scrambling a quantum state

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Scrambling a quantum state

Kac's Walk (on $x \in \mathcal{S}(\mathbb{R}^{2^n})$)

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Kac's Walk (on $x \in \mathcal{S}(\mathbb{R}^{2^n})$)

$$(i, j) \quad \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_{2^n} \end{bmatrix}$$

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$$\begin{pmatrix} x_i' \\ x_j' \end{pmatrix} \longleftarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x_i \\ x_j \end{pmatrix}$$

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- Kac's walk on $\mathrm{SO}(N)$:
  - Spectral gap [DSC00, Jan03, CCL03]
  - L^2 W-distance: $O(N^2 \log N)$ [Oli09]
  - TV distance: $O(N^2) \sim O(N^4 \log N)$ [PS18]

Constructing RSS ($\mathscr{H} = \mathbb{R}^{2^n}$)

Scrambling a quantum state

Our construction: Parallel Kac's Walk

1. Select a random perfect matching of $\{1,\cdots,2^n\}$

$$P = \left\{(i_1,j_1), \ldots, (i_{2^{n-1}},j_{2^{n-1}})\right\}$$

2. For each pair $(i_k,j_k) \in P$, sample an angle

$$\theta_k \in [0,2\pi)$$

and set

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One step of the parallel Kac's walk

$\approx$

2^{n-1} steps of the original Kac's walk

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Theorem

Let $\{|\phi_t\rangle \in \mathbb{R}^{2^n}\}_{t\geq 0}$ be a parallel Kac's walk. For $T = 10(\lambda+1)n$,

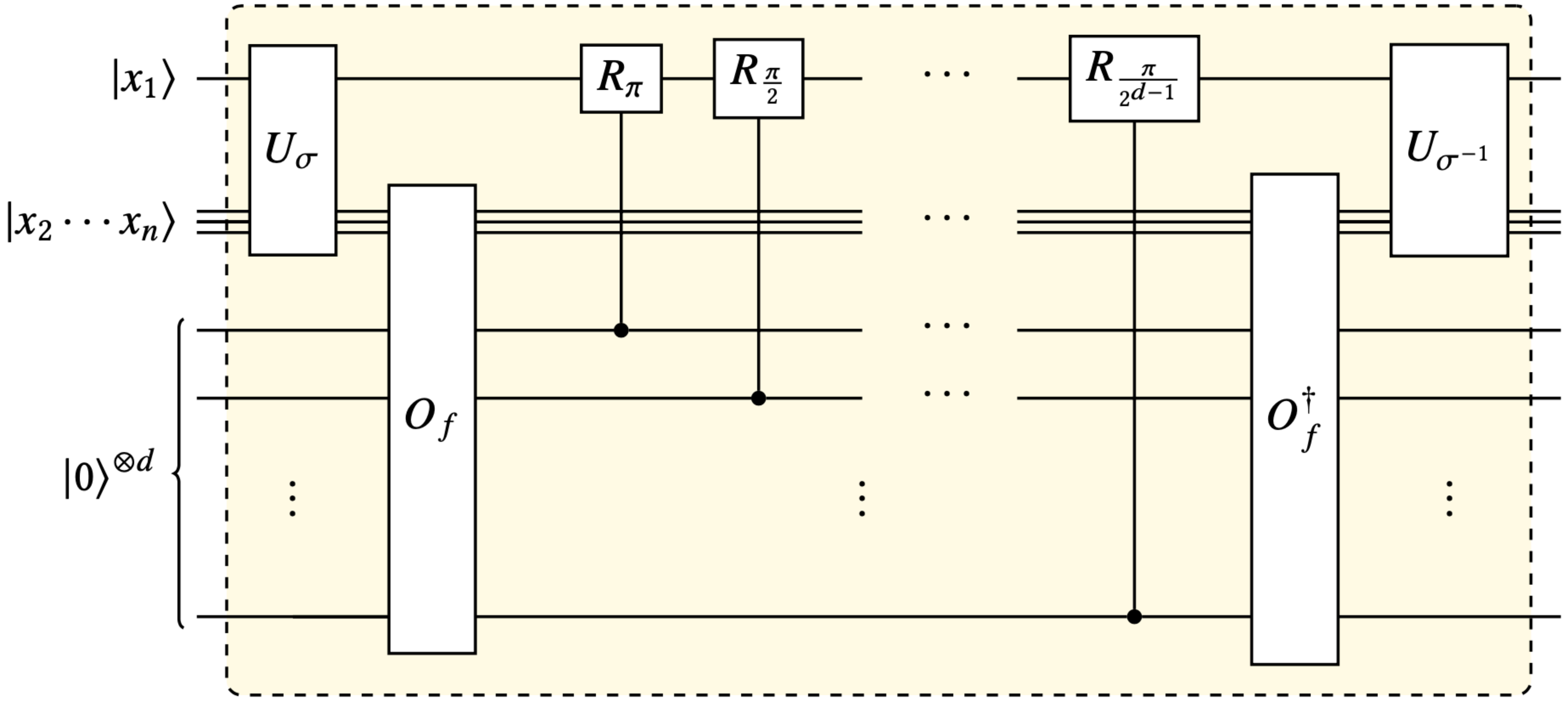
$$\left(|\phi_T\rangle\langle\phi_T|\right)^{\otimes l} \approx_s \left(|\psi\rangle\langle\psi|\right)^{\otimes l}$$

Constructing RSS ($\mathscr{H} = \mathbb{R}^{2^n}$)

Implementing via quantum circuit

$K_{\sigma,f}$ simulates one step of Kac's walk.

$\sigma : \{0,1\}^n \to \{0,1\}^n \quad f : \{0,1\}^{n-1} \to \{0,1\}^d$



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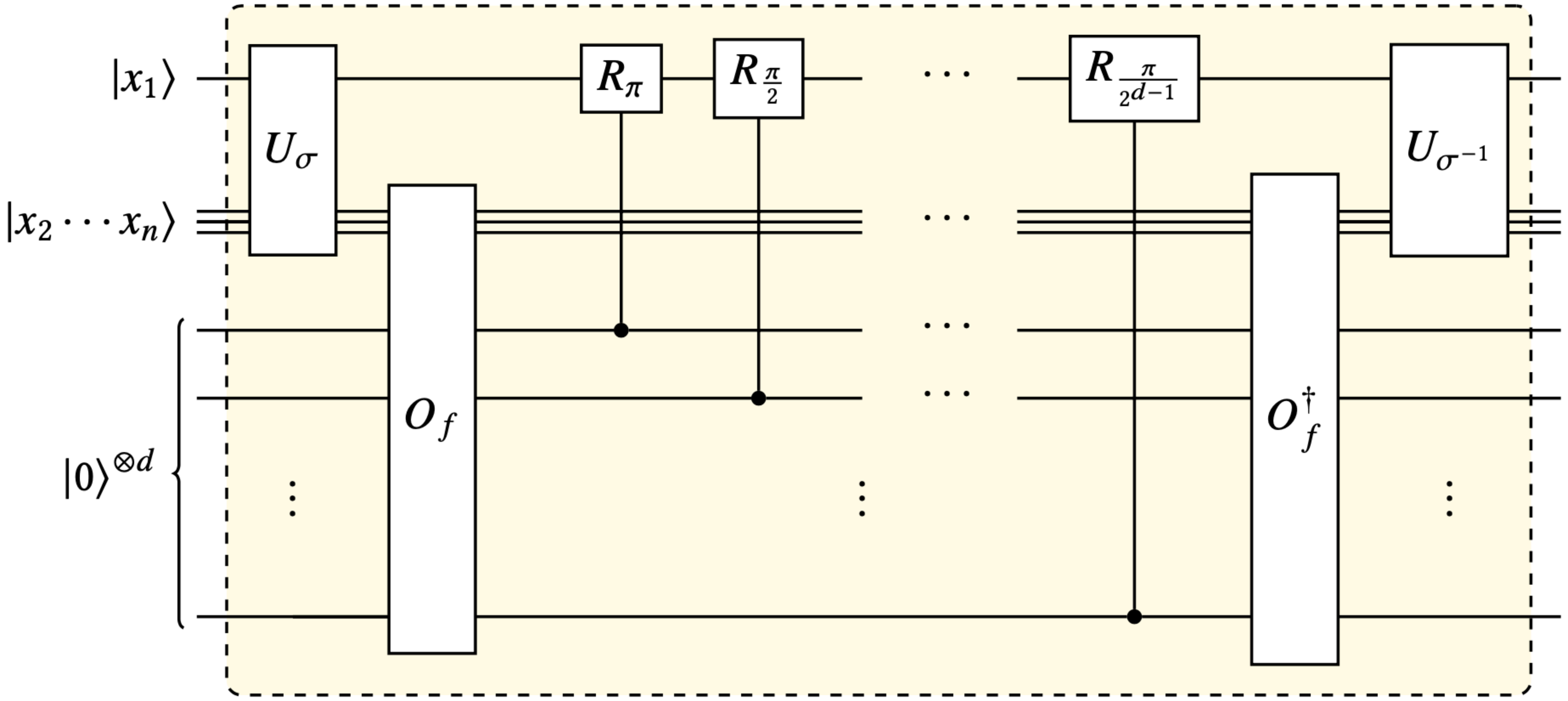
$\sigma : \{0,1\}^n \to \{0,1\}^n \quad f : \{0,1\}^{n-1} \to \{0,1\}^d$

σ: partition the basis into 2^{n-1} pairs

f: choose an independently random angle



Constructing RSS ($\mathscr{H} = \mathbb{R}^{2^n}$)

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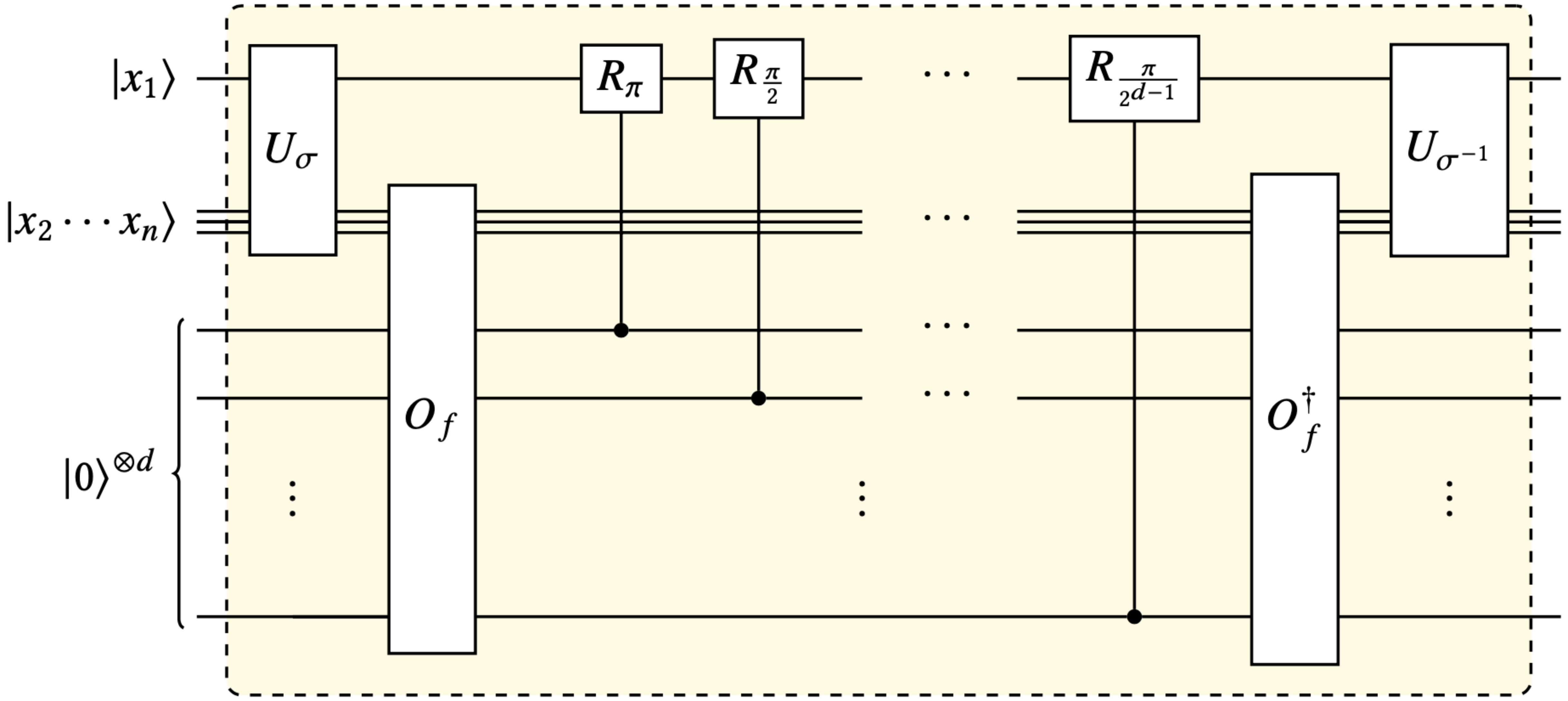
$$C_{\sigma_1,\ldots,\sigma_T,f_1,\ldots,f_T} = K_{\sigma_T,f_T} \cdots K_{\sigma_2,f_2} K_{\sigma_1,f_1}$$



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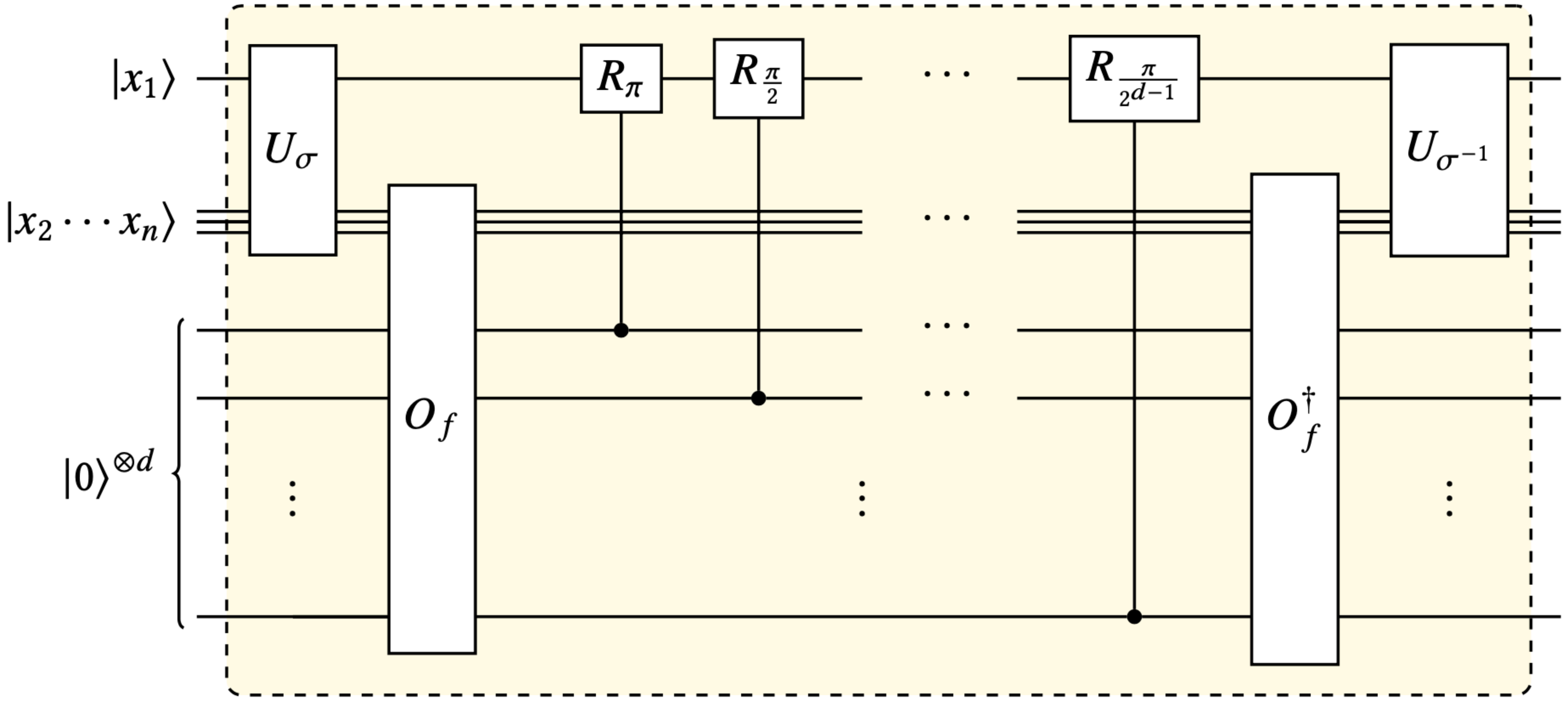

Theorem

Let $d = \log^2 \lambda + \log^2 n$ and $T = 10(\lambda + 1)n$. $\left\{ C_{\sigma_1,\ldots,\sigma_T,f_1,\ldots,f_T} \right\}$ is an RSS.

A Dispersing Property

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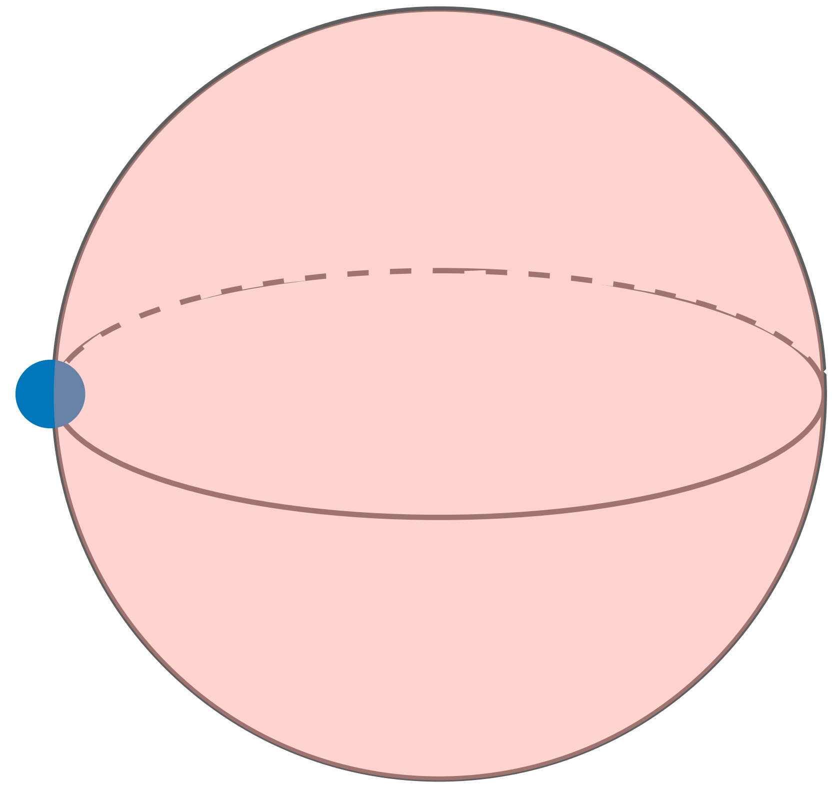

Parallel Kac's walk

Close to Haar in total variation distance

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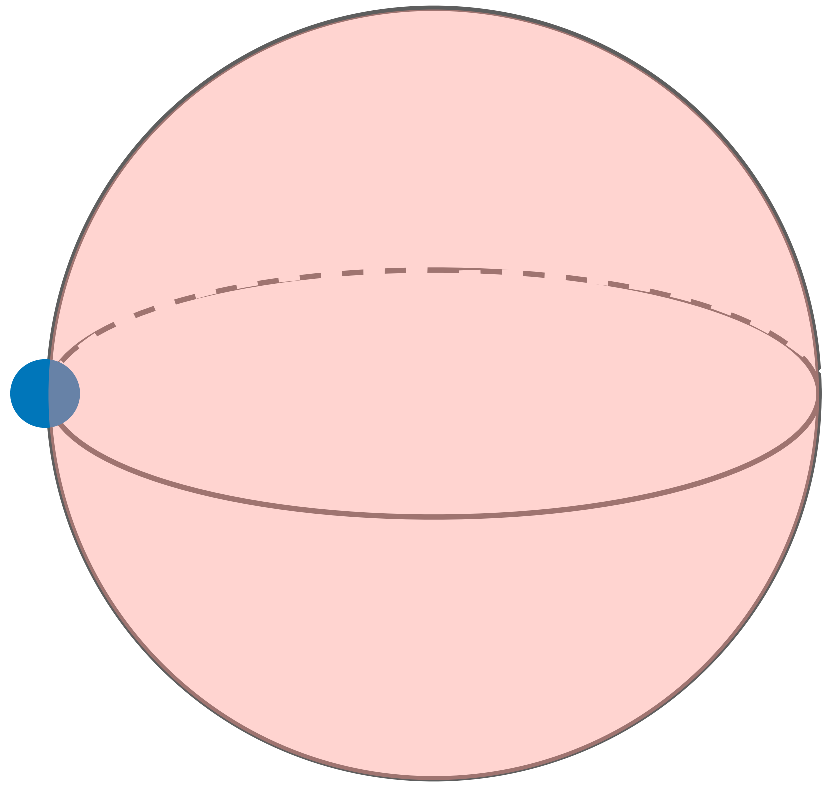

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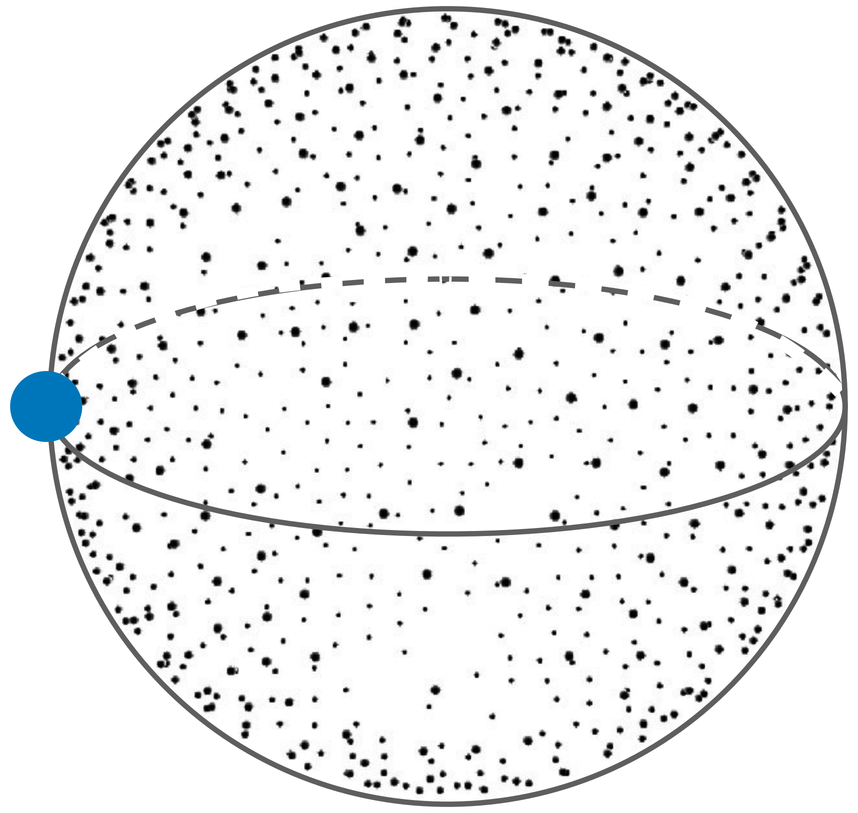

Our RSS

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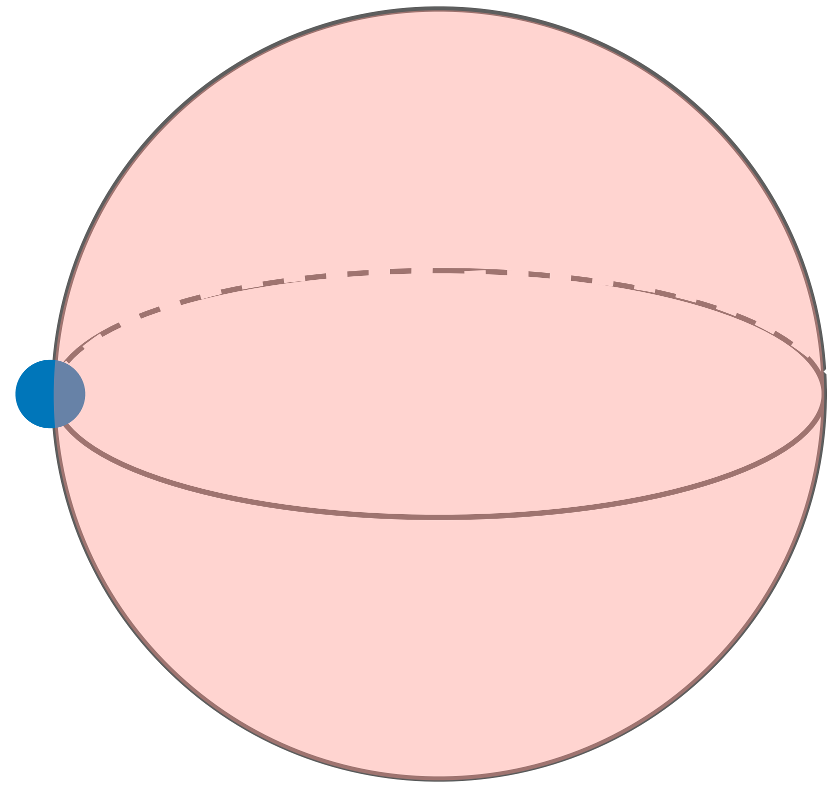

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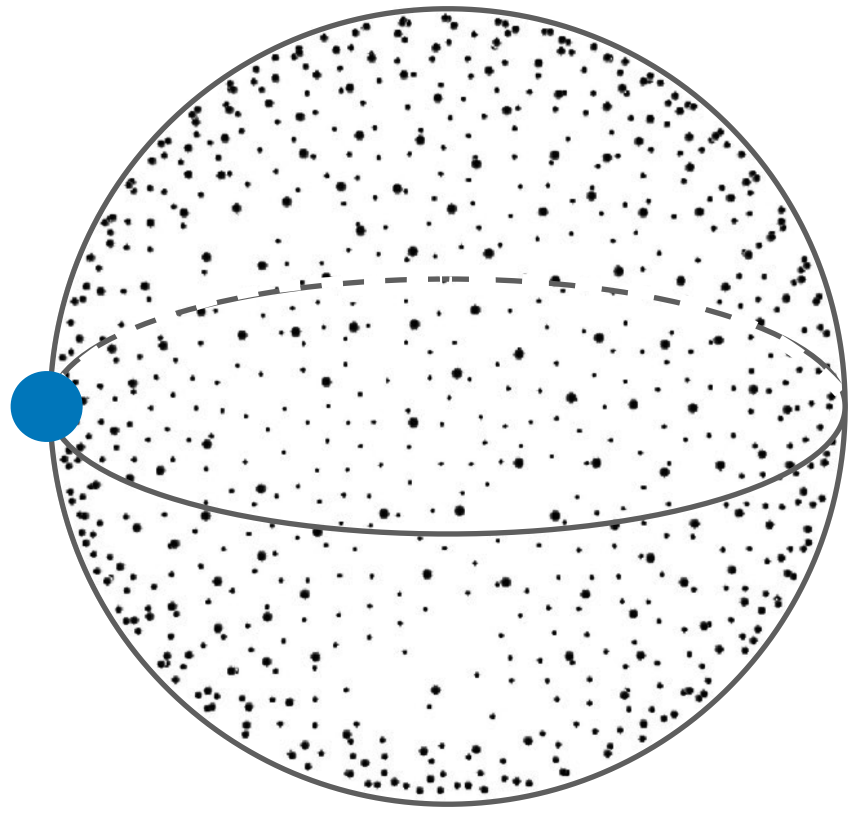

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Output states span an ε-net of the state space

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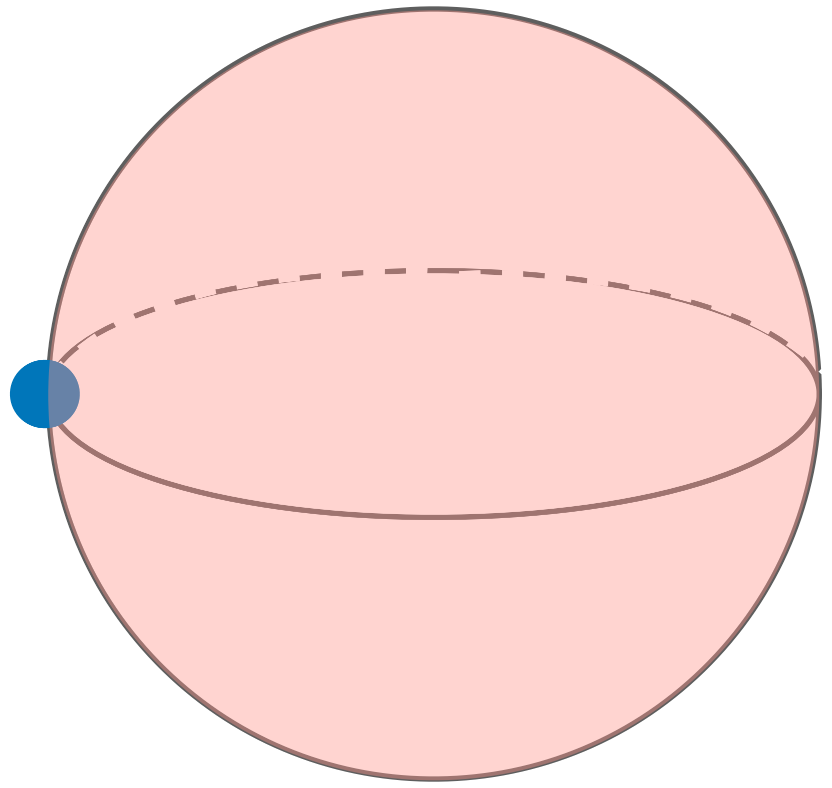

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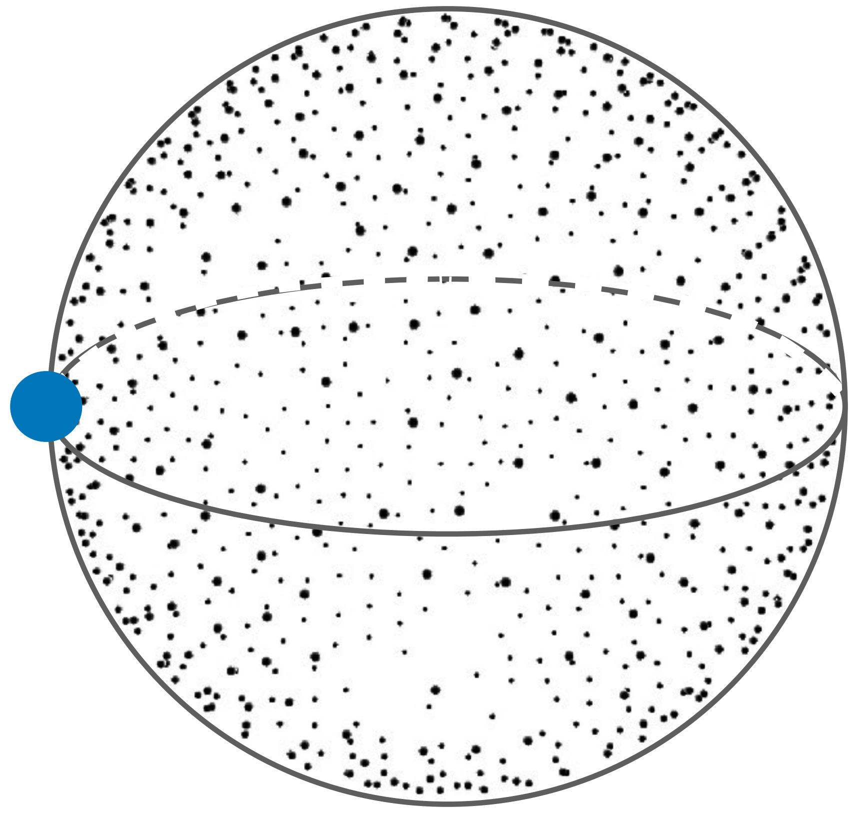

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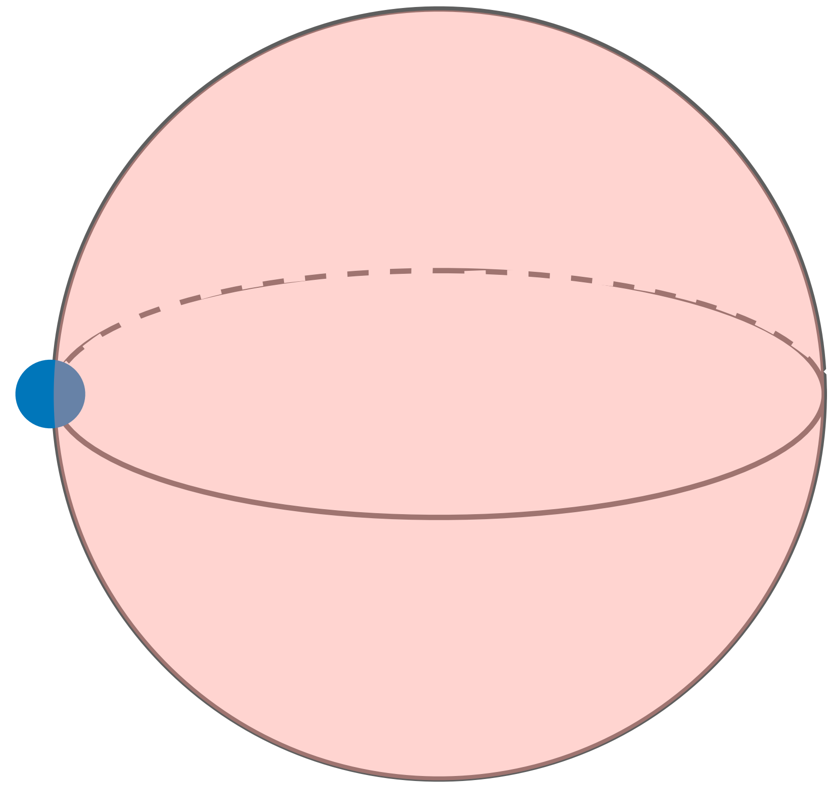

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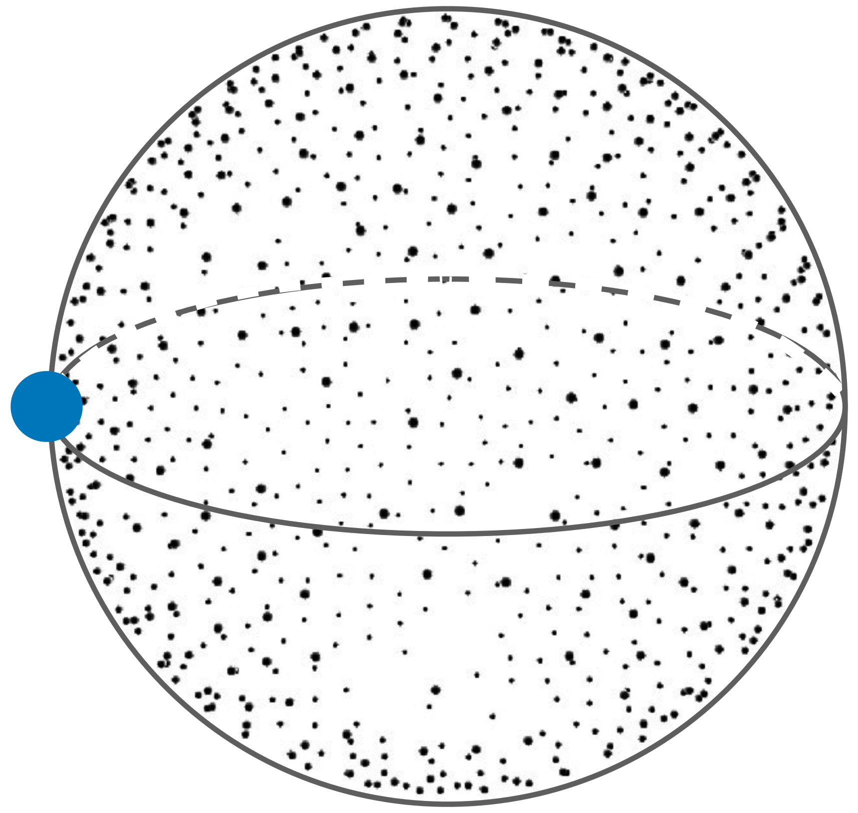

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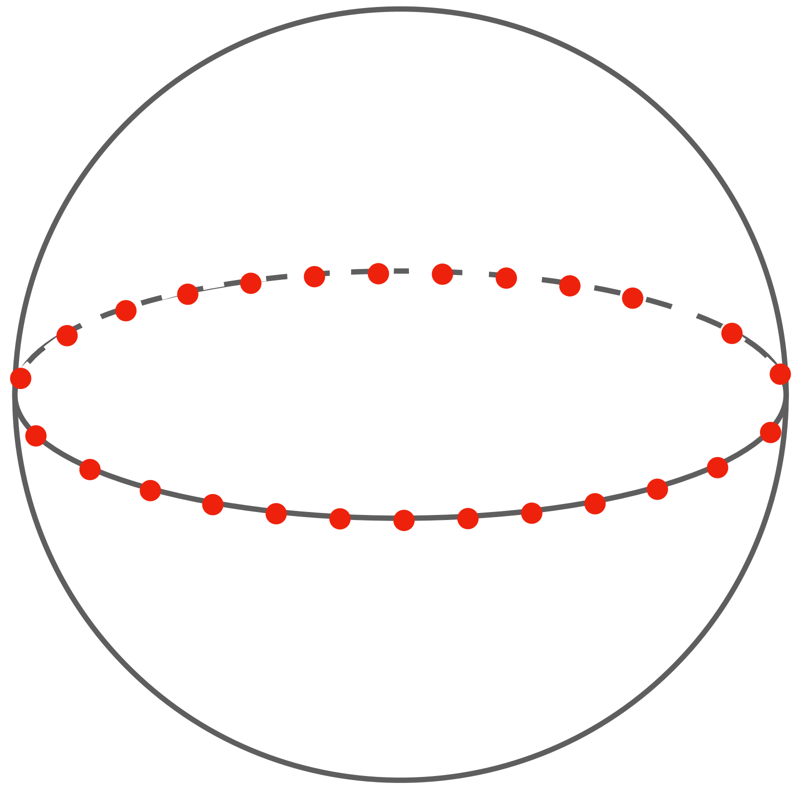

Random phase states

$$|\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} \omega_N^{f_k(x)} |x\rangle$$

$$|\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} (-1)^{f_k(x)} |x\rangle$$

Close to Haar in average

Final Remarks and Open Questions

$U_k \approx$ Haar random unitary

Able to scramble an arbitrary pure state

PRU $\{U_k\}$

Scramble an arbitrary pure state

$$\left(R_k|\phi\rangle\right)^{\otimes l(n)} \approx |\psi\rangle^{\otimes l(n)}$$

Haar random state

PRSS $\{R_k\}$

Scramble an fixed initial state, e.g. $|0^n\rangle$

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Haar random state

PRSG G

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Q3: Simplify the construction?

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PRSG G