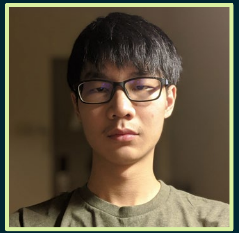
Worst-case to Average-case Hardness of LWE:

An Alternative Perspective


Divesh Aggarwal


Leong Jin Ming


Alexandra Veliche

Cryptographic from LWE

LWE

IND-CPA-secure
Public-Key Encryption
[Regev, 2005]

CCA-secure
Encryption
[Peikert, 2009]

…

Key Exchange Scheme
[Ding-Xie-Lin, 2012],
[Peikert, 2014]

Cryptographic Significance

worst-case problems → average-case problem: **LWE**

LWE →

- IND-CPA-secure Public-Key Encryption [Regev, 2005]
- CCA-secure Encryption [Peikert, 2009]
- …
- Key Exchange Scheme [Ding-Xie-Lin, 2012], [Peikert, 2014]

Cryptographic Significance

worst-case hardness $\Longrightarrow$ average-case hardness $\Longrightarrow$ cryptosystems

GapSVP BDD $\longrightarrow$ LWE

Cryptographic Significance

worst-case hardness $\Longrightarrow$ average-case hardness $\Longrightarrow$ Post-quantum cryptosystems

GapSVP BDD $\longrightarrow$ LWE

no known efficient quantum algorithms for these problems

Learning With Errors

$\mathbf{LWE}_{n,p,\phi}$: n dimension, p modulus, $\phi \sim \mathbb{R}/\mathbb{Z}$ error distribution

Given noisy samples $(\mathbf{a},\ \langle \mathbf{a}, \mathbf{s} \rangle + e)$, where

$\mathbf{a} \leftarrow \mathbb{Z}_p^n$ uniformly random, $\mathbf{s} \in \mathbb{Z}_p^n$ unknown, $e \leftarrow \phi$ small error,

(search-LWE) output $\mathbf{s}$.

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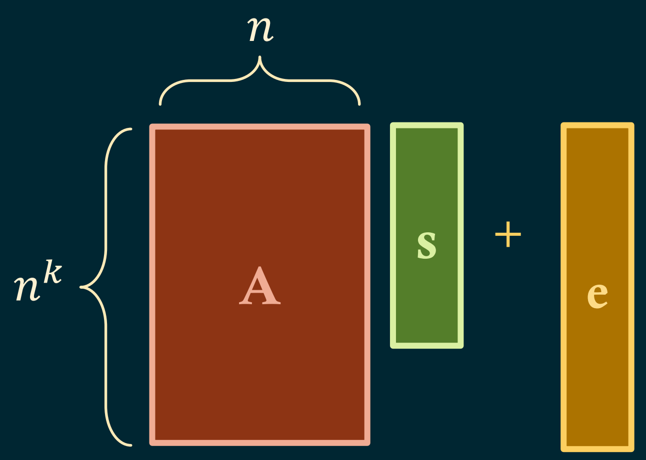


Learning With Errors

$\mathbf{LWE}_{n,p,\phi}$: n dimension, p modulus, $\phi \sim \mathbb{R}/\mathbb{Z}$ error distribution

Given noisy samples $(\mathbf{a}, b)$, where

$\mathbf{a} \leftarrow \mathbb{Z}_p^n$ uniformly random and $b \in \mathbb{Z}_p$,

(decision-LWE) output

- YES if samples are from the LWE distribution for $\mathbf{s}$ and ϕ,
- NO if samples are uniformly random.

Lattices

Lattice:

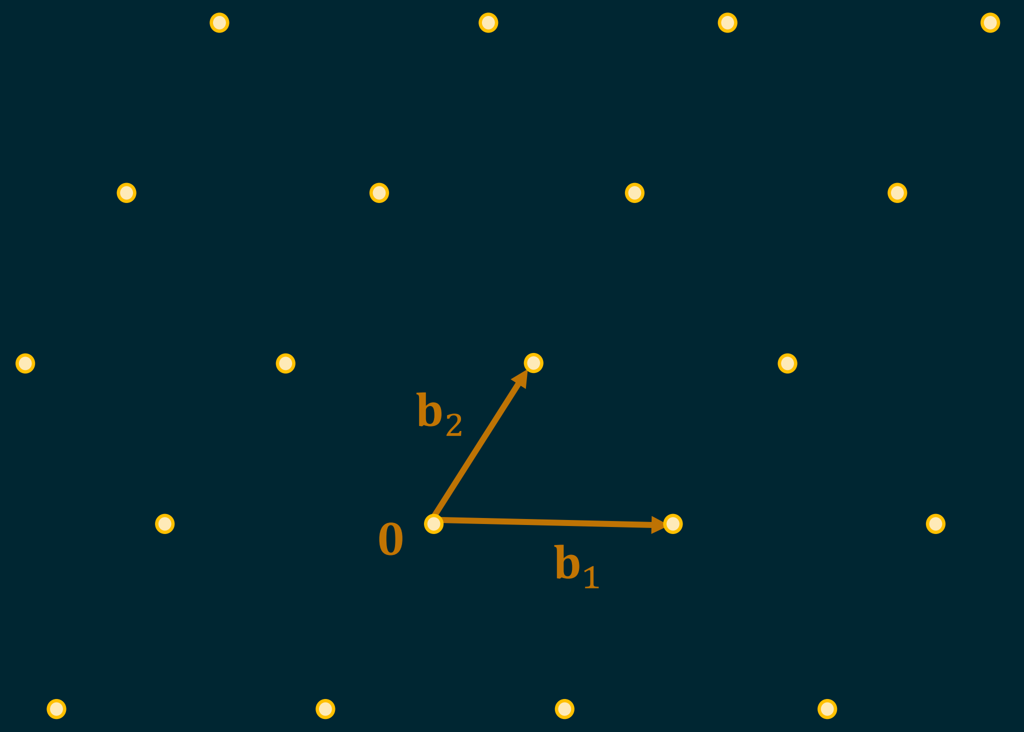
An infinite discrete set of vectors in $\mathbb{R}^n$

consisting of all integer linear combinations

$$\mathcal{L} = \{a_1\mathbf{b}_1 + \cdots + a_k\mathbf{b}_k \;:\; a_1, \dots, a_k \in \mathbb{Z}\}$$

of some linearly independent vectors $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathbb{R}^n$.

The set $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is called a *basis*.



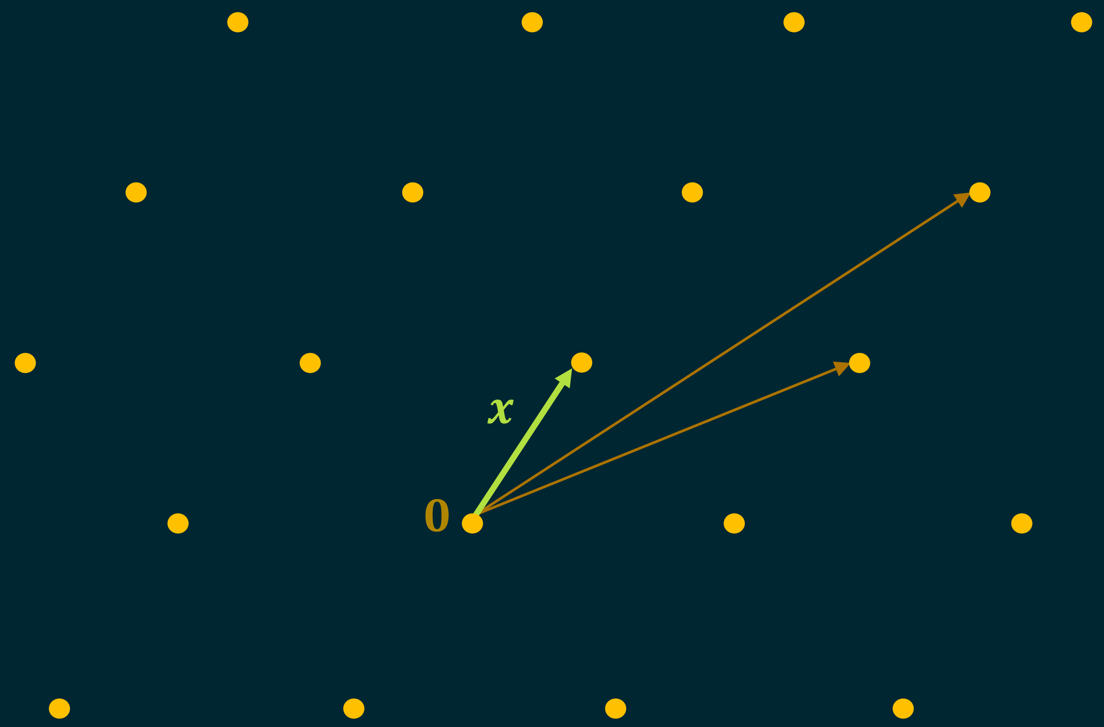
Shortest Vector Problem

SVP :

Given a basis $\mathcal{B}$ for lattice $\mathcal{L} \subset \mathbb{R}^n$,

find a shortest non-zero lattice vector $\boldsymbol{x}$,

i.e. $\boldsymbol{x} \in \mathcal{L} \setminus \{\mathbf{0}\}$, such that $\|\boldsymbol{x}\| = \lambda_1(\mathcal{L})$.



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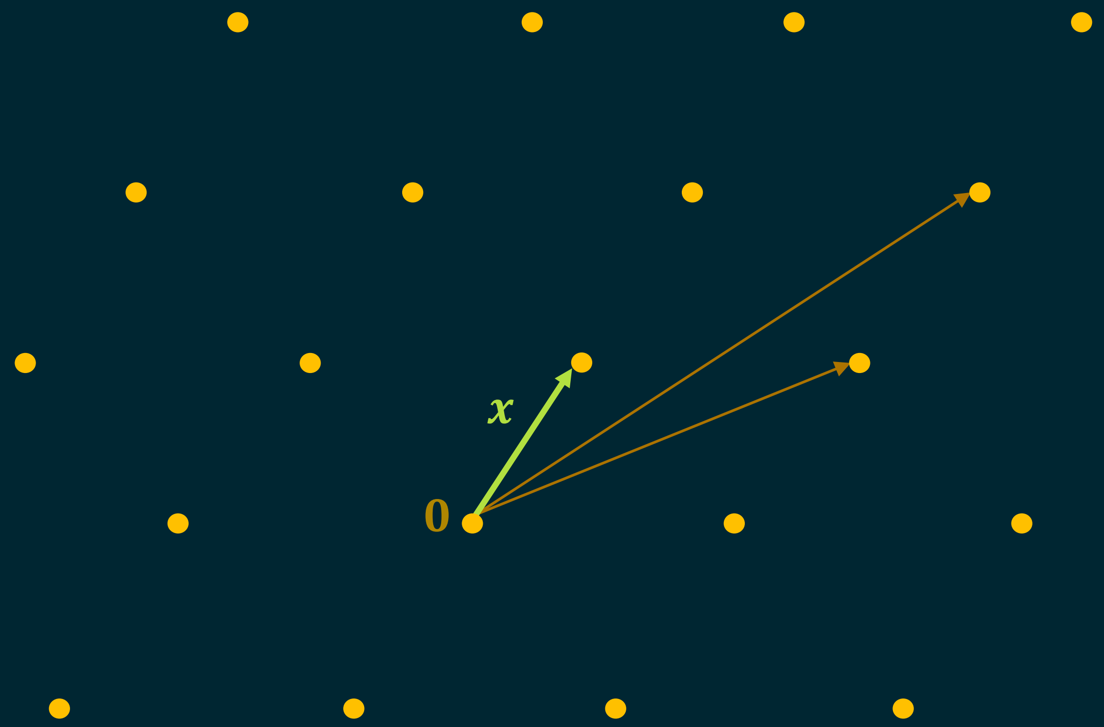


GapSVP$_\gamma$ is an approximate decision variant.

Bounded Distance Decoding

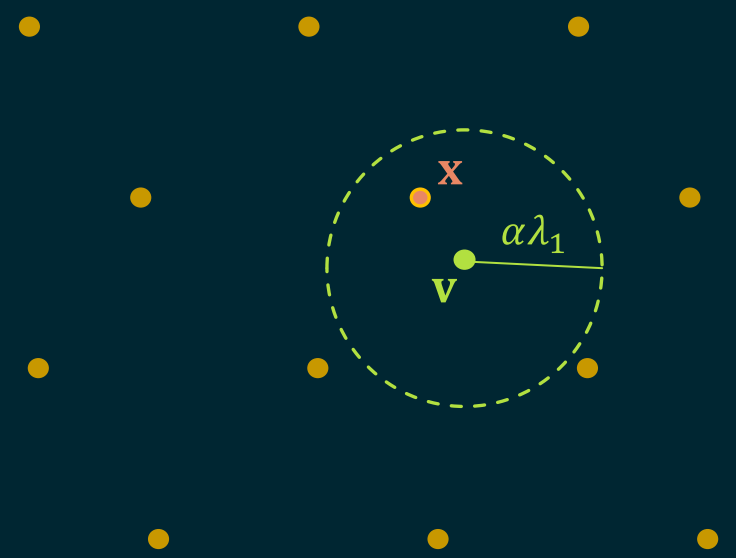
$\mathbf{BDD}_{\alpha}$: $\alpha > 0$ distance approximation factor

Given a basis $\mathcal{B}$ for a full-rank lattice $\mathcal{L} \subseteq \mathbb{R}^n$

and a target vector $\mathbf{v} \in \mathbb{R}^n$ close to the lattice,

find a lattice vector $x \in \mathcal{L}$ closest to $\mathbf{v}$,

i.e. $\mathbf{x} \in \mathcal{L}$ such that $\|\mathbf{v} - \mathbf{x}\|_2 < \alpha \cdot \lambda_1(\mathcal{L})$.



Hardness of LWE

[Regev, 2009] – quantum reduction from worst-case lattice problems to decision-LWE

Hardness of LWE

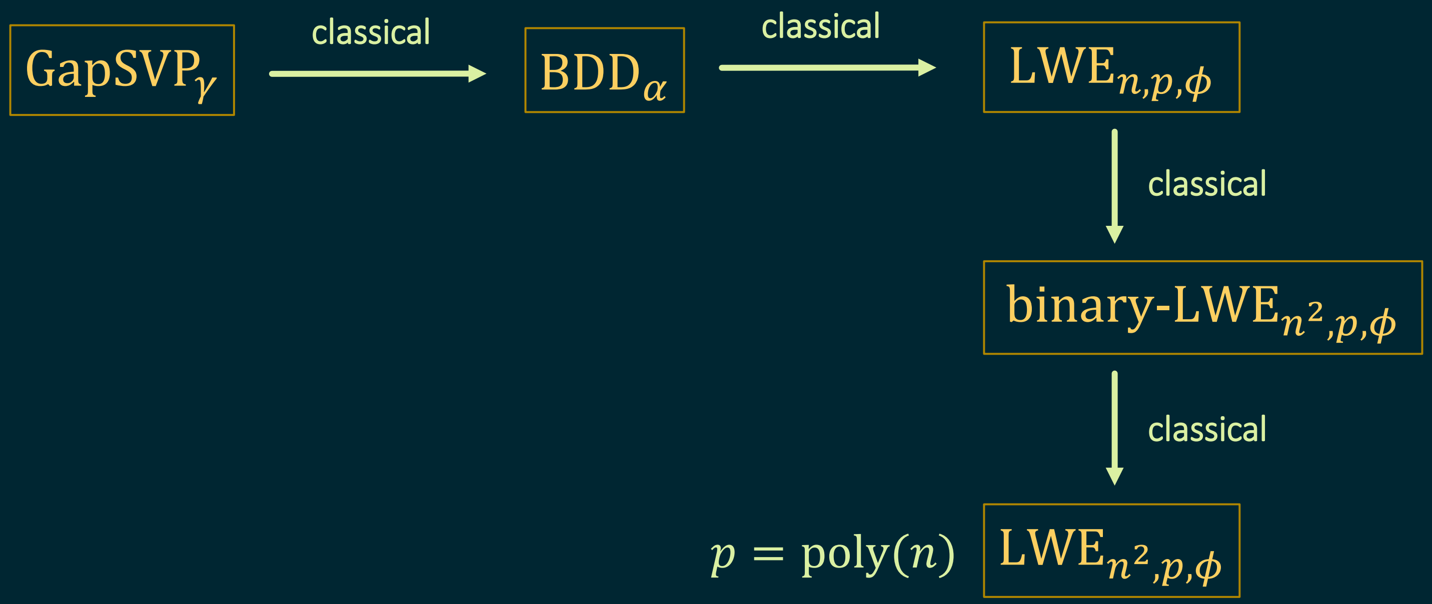
[Peikert, 2009] – classical reduction, but modulus becomes exponential

$$\mathrm{GapSVP}_{\gamma} \xrightarrow{\text{classical}} \mathrm{BDD}_{\alpha} \xrightarrow{\text{classical}} \mathrm{LWE}_{n,p,\phi}$$

$p = \exp(n)$

Hardness of LWE

[Brakerski, Peikert, Langlois, Regev, Stehle, 2013] – classical reduction with polynomial modulus



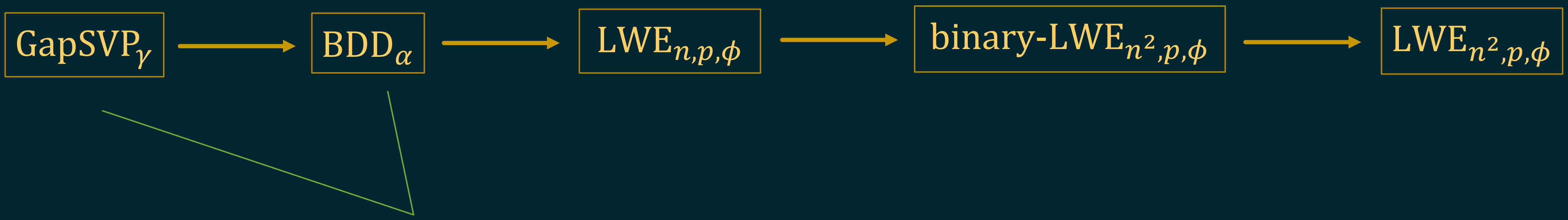
Hardness of LWE

$\mathrm{GapSVP}_\gamma \rightarrow \mathrm{BDD}_\alpha \rightarrow \mathrm{LWE}_{n,p,\phi} \rightarrow \text{binary-}\mathrm{LWE}_{n^2,p,\phi} \rightarrow \mathrm{LWE}_{n^2,p,\phi}$

$\mathrm{LWE}_{n,p,\phi}$: $p = \exp(n)$

$\mathrm{LWE}_{n^2,p,\phi}$: $p = \mathrm{poly}(n)$

Algorithms for Lattice Problems



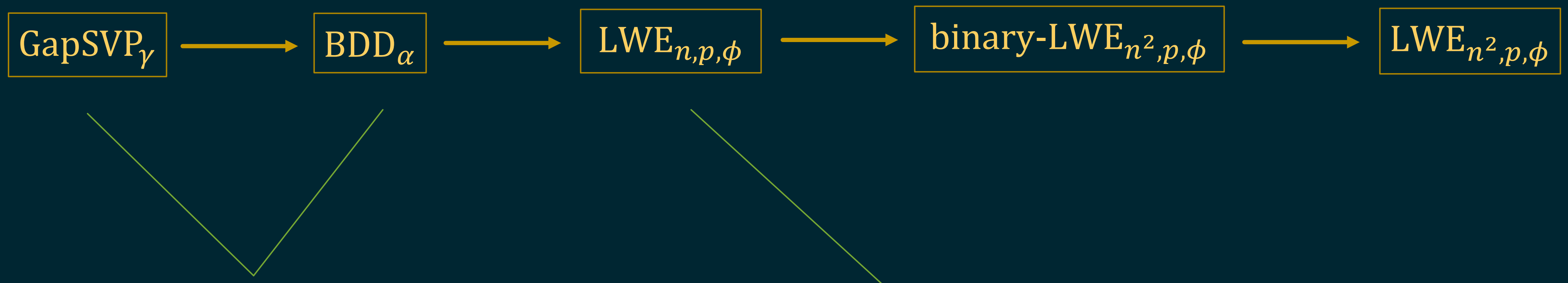
Fastest algorithms for these
problems run in $2^{\Theta(n)}$ time
(for polynomial approximation factor).

What the Reduction says about LWE Algorithms

$\text{GapSVP}_\gamma \longrightarrow \text{BDD}_\alpha \longrightarrow \text{LWE}_{n,p,\phi} \longrightarrow \text{binary-LWE}_{n^2,p,\phi} \longrightarrow \text{LWE}_{n^2,p,\phi}$

Conjecture: known algorithms are the best possible

What the Reduction says about LWE Algorithms



Conjecture: known algorithms are the best possible $\Longrightarrow$ LWE cannot be solved in less than $2^{\Omega(\sqrt{n})}$ time.

What the Reduction says about LWE Algorithms

[Blum-Kalai-Wasserman, 2000] – Best known algorithm for $\mathrm{LWE}_{n,p,\phi}$ runs in $2^{O(n \log p \,/\, \log n)}$ time.

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Big Gap!

Conjecture: known algorithms are the best possible $\Longrightarrow$ LWE cannot be solved in less than $2^{\Omega(\sqrt{n})}$ time.

Closing the Gap

How to close this gap?

We change our perspective!

Security in Practice

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Security in Practice

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(b) No reasonably efficient algorithm can break the cryptosystem with probability $> 2^{-256}$.

This is what we usually want for cryptographic security

An Alternative Perspective

An alternative measure of computational hardness:

The maximum success probability of any PPT algorithm that finds a solution.

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Yes (this talk!)

Success Probability of Solving LWE

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All other algorithms are not PPT, so it is unlikely that we can achieve better than this.

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LLL / Slide Reduction + guess coefficients: Success probability of solving GapSVP_γ is $2^{-\Theta(n^2/\log n)}$.

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BDD_α is closely related to GapSVP_γ for $\gamma = \mathrm{poly}(n) = \frac{1}{\alpha}$,

so it is unlikely we can achieve better than known algorithms.

A Natural Conjecture

Conjecture:

(informal) No algorithm can solve BDD_α on an arbitrary n-rank lattice for $\alpha = 1/\mathrm{poly}(n)$ in polynomial time with success probability better than $2^{-n^2/\log n}$.

What We Show

Trivial algorithm: Success probability for efficiently solving $\mathrm{LWE}_{n,p,\phi}$ is $p^{-\Omega(n)}$.

Conjecture $\Longrightarrow$ Maximum success probability for efficiently solving $\mathrm{LWE}_{n,p,\phi}$ is $p^{-\Omega(n/\log^2 n)}$.

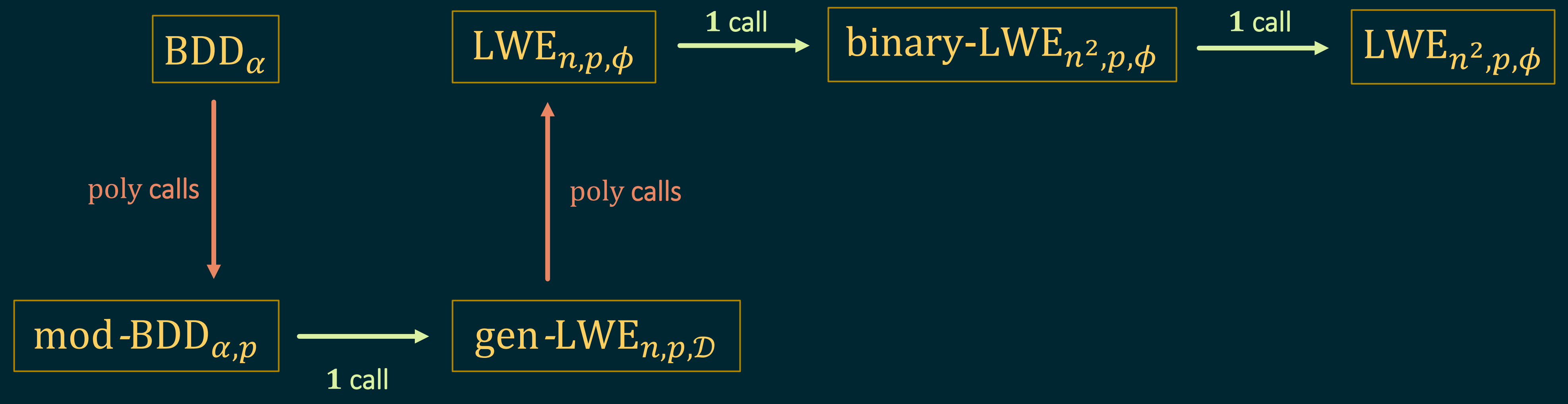
What We Show

Trivial algorithm: Success probability for efficiently solving $\mathrm{LWE}_{n,p,\phi}$ is $p^{-\Omega(n)}$.

Tight!

Conjecture $\Longrightarrow$ Maximum success probability for efficiently solving $\mathrm{LWE}_{n,p,\phi}$ is $p^{-\Omega(n/\log^2 n)}$.

Limitations of the Original Reduction



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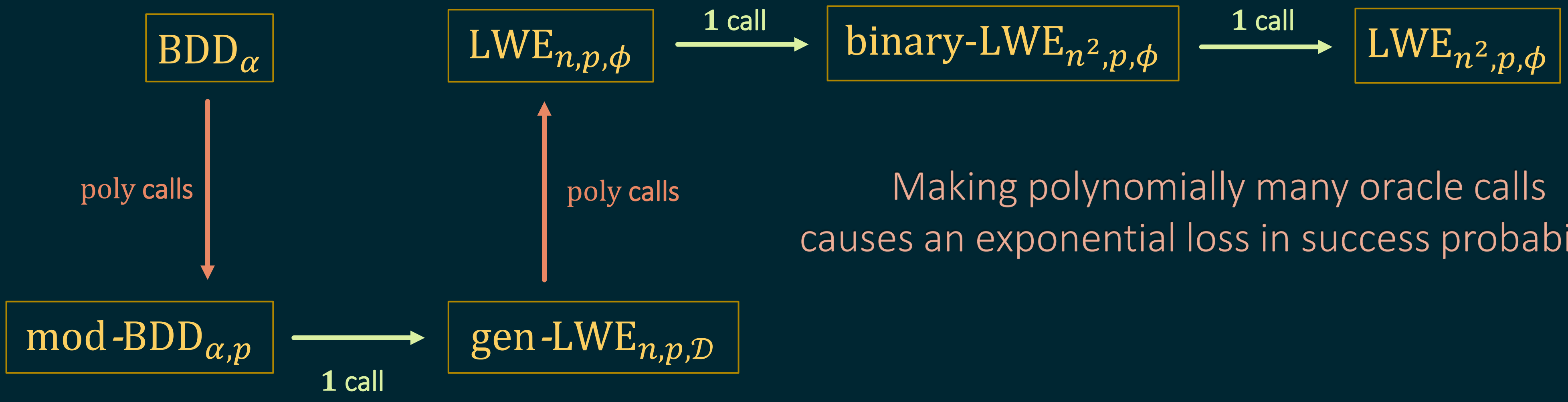

Making polynomially many oracle calls
causes an exponential loss in success probability!

Limitations of the Original Reduction

Reduction algorithm for $\mathcal{P} \to \mathcal{Q}$ makes k calls to oracle for $\mathcal{Q}$.

Success probability of solving $\mathcal{Q}$ is $\geq \epsilon \implies$ success probability of solving $\mathcal{P}$ is $\geq \epsilon^k$.

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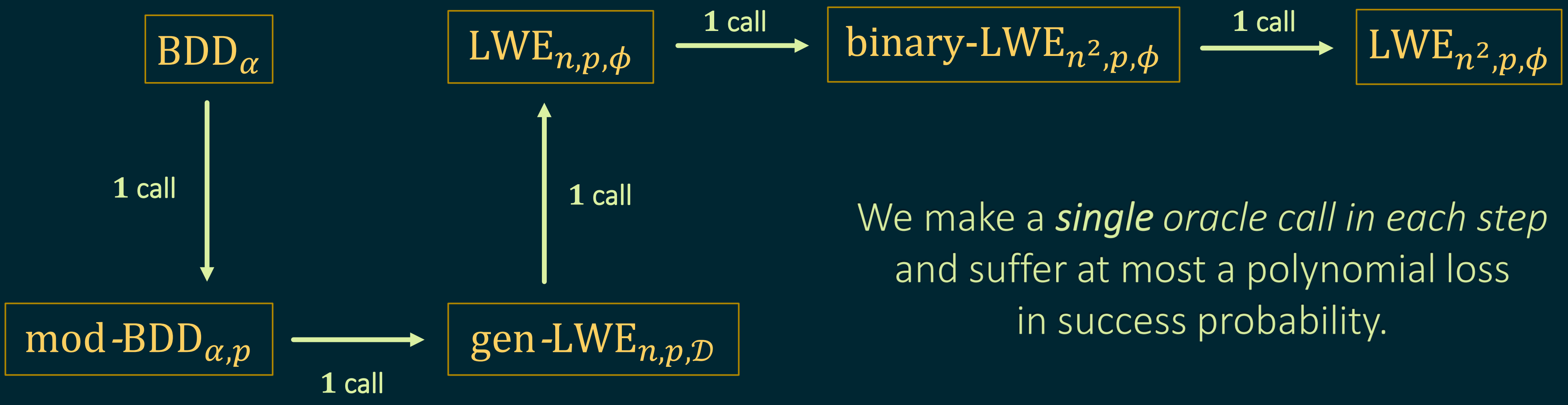
Success probability of solving $\mathcal{P}$ is $\leq \delta \implies$ success probability of solving $\mathcal{Q}$ is $\leq \delta^{1/k}$.

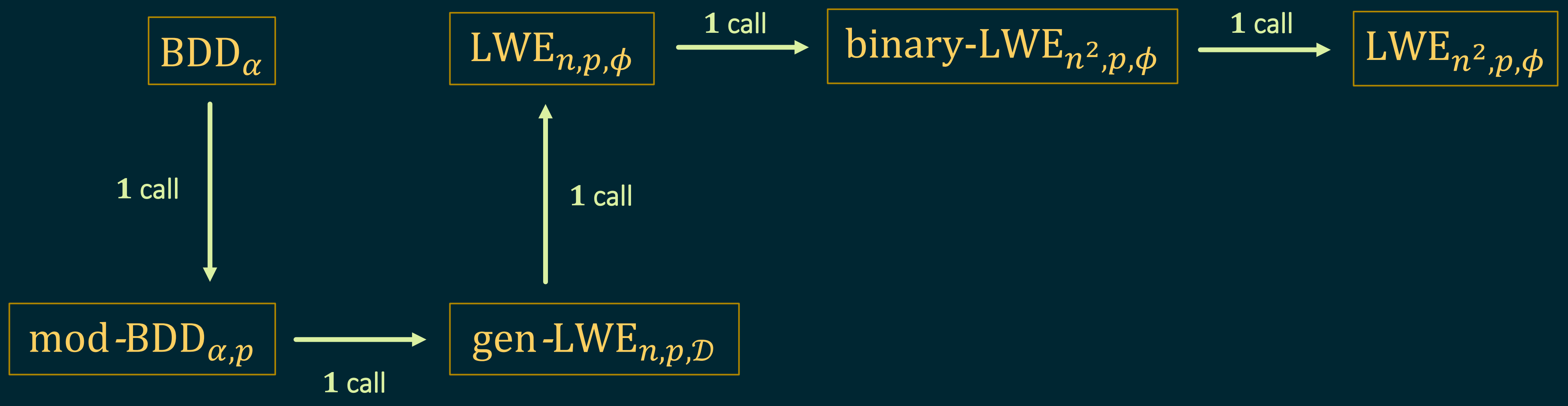
We want just $O(1)$ oracle calls to get a meaningful conclusion.

Our Reduction



We make a ***single*** *oracle call in each step* and suffer at most a polynomial loss in success probability.

Our Reduction



We use the same techniques as [Regev, 2005] and [Brakerski+, 2013],
but with great care to the *explicit loss in success probability* and *number of oracle calls*.

Our Main Result

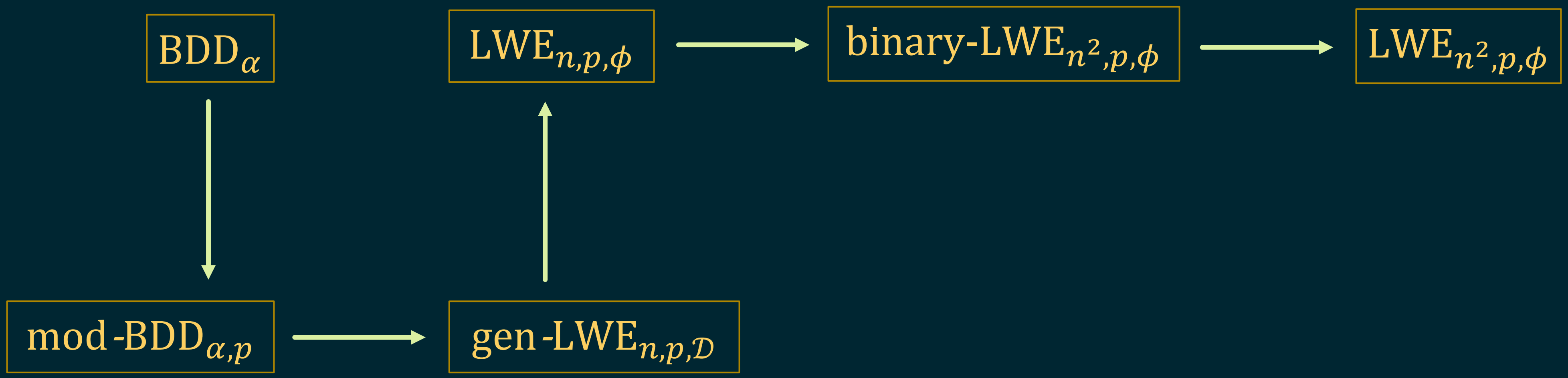
Theorem 1: *(informal)* If no PPT algorithm can solve BDD_{α} for $\alpha \in (0, 1/2)$

with success probability greater than $2^{-\Omega\left(\frac{n^2}{\log n}\right)}$,

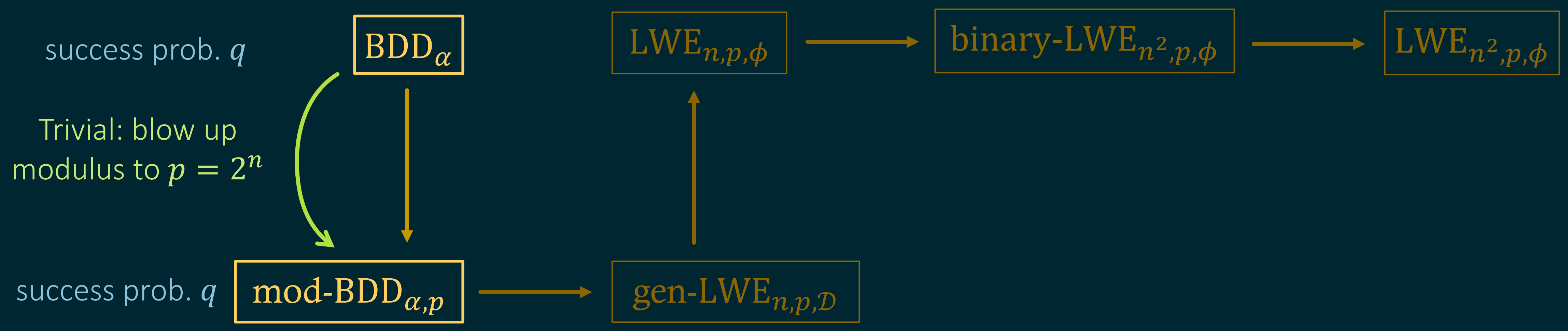
then no PPT algorithm can solve search-$\mathrm{LWE}_{n,p,\phi}$ (even for binary secret)

for dimension n, and modulus $p = \mathrm{poly}(n)$ with success probability $2^{-\frac{n}{\log n}}$.

Our Reduction



Our Proof Techniques



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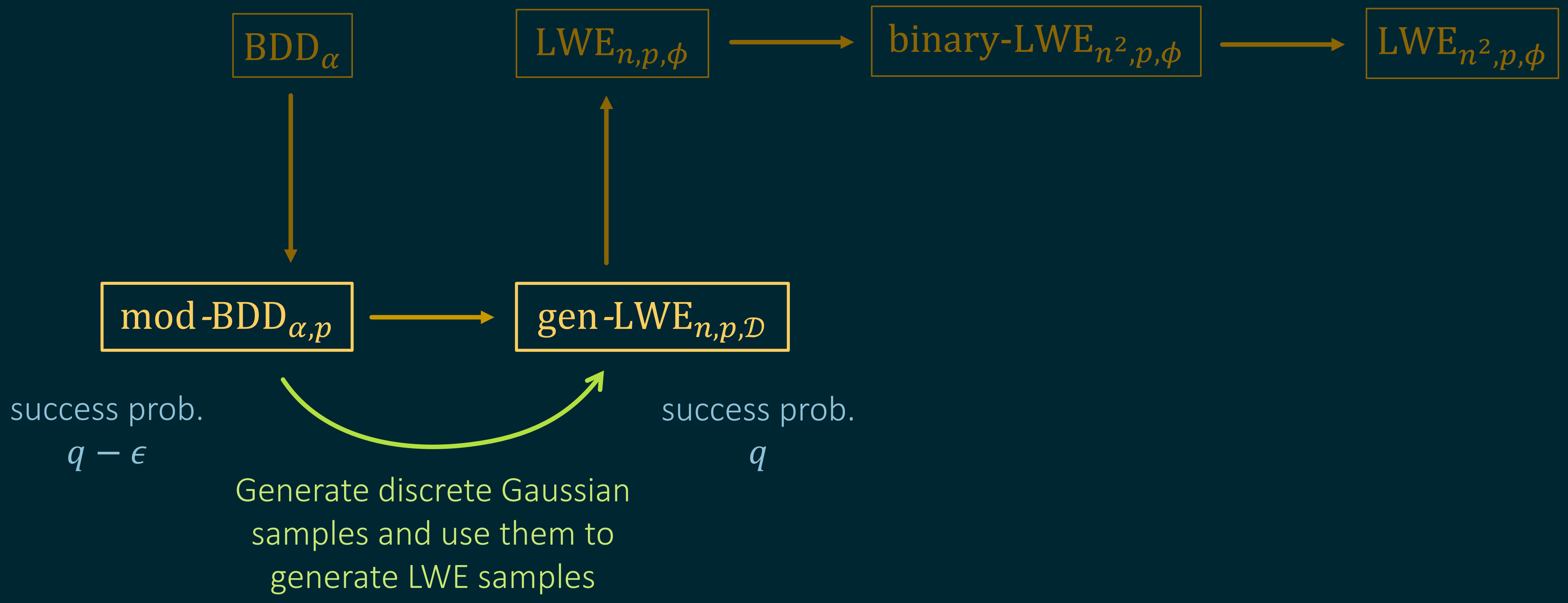

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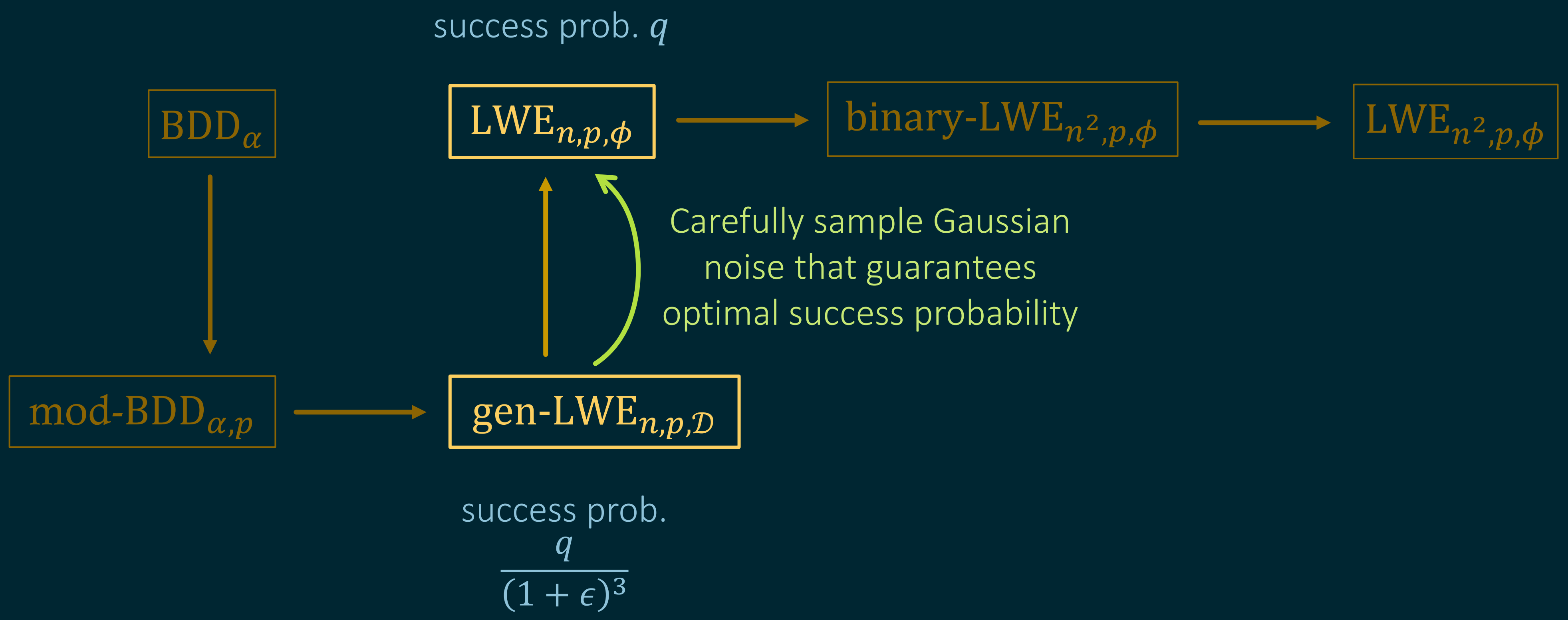

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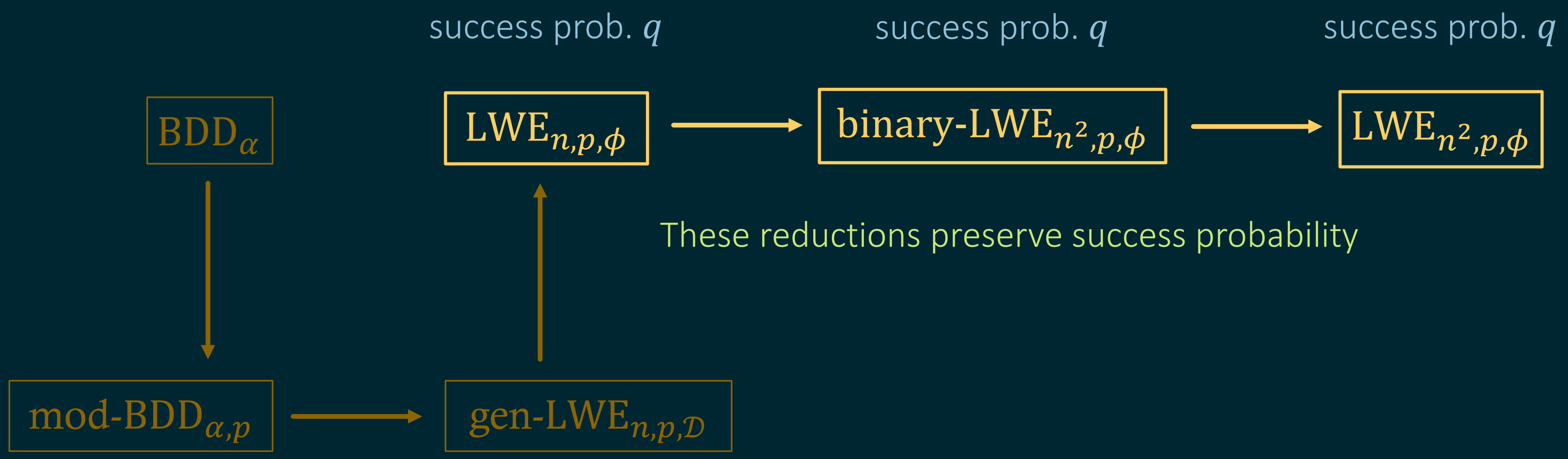

Our Second Result

Theorem 2: *(informal)* If no algorithm can solve search-$\mathrm{LWE}_{n,p}$ for polynomial modulus

with success probability $\boldsymbol{\alpha}$ in *expected* polynomial time,

then no PPT algorithm can "solve" decision-$\mathrm{LWE}_{n,p}$

with probability $\approx \boldsymbol{\alpha}$.

Future Directions

- Establish a similar result for $\mathrm{GapSVP} \to \mathrm{BDD}$ (or prove impossibility).
- Reductions $\mathrm{BDD} \to$ search-LWE and search-$\mathrm{LWE} \to$ decision-LWE are disconnected, because *expected* polynomial-time is a fundamental part of the second reduction.

  Is a workaround possible?
- Use this alternative framework to study the complexity of other computational problems relevant to cryptography or learning.

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Thank you! Questions?