Verifiable Computation for Approximate Homomorphic Encryption Schemes

Ignacio Cascudo, Anamaria Costache, **Daniele Cozzo**, Dario Fiore,
Antonio Guimarães, Eduardo Soria-Vazquez

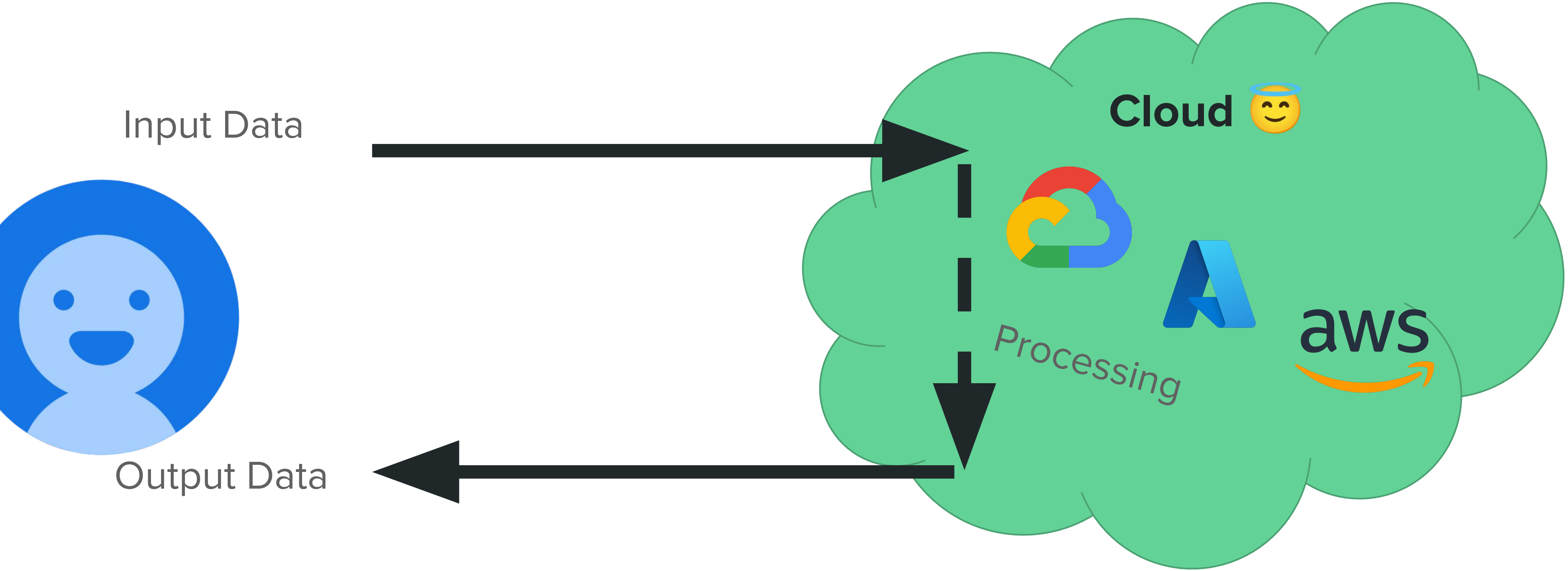
institute IMDEA software

NTNU Norwegian University of Science and Technology

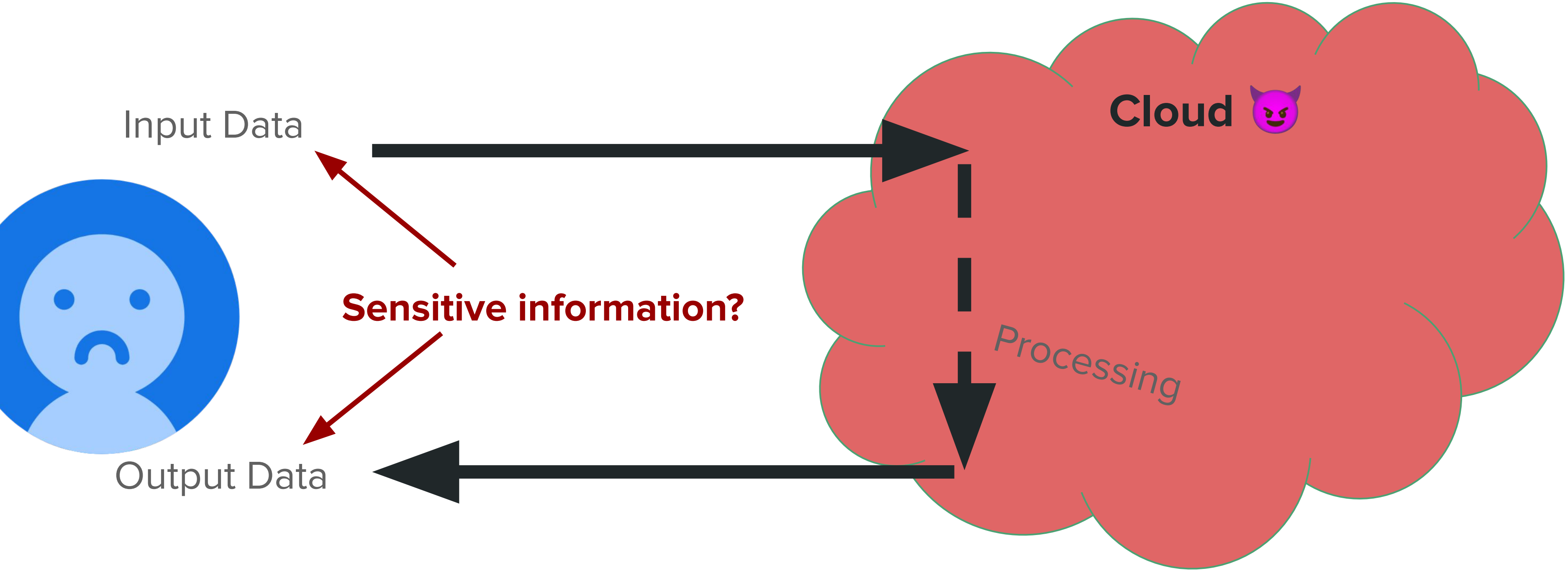
TII Technology Innovation Institute

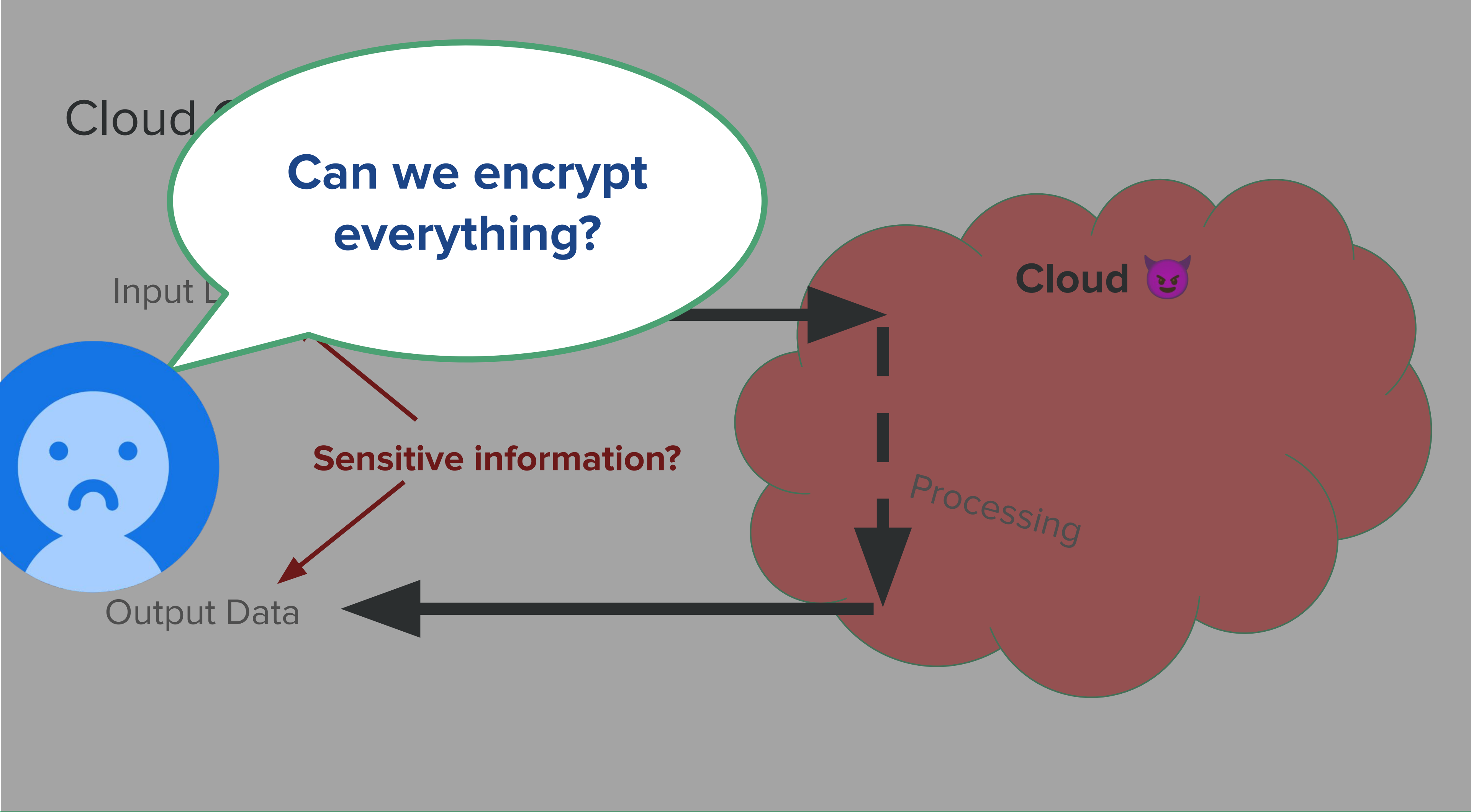
Context

Cloud Computing

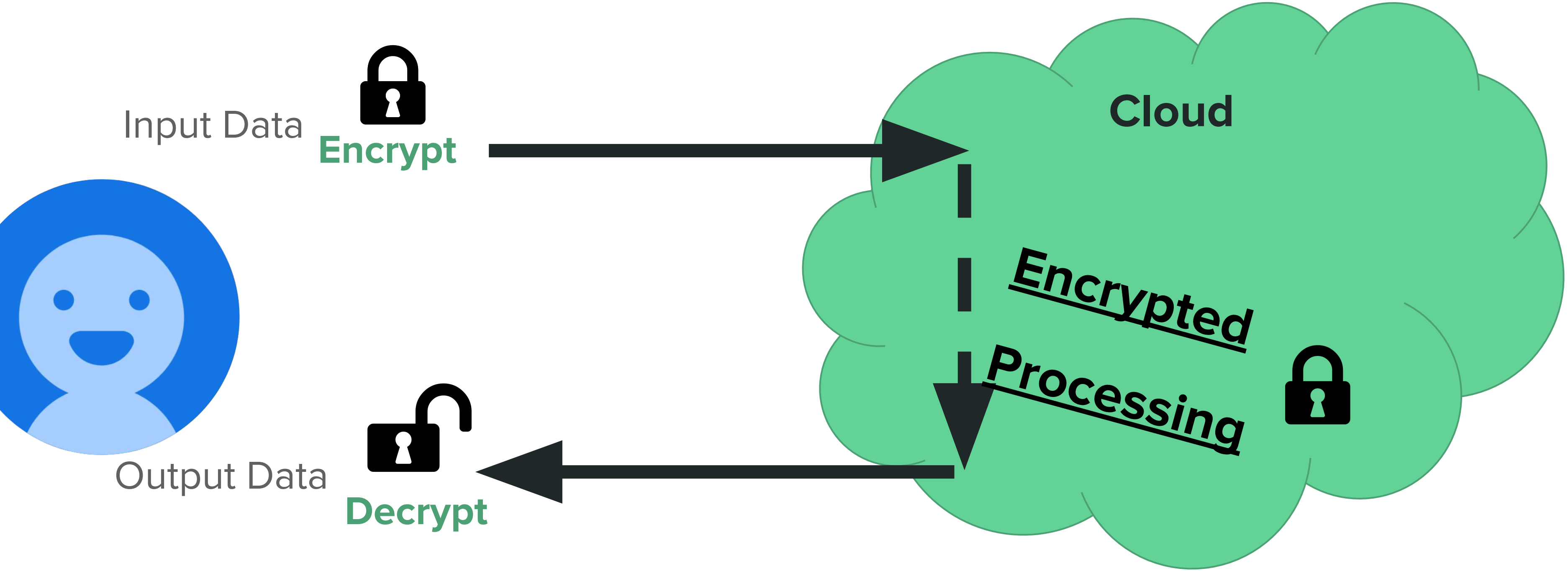


Cloud Computing



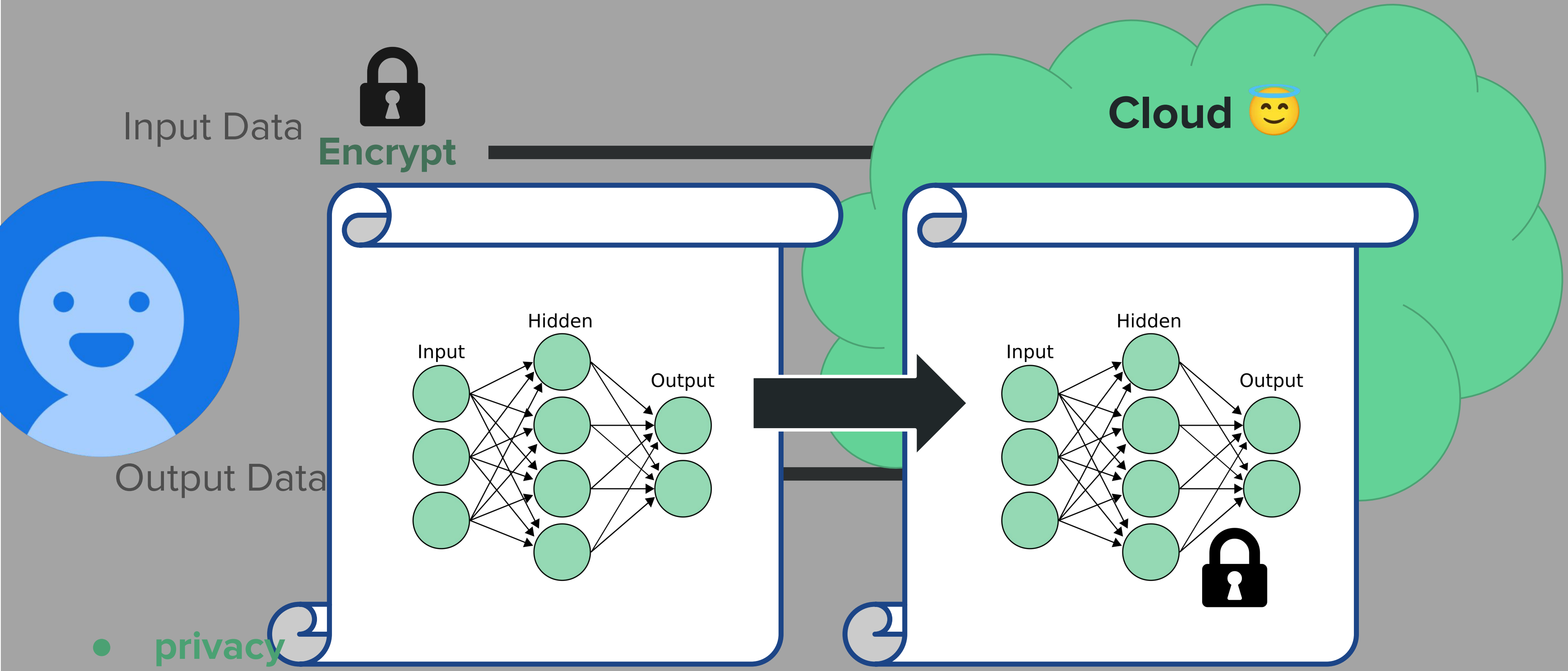

Cloud
Can we encrypt everything?
Input
Cloud 😈
Sensitive information?
Processing
Output Data

Homomorphic Encryption (HE)



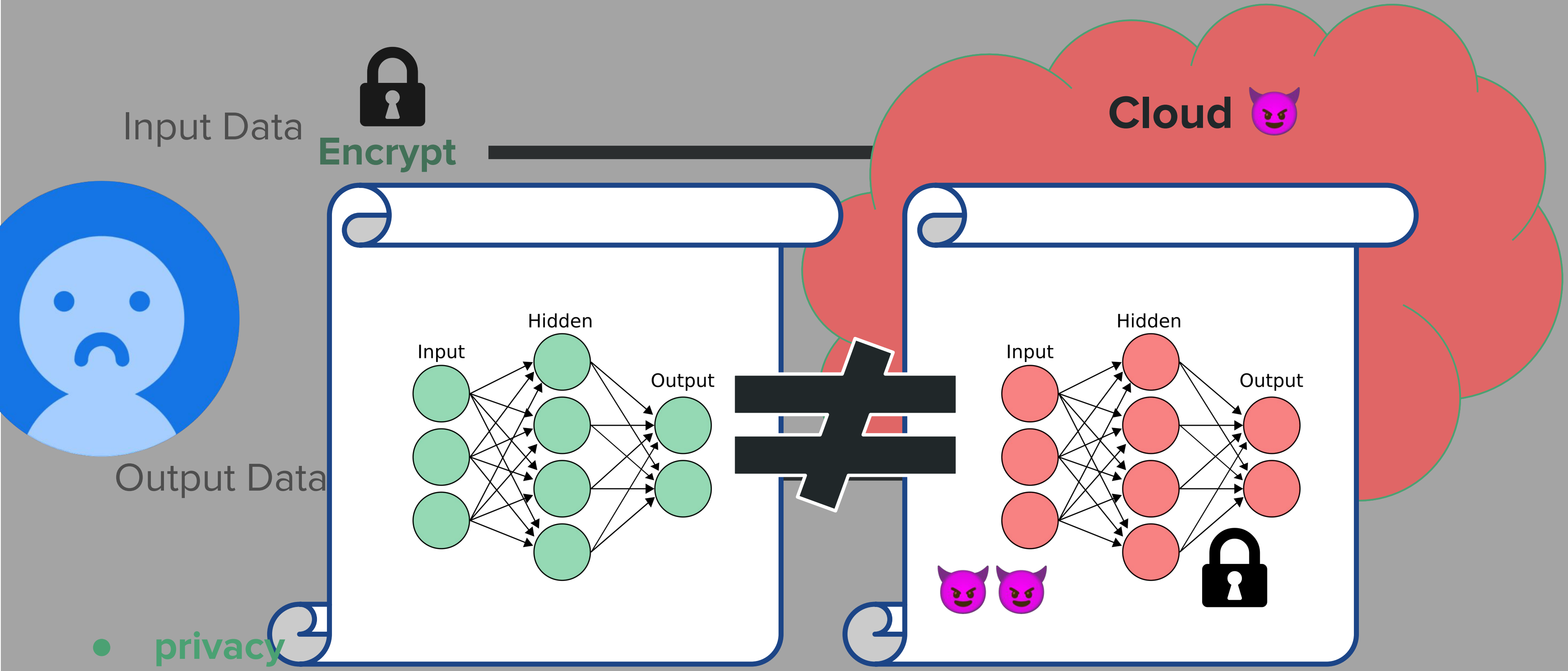
- **privacy**

Homomorphic Encryption (HE)

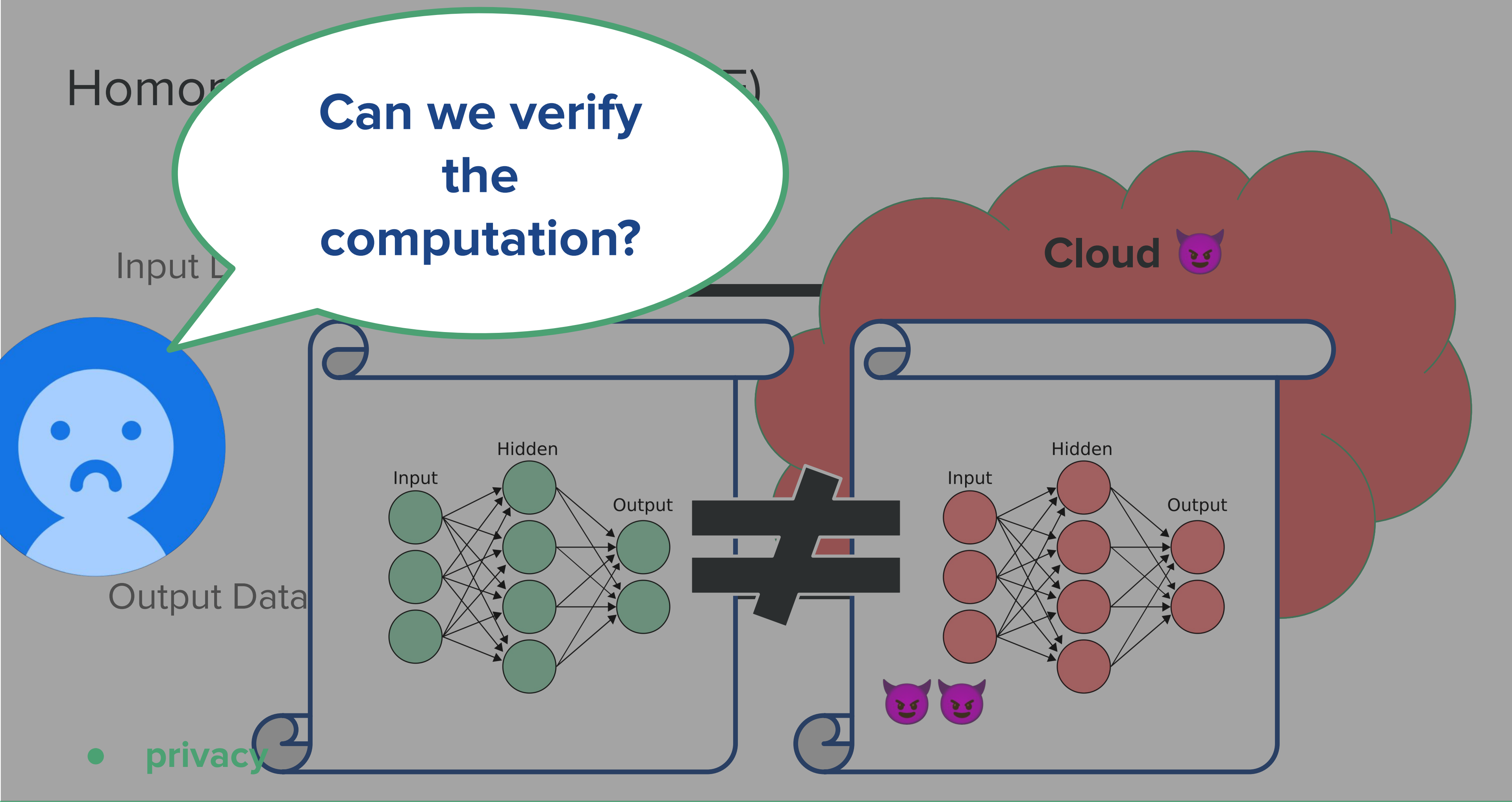


- privacy

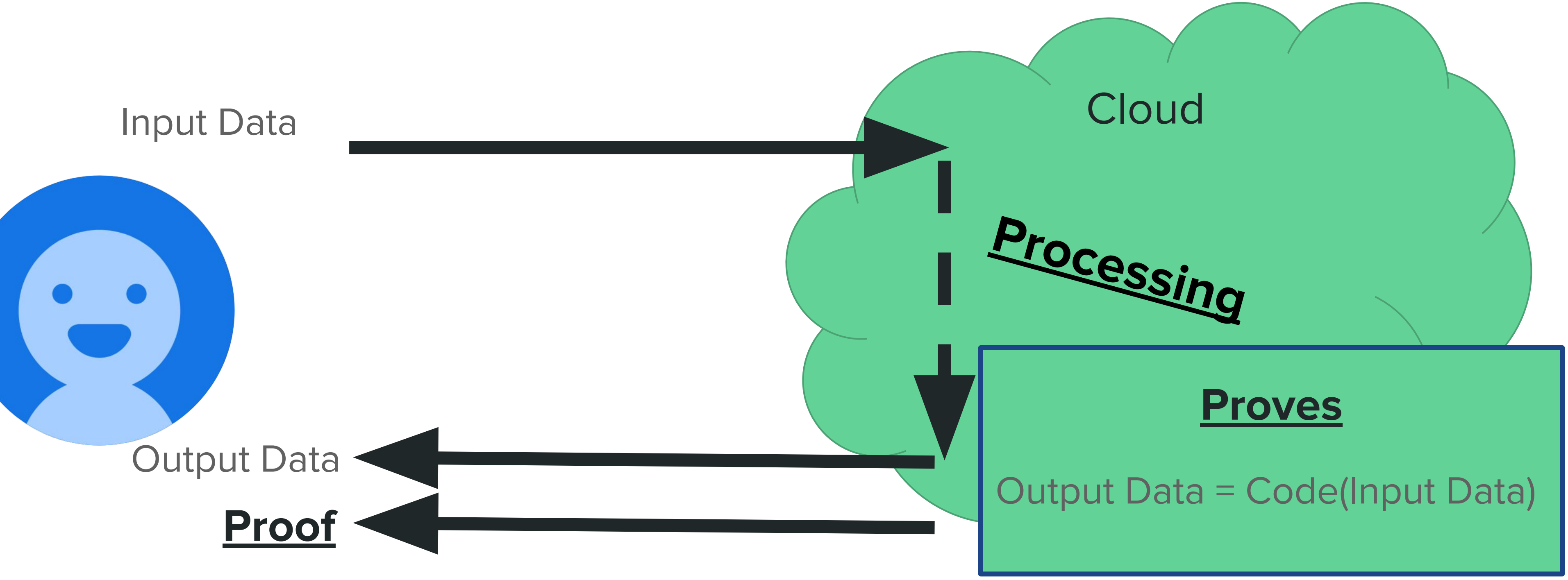
Homomorphic Encryption (HE)



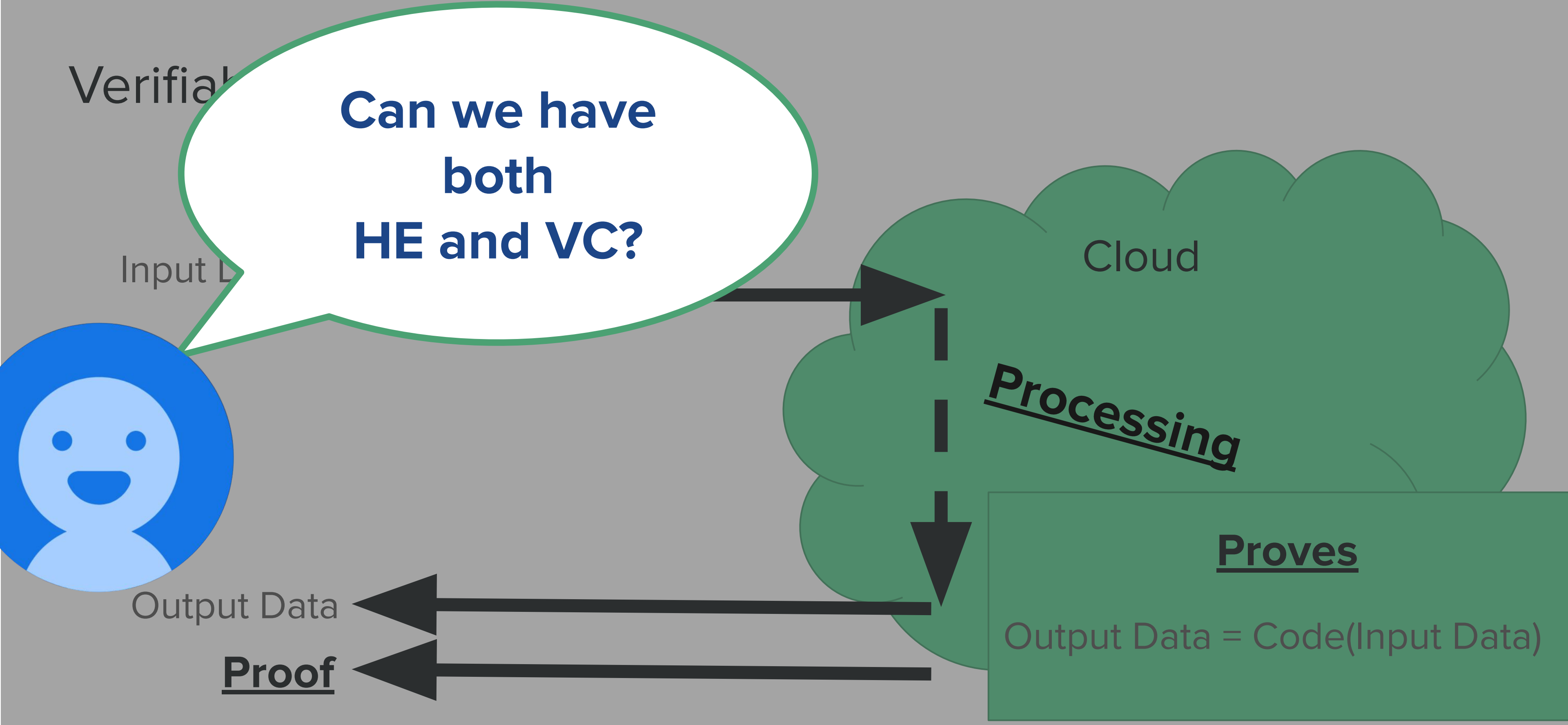
- privacy


Homor
E)
Input D
Output Data
privacy
Can we verify the computation?
Cloud
Input
Hidden
Output
Input
Hidden
Output

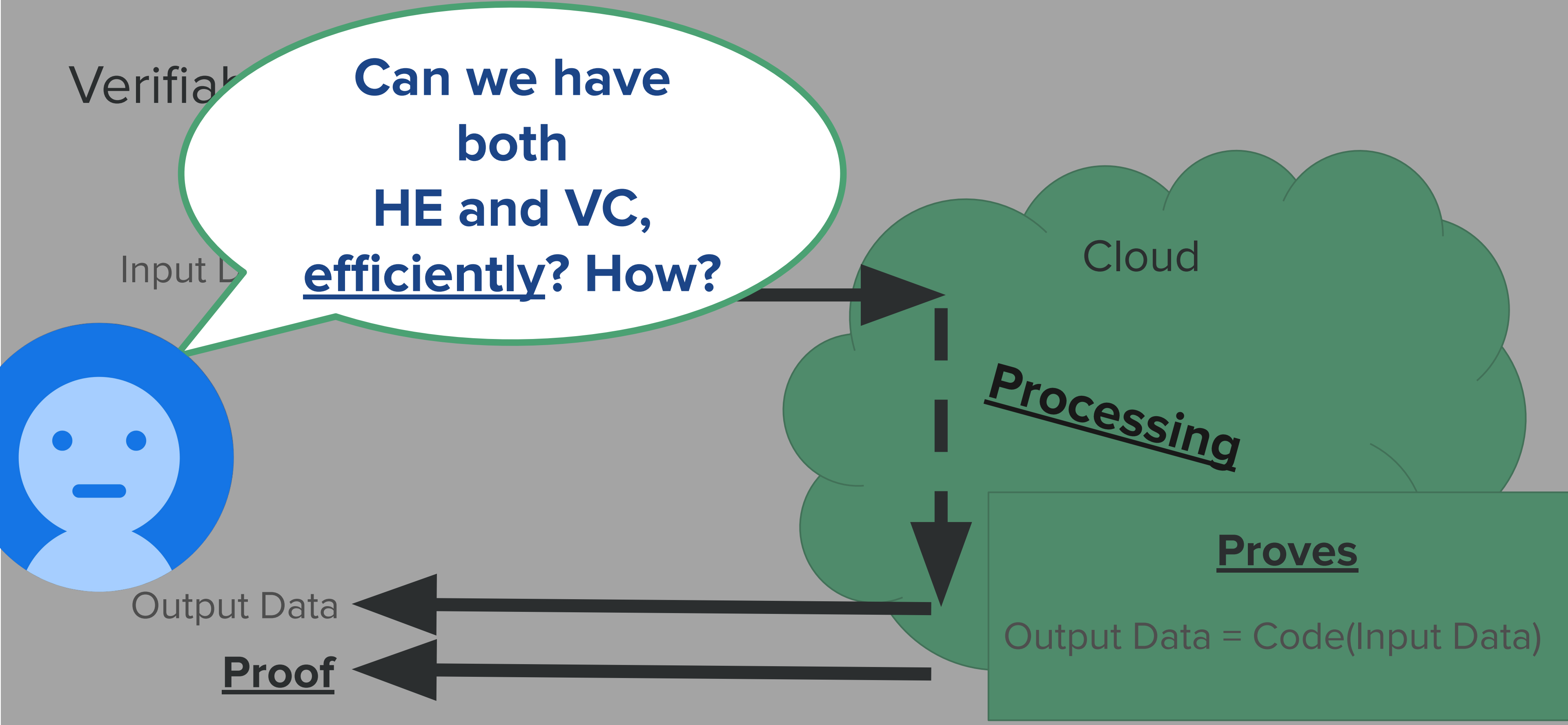
Verifiable computation (VC)



- **Integrity**
- **Succinctness**



- Privacy
- Integrity
- Succinctness



- Privacy
- Integrity
- Succinctness

vHE

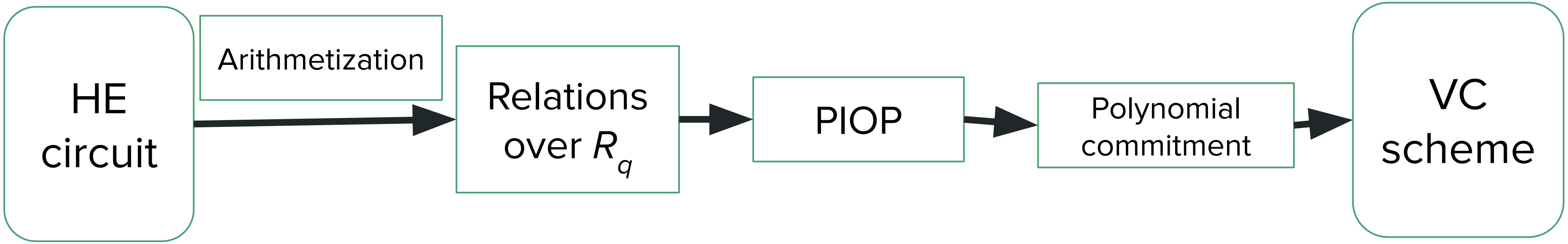
vHE

	Native R_q Arithmetic	Efficient Key Switching / Rescale	Efficient Bootstrapping	Public Verification	CKKS (approximate schemes)
Generic SNARK[1]	❌	❌	❌	✓	✓
Rinocchio[2]	✓	❌	❌	❌	✓
HE-IOPs[3]	✓	✓	✓	❌	❌
Our Work	✓	✓	?	✓	✓

[1] A. Viand, C. Knabenhans, and A. Hithnawi, "*Verifiable Fully Homomorphic Encryption*" arXiv:2301.07041
[2] C. Ganesh, A. Nitulescu, and E. Soria-Vazquez, "*Rinocchio: SNARKs for Ring Arithmetic*" Journal of Cryptology, 2023
[3] D. F. Aranha, A. Costache, A. Guimarães, and E. Soria-Vazquez, "*HELIOPOLIS: Verifiable Computation over Homomorphically Encrypted Data from Interactive Oracle Proofs is Practical*" ASIACRYPT 2024

Our contributions

- vHE for **CKKS**
- **Modular** solution



- **Proof-friendly CKKS**
- **AC satisfiability**
- **Range check**
- **Custom GKR**
- **Range proofs for polynomial rings**
- **Improved Brakedown**

Setting up the ring

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$q \approx 2^{300} \qquad N \approx 2^{14}$$

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$R_q$$

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$$R_q$$

- Efficient HE computations
  - RNS

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$R_q$$

- Efficient HE computations
  - RNS
- Soundness
  - Large exceptional set

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$q = \prod_{i=1}^{L} p_i$$

$$R_q \cong \begin{matrix} R_{p_1} \\ R_{p_2} \\ R_{p_3} \end{matrix}$$

- Efficient HE computations
  - RNS
- Soundness
  - Large exceptional set

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$R_q \underset{q = \prod_{i=1}^{L} p_i}{\cong} R_{p_1} \times R_{p_2} \times R_{p_3}$$

$$R_{p_1} \underset{X^N + 1 = \prod_{i=1}^{k}(X^d - \zeta^{2i-1}) \mod p_1}{\cong} R_{11} \; R_{12} \; R_{13} \; R_{14}$$

$$R_{p_2} \underset{X^N + 1 = \prod_{i=1}^{k}(X^d - \zeta^{2i-1}) \mod p_2}{\cong} R_{21} \; R_{22} \; R_{23} \; R_{24}$$

$$R_{p_3} \underset{X^N + 1 = \prod_{i=1}^{k}(X^d - \zeta^{2i-1}) \mod p_3}{\cong} R_{31} \; R_{32} \; R_{33} \; R_{34}$$

- Efficient HE computations
  - RNS
- Soundness
  - Large exceptional set

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$R_q \cong R_{p_1} \times R_{p_2} \times R_{p_3}, \quad q = \prod_{i=1}^{L} p_i$$

$$X^N + 1 = \prod_{i=1}^{k} (X^d - \zeta^{2i-1}) \mod p_1 \qquad R_{p_1} \cong \mathbb{F}_{p_1^d} \times \mathbb{F}_{p_1^d} \times \mathbb{F}_{p_1^d} \times \mathbb{F}_{p_1^d}$$

$$X^N + 1 = \prod_{i=1}^{k} (X^d - \zeta^{2i-1}) \mod p_2 \qquad R_{p_2} \cong \mathbb{F}_{p_2^d} \times \mathbb{F}_{p_2^d} \times \mathbb{F}_{p_2^d} \times \mathbb{F}_{p_2^d}$$

$$X^N + 1 = \prod_{i=1}^{k} (X^d - \zeta^{2i-1}) \mod p_3 \qquad R_{p_3} \cong \mathbb{F}_{p_3^d} \times \mathbb{F}_{p_3^d} \times \mathbb{F}_{p_3^d} \times \mathbb{F}_{p_3^d}$$

- Efficient HE computations
  - RNS
- Soundness
  - Large exceptional set

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$q = \prod_{i=1}^{L} p_i$$

$$R_q \cong R_{p_1},\ R_{p_2},\ R_{p_3}$$

$$X^N + 1 = \prod_{i=1}^{k} (X^d - \zeta^{2i-1}) \mod p_1$$

$$\cong \quad \mathbb{F}_{p_1^d}\ \mathbb{F}_{p_1^d}\ \mathbb{F}_{p_1^d}\ \mathbb{F}_{p_1^d}$$

X^N … $\mathbb{F}_{p_2^d}$

X^N … $\mathbb{F}_{p_3^d}$

Size of our exceptional set!

- Efficient HE computations
  - RNS
- Soundness
  - Large exceptional set

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$R_q \quad q = \prod_{i=1}^{L} p_i \quad \cong \quad R_{p_1},\ R_{p_2},\ R_{p_3}$$

$$X^N + 1 = \prod_{i=1}^{k} (X^d - \zeta^{2i-1}) \mod p_1$$

$$\cong \quad \mathbb{F}_{p_1^d}\ \mathbb{F}_{p_1^d}\ \mathbb{F}_{p_1^d}\ \mathbb{F}_{p_1^d}$$

X^N

$\mathbb{F}_{p_2^d}$

X^N

$\mathbb{F}_{p_3^d}$

Size of our exceptional set!

Optimal performance/security:
$|p_i| = 49$, $d = 4$

- Efficient HE computations
  - RNS
- Soundness
  - Large exceptional set

The polynomial ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

R_q

$q = \prod_{i=1}^{L}$

$\cong$

$\mathbb{F}_{p_1^d}$ $\mathbb{F}_{p_1^d}$ $\mathbb{F}_{p_1^d}$

$\mathbb{F}_{p_2^d}$

$\mathbb{F}_{p_3^d}$

al set!

curity:

- Efficient HE comp
  - RNS
- Soundness
  - Large exceptional set

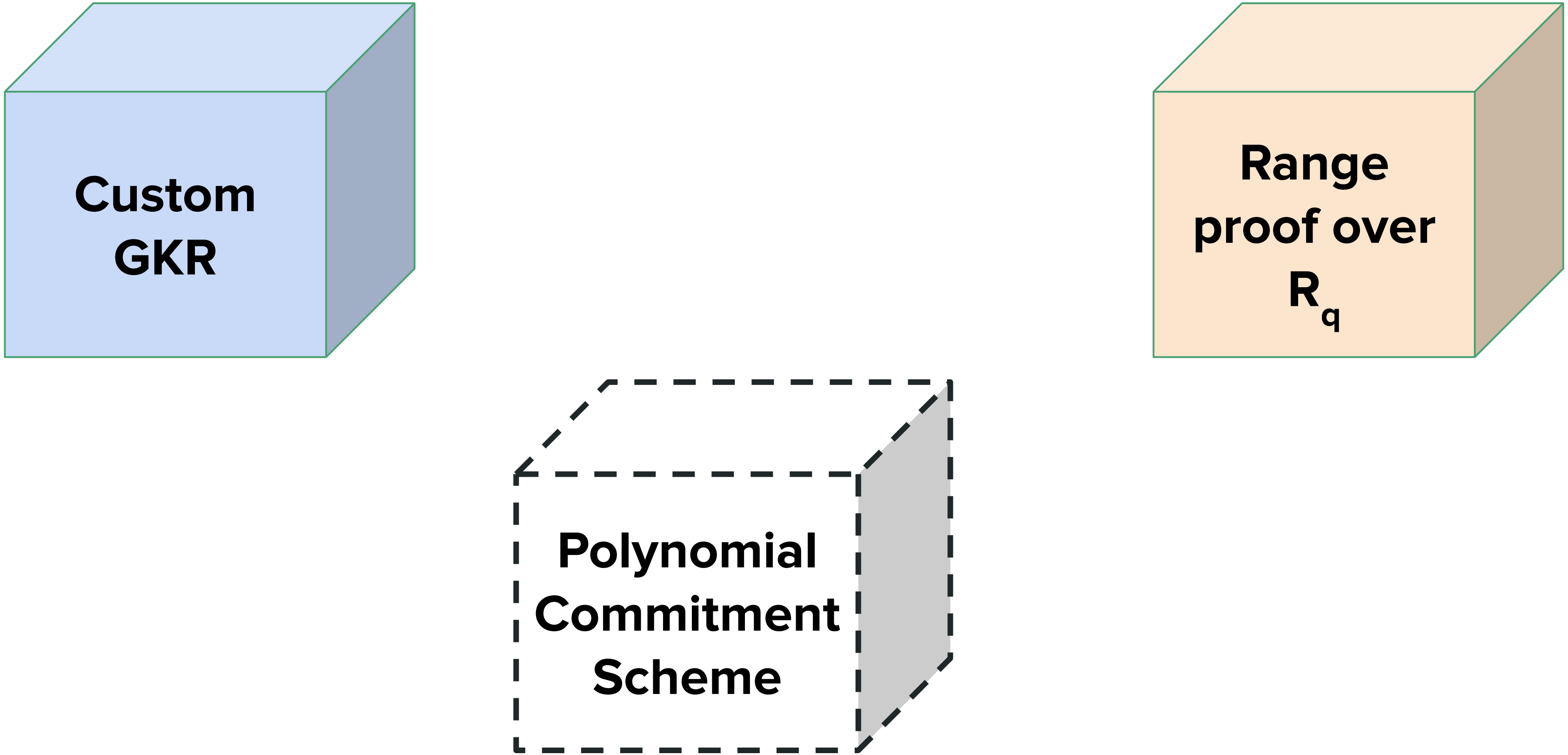
Efficient arithmetic for almost-fully-splitting rings:

- Incomplete NTTs[1]
- Cost:
  - d = 2 -> ~5%
  - d = 4 -> 20%

[1] V. Lyubashevsky and G. Seiler, "NTTRU: Truly Fast NTRU Using NTT," IACR Transactions on Cryptographic Hardware and Embedded Systems, pp. 180–201, May 2019, doi: 10.13154/tches.v2019.i3.180-201.

Proof-friendly CKKS

Proof components



CKKS

- An approximate scheme:

- RLWE ciphertext:

$$(a_0, a_1) \in R_q^2$$

- RNS representation (with e.g. 3 components):

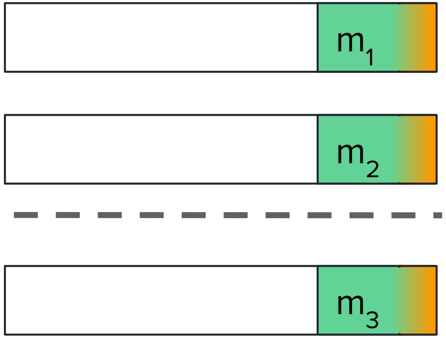
CKKS Addition

$a = (a_{01}, a_{02}, a_{03}) \; (a_{11}, a_{12}, a_{13})$

$+$

$b = (b_{01}, b_{02}, b_{03}) \; (b_{11}, b_{12}, b_{13})$

$c = (c_{01}, c_{02}, c_{03}) \; (c_{11}, c_{12}, c_{13})$



$m_3 = m_1 + m_2$

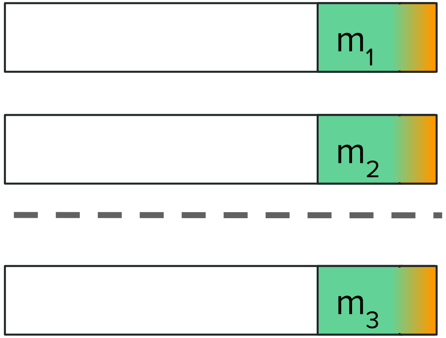
CKKS Addition

$a = [a_{01}\ a_{02}\ a_{03}]\ [a_{11}\ a_{12}\ a_{13}]$

$+$

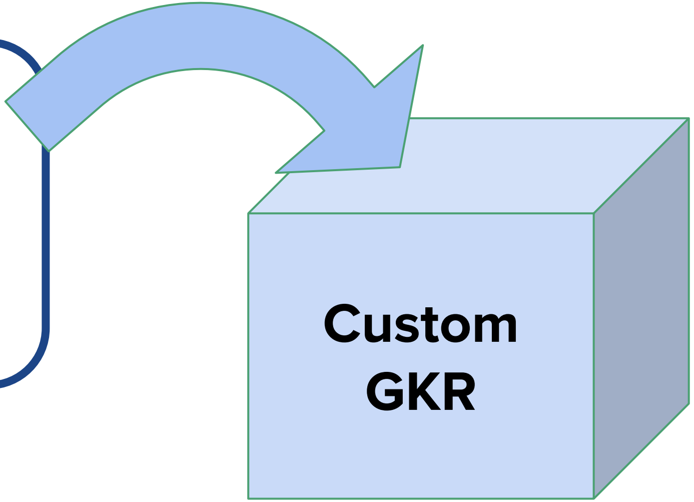
$b = [b_{01}\ b_{02}\ b_{03}]\ [b_{11}\ b_{12}\ b_{13}]$

$c = [c_{01}\ c_{02}\ c_{03}]\ [c_{11}\ c_{12}\ c_{13}]$



$m_3 = m_1 + m_2$

Pure arithmetic!



CKKS Multiplication *Level 1*

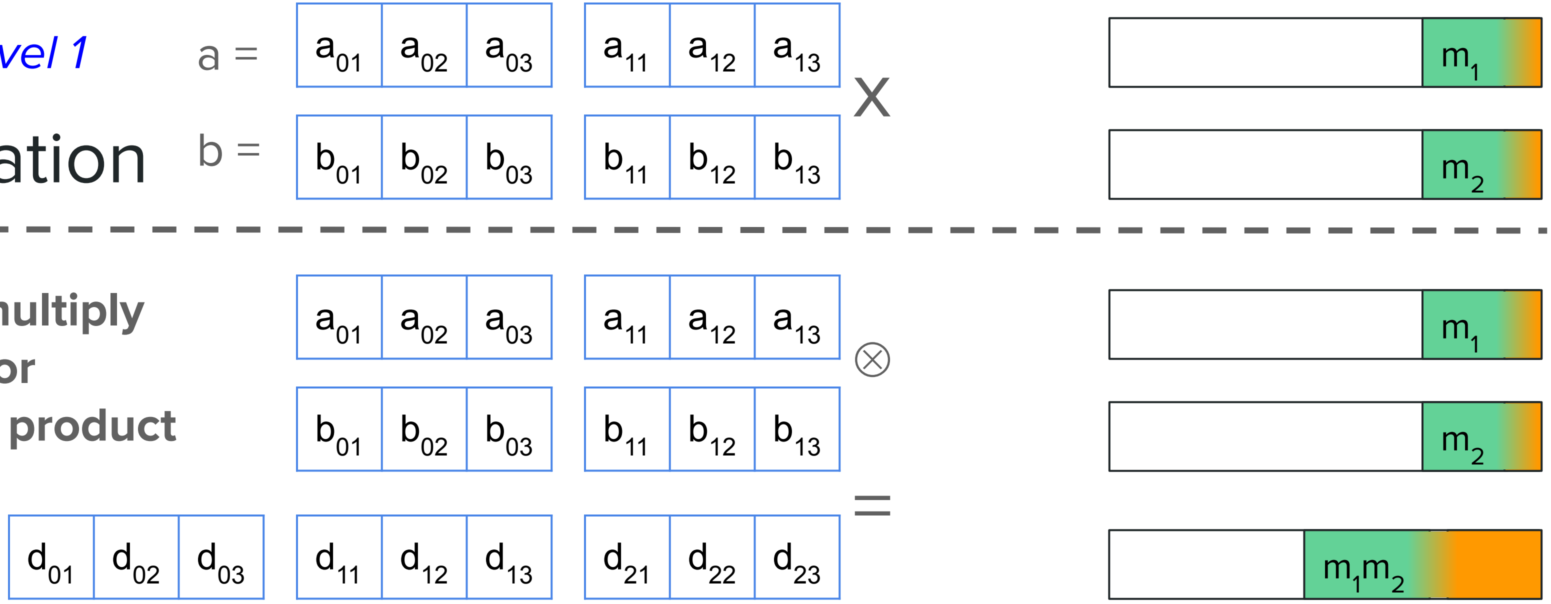
$a = [a_{01} \; a_{02} \; a_{03}] \; [a_{11} \; a_{12} \; a_{13}]$

x

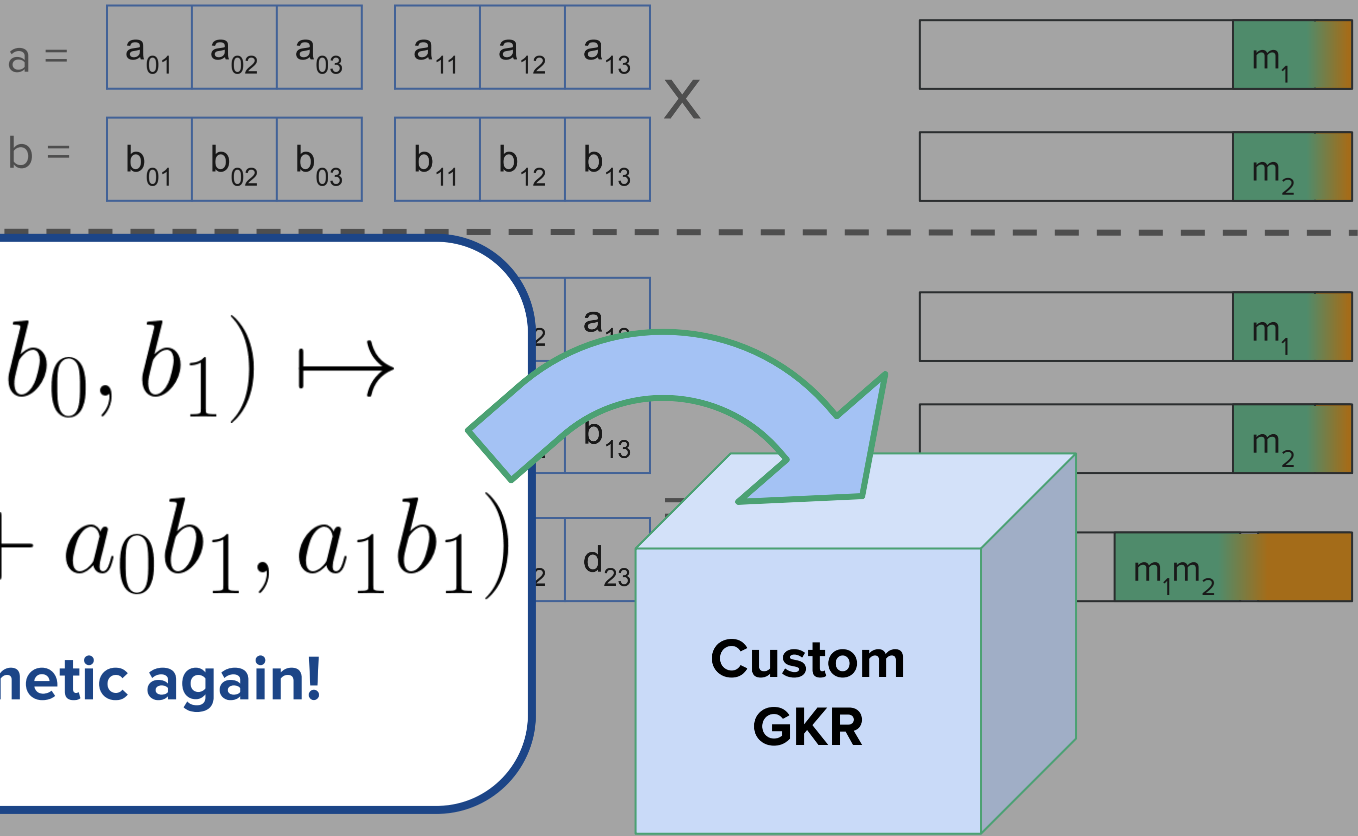
$b = [b_{01} \; b_{02} \; b_{03}] \; [b_{11} \; b_{12} \; b_{13}]$

CKKS *Level 1* Multiplication

Pre-multiply or Tensor product



CKKS *Level 1* Multiplication



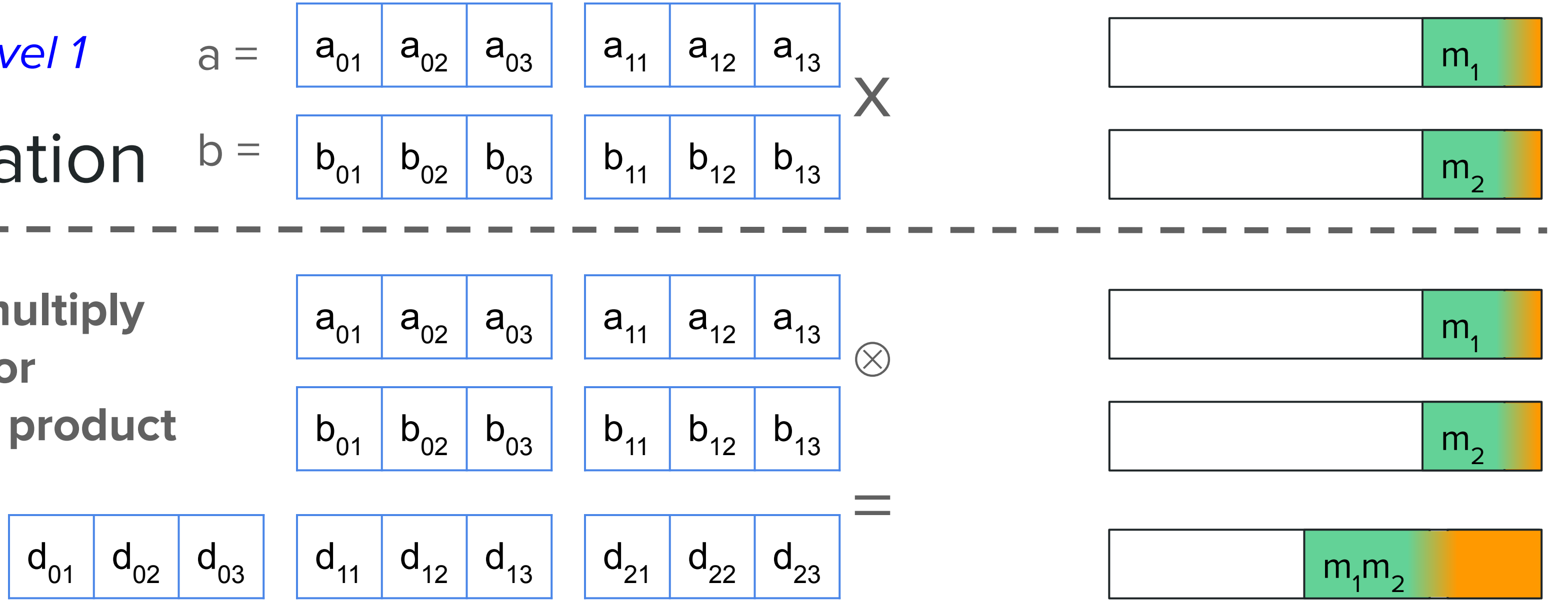
$$(a_0, a_1) \otimes (b_0, b_1) \mapsto$$
$$(a_0 b_0, a_1 b_0 + a_0 b_1, a_1 b_1)$$

Pure arithmetic again!

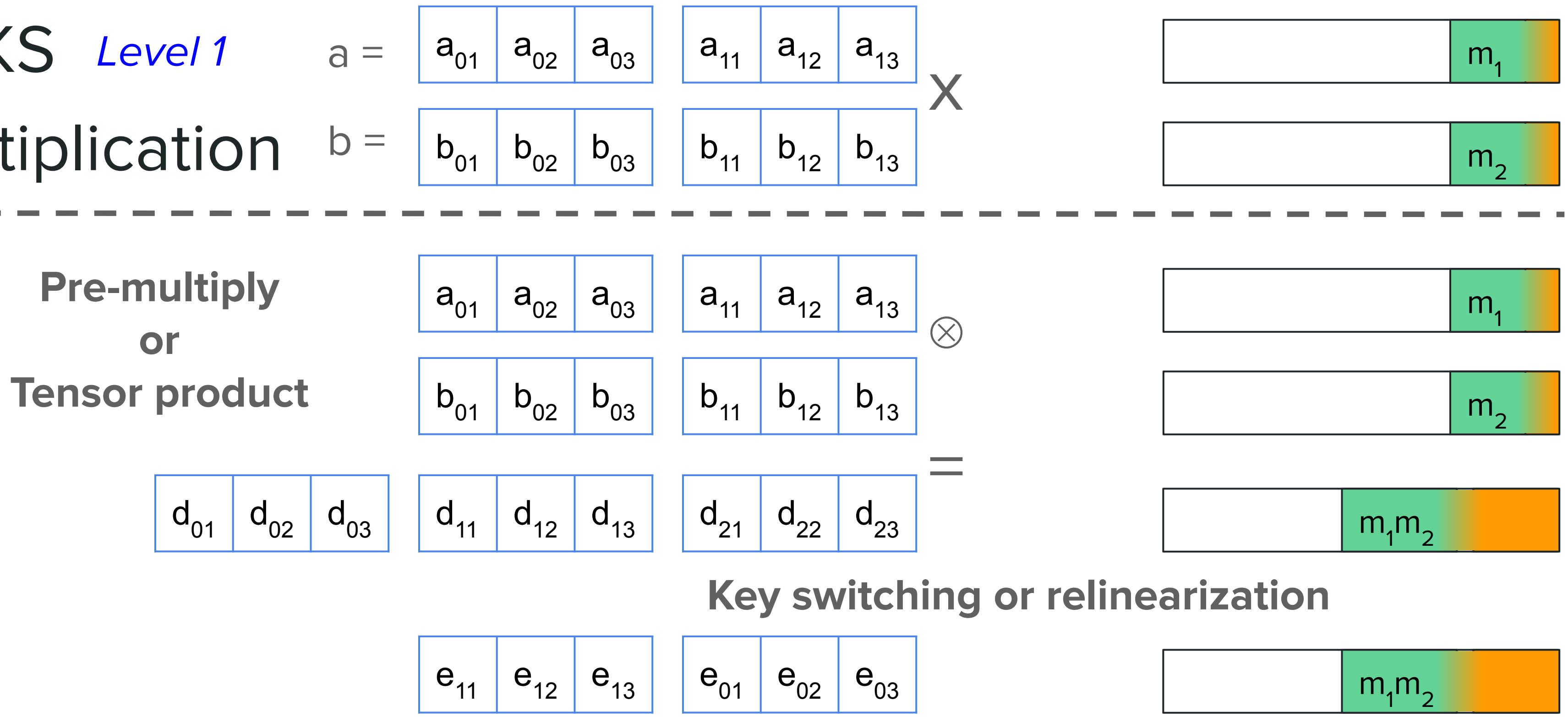
CKKS Multiplication

Level 1

Pre-multiply or Tensor product



CKKS *Level 1* Multiplication



CKKS *Level 1*

Multi

$a = $ | a_{01} | a_{02} | a_{03} | | a_{11} | a_{12} | a_{13} | × | m_1
b_{11} | b_{12} | b_{13} | | m_2

Key Switching:

d_0, d_1, d_2

$$e_0 = d_0 + \langle \text{evk}, CRT^{-1}(d_2) \rangle$$
$$e_1 = d_1 + \langle \text{evk}, CRT^{-1}(d_2) \rangle$$

e_0, e_1

a_{11}	a_{12}	a_{13}	m_1

$\otimes$

b_{11}	b_{12}	b_{13}	m_2

=

d_{21}	d_{22}	d_{23}	$m_1 m_2$

Key switching or relinearization

e_{01}	e_{02}	e_{03}	$m_1 m_2$

CKKS *Level 1*

Multi…

a = a_{01} a_{02} a_{03} | a_{11} a_{12} a_{13} — m_1

X

b_{11} b_{12} b_{13} — m_2

=

d_{21} d_{22} d_{23} — m_1m_2

Key switching or relinearization

e_{01} e_{02} e_{03} — m_1m_2

Key Switching:

d_0 d_1 d_2

$$e_0 = d_0 + \langle \text{evk}, CRT^{-1}(d_2) \rangle$$

$$e_1 = d_1 + \langle \text{evk}, CRT^{-1}(d_2) \rangle$$

e_0 e_1

Not algebraic!

We don't verify:

Inputs: d_0, d_1, d_2

$$e_0 = d_0 + \langle \text{evk}, CRT^{-1}(d_2)\rangle$$
$$e_1 = d_1 + \langle \text{evk}, CRT^{-1}(d_2)\rangle$$

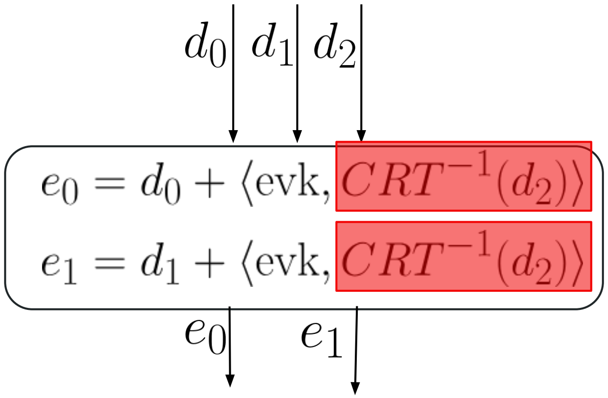
Outputs: e_0, e_1

Instead, we:

1. Ask the prover to provide $\mathbf{w} = CRT^{-1}(d_2)$
2. Check:
   a. $d_2 - CRT(\mathbf{w}) = 0$
   b. $\|\mathbf{w}[i]\| < p_i$
3. Compute and check:

$$e_0 = d_0 + \langle evk, \mathbf{w}\rangle$$
$$e_1 = d_1 + \langle evk, \mathbf{w}\rangle$$

We don't verify:



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3. Compute and check:

$$e_0 = d_0 + \langle evk, \mathbf{w} \rangle$$
$$e_1 = d_1 + \langle evk, \mathbf{w} \rangle$$

Instead, we:

1. Ask the prover to provide **w** = $CRT^{-1}(d_2)$
2. Check:
   a. $d_2 - CRT(\mathbf{w}) = 0$
   b. $||\mathbf{w}[i]|| < p_i$
3. Compute and check:

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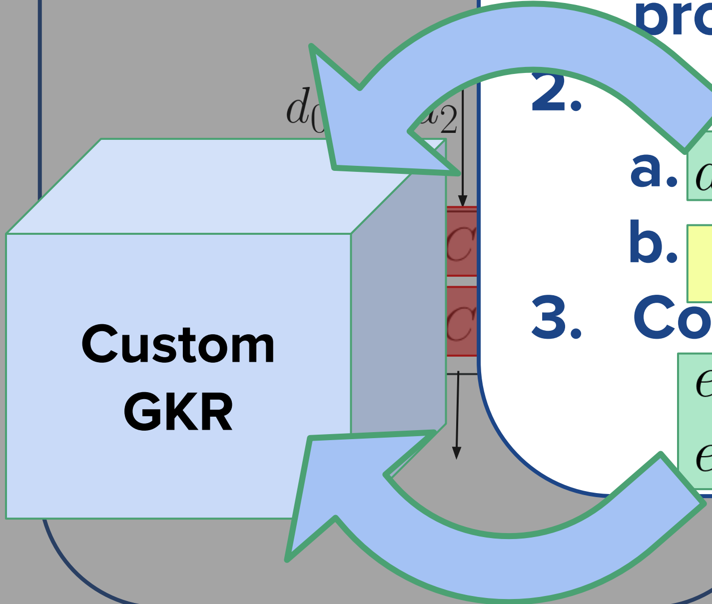
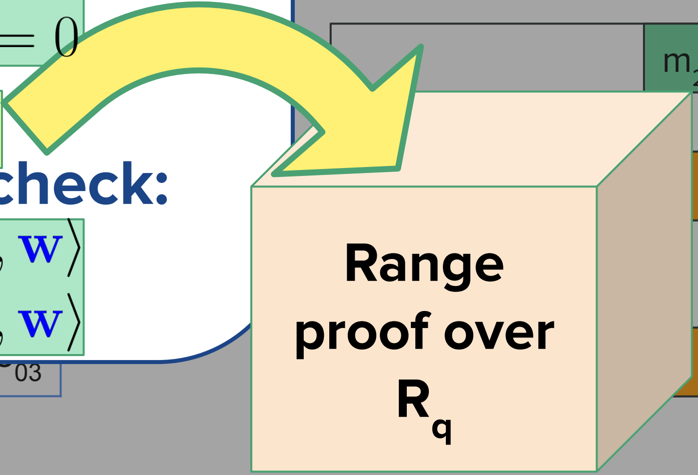
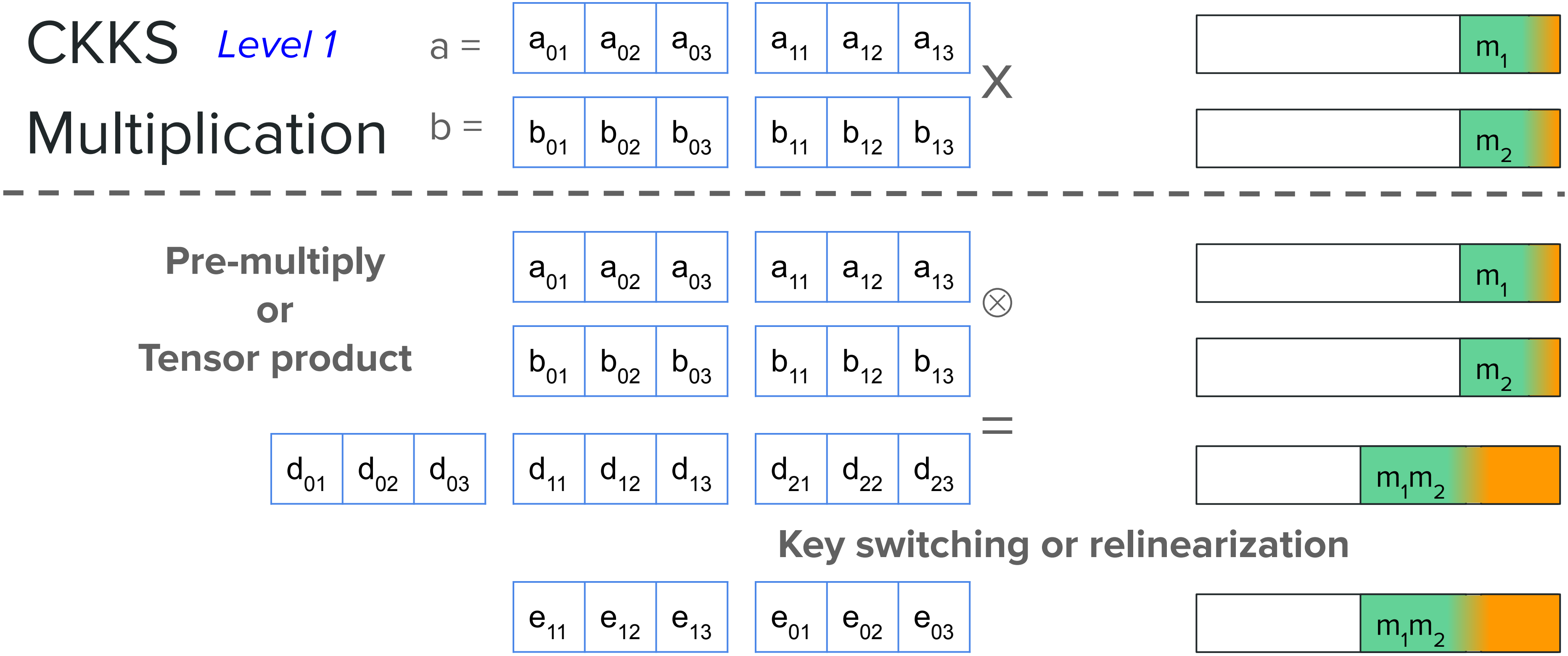
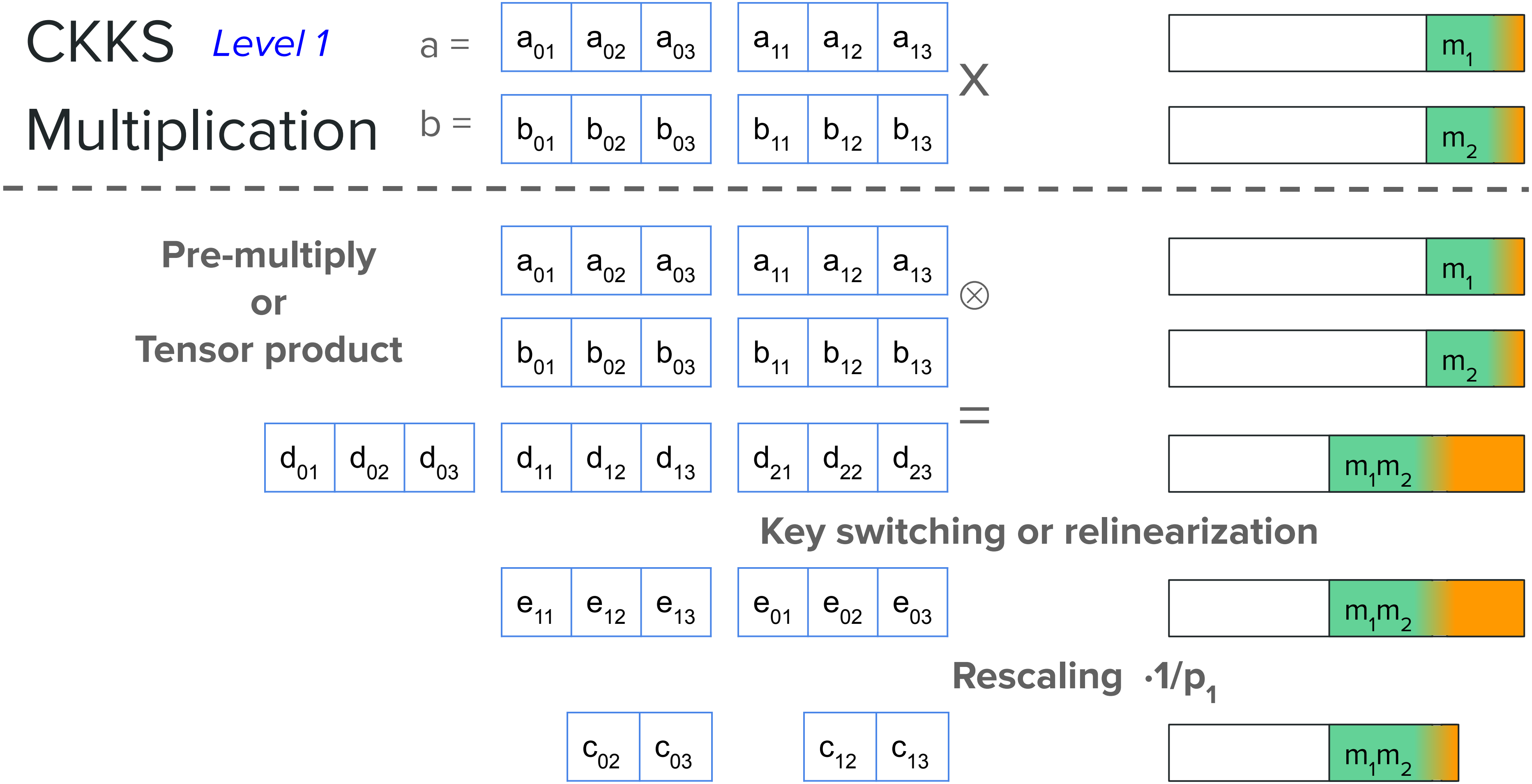




CKKS *Level 1* Multiplication



CKKS Multiplication *Level 1*



CKKS *Level 1*

Multi

$a =$ a_{01} a_{02} a_{03} a_{11} a_{12} a_{13} — m_1

X

b_{11} b_{12} b_{13} — m_2

Rescaling:

$$c_0 = (e_0 - [e_0]_{p_l}) \cdot p_l^{-1}$$
$$c_1 = (e_1 - [e_1]_{p_l}) \cdot p_l^{-1}$$

Not algebraic!

=

d_{21} d_{22} d_{23} — $m_1 m_2$

Key switching or relinearization

e_{01} e_{02} e_{03} — $m_1 m_2$

Rescaling $\cdot 1/p_1$

c_{02} c_{03} c_{12} c_{13} — $m_1 m_2$

CKKS *Level 1*

a = a_{01} a_{02} a_{03} a_{11} a_{12} a_{13} ... m_1

Rescaling:

$$c_0 = (e_0 - [e_0]_{p_l}) \cdot p_l^{-1}$$
$$c_1 = (e_1 - [e_1]_{p_l}) \cdot p_l^{-1}$$

Can be rewritten as **Euclidean division**

$$e_i = c_i \cdot p_l + [e_i]_{p_l}$$

Prover inputs $w_{\mathrm{quo},i}$ $w_{\mathrm{rmd},i}$

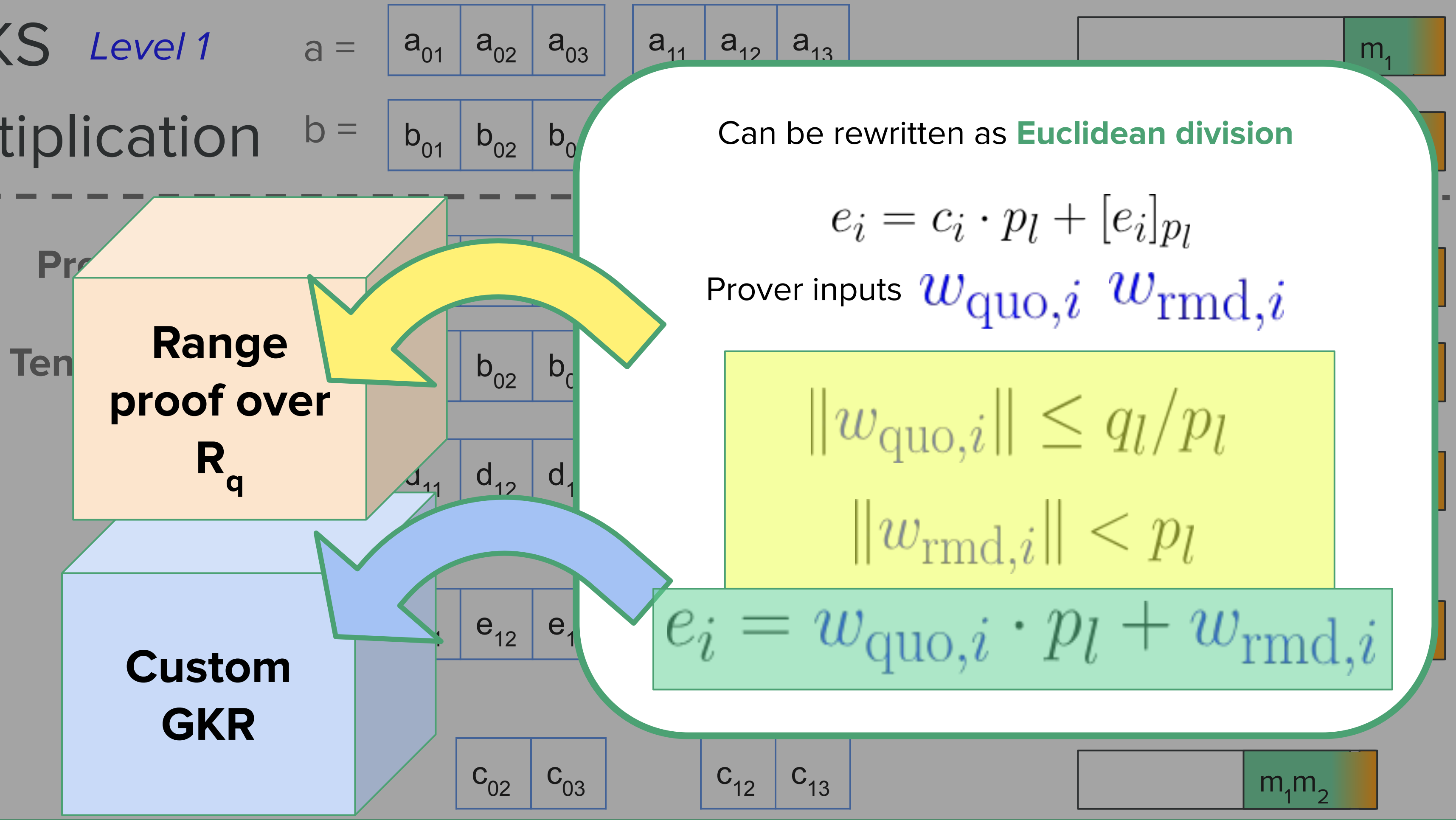
$$||w_{\mathrm{quo},i}|| \leq q_l / p_l$$

$$||w_{\mathrm{rmd},i}|| < p_l$$

$$e_i = w_{\mathrm{quo},i} \cdot p_l + w_{\mathrm{rmd},i}$$

c_{02} c_{03} c_{12} c_{13} $m_1 m_2$

CKKS *Level 1* Multiplication



Can be rewritten as **Euclidean division**

$$e_i = c_i \cdot p_l + [e_i]_{p_l}$$

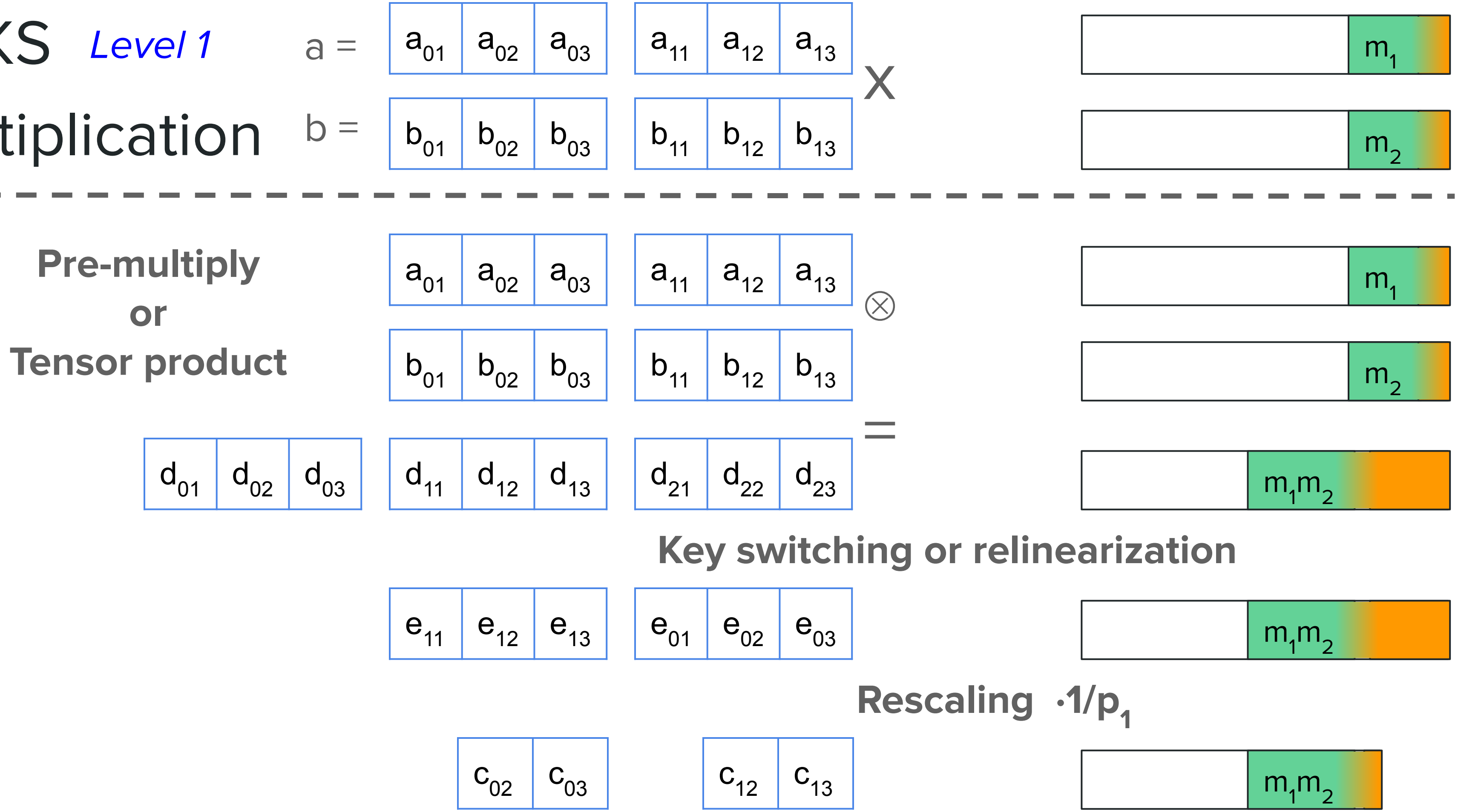
Prover inputs $w_{\text{quo},i}$ $w_{\text{rmd},i}$

$$\|w_{\text{quo},i}\| \le q_l / p_l$$

$$\|w_{\text{rmd},i}\| < p_l$$

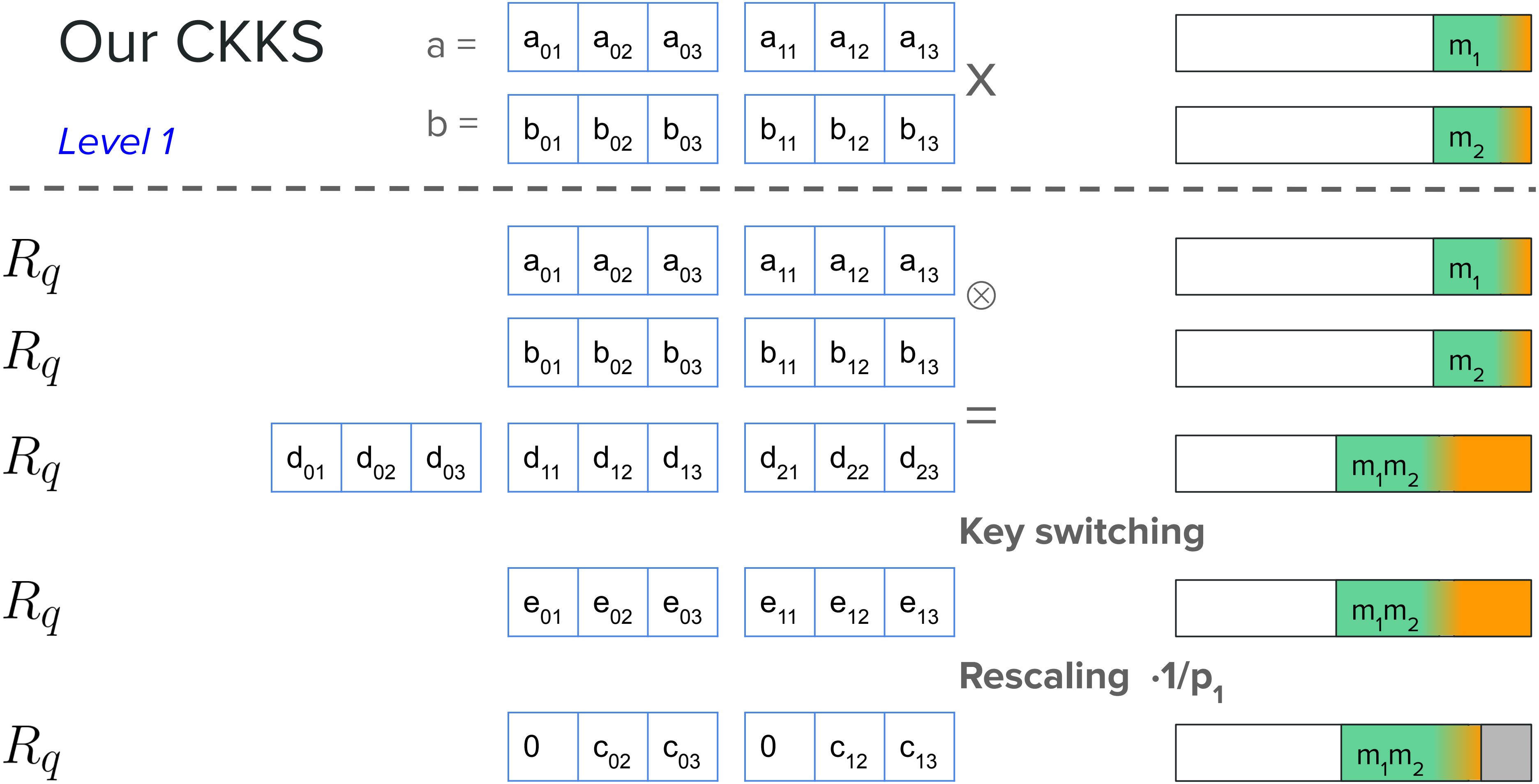
$$e_i = w_{\text{quo},i} \cdot p_l + w_{\text{rmd},i}$$

CKKS Multiplication *Level 1*



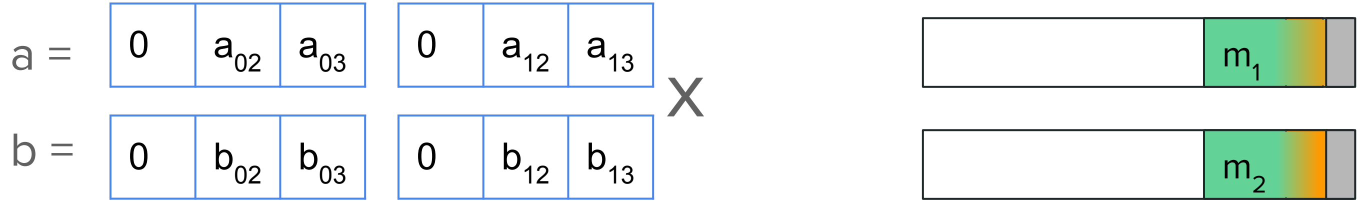
Our CKKS

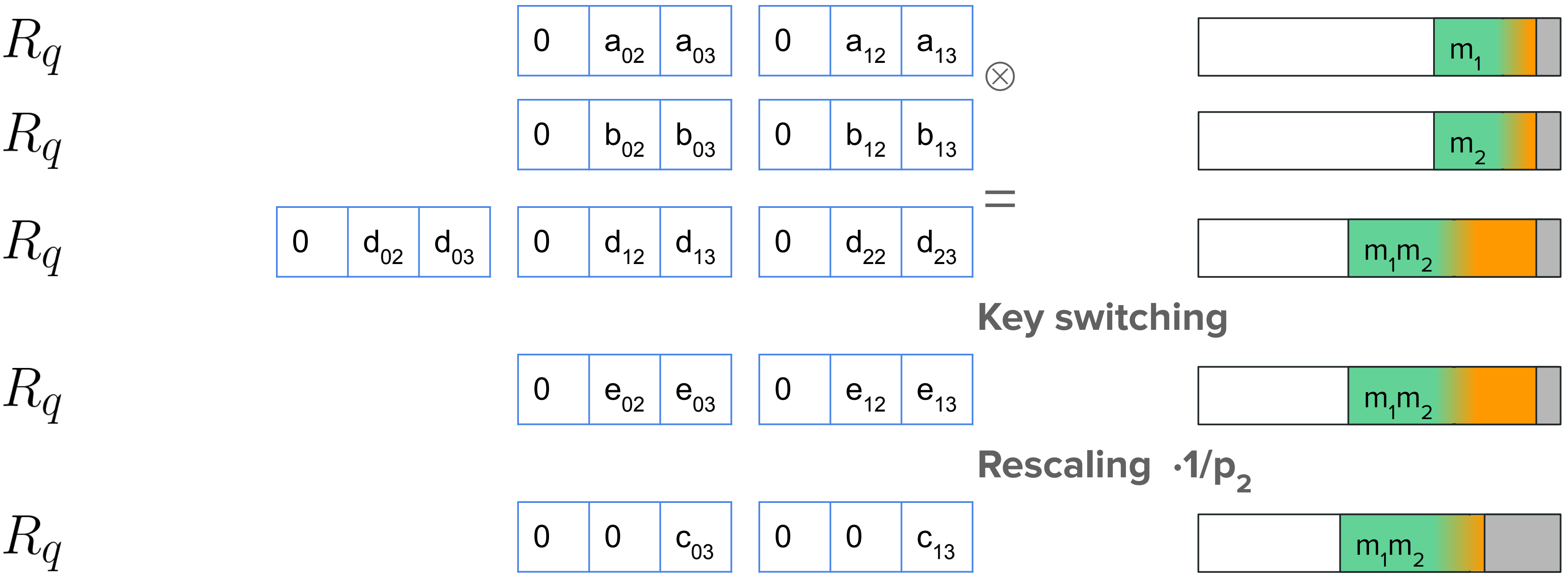
Level 1



Our CKKS

Level 2





Proof-friendly CKKS vs CKKS

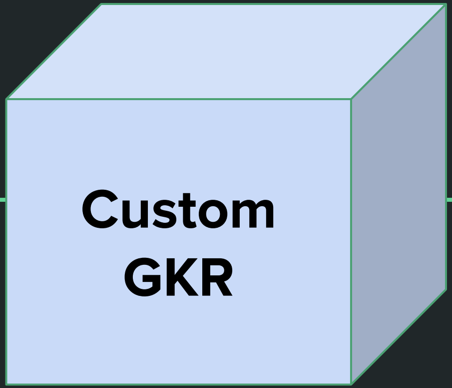
	Proof-friendly CKKS		CKKS
	d = 2	d = 4	HEXL
CKKS multiplication	7.394ms	8.457ms	7.197ms

N = 16384
#RNS components (L) = 6

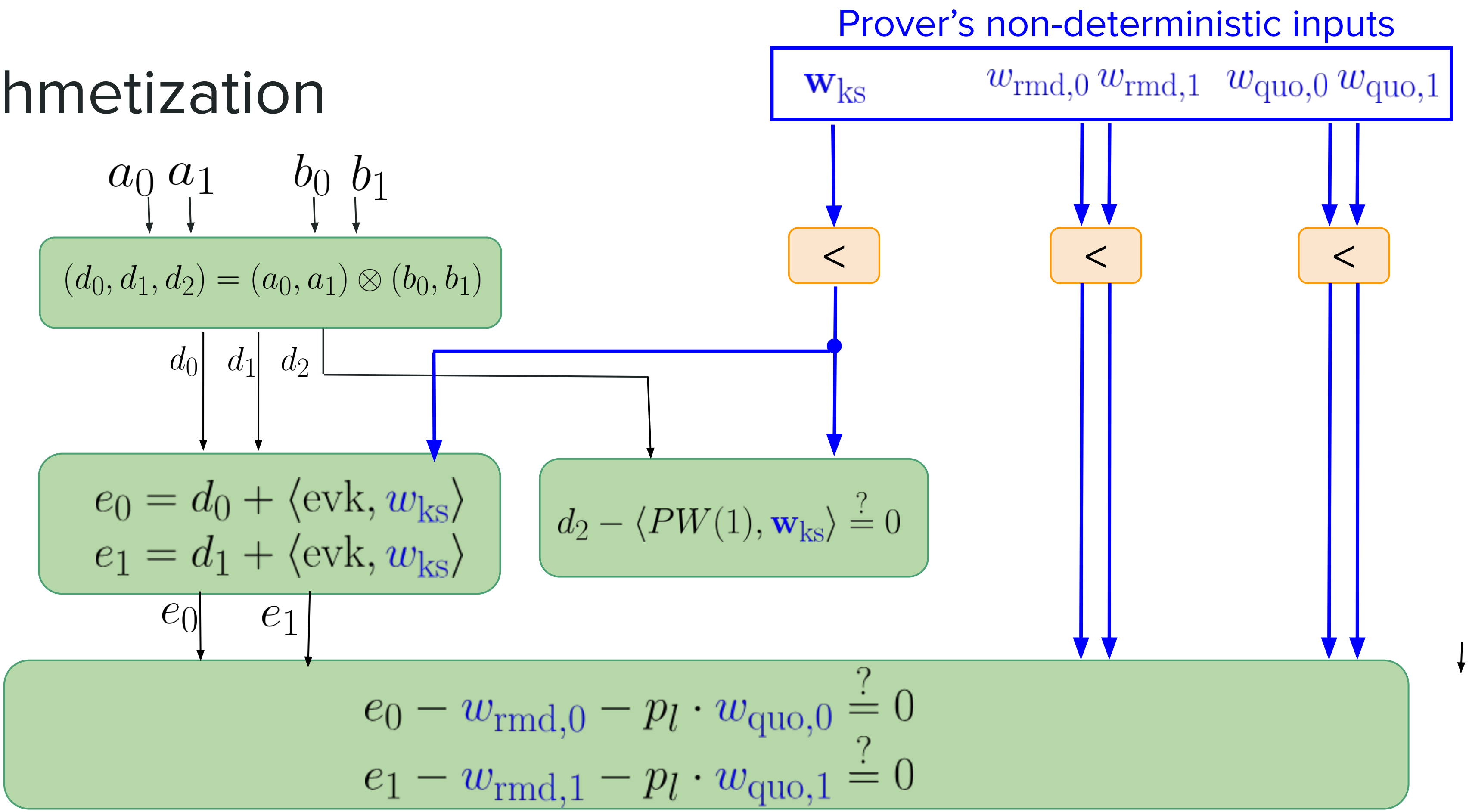
Proof-friendly CKKS in summary

- Carefully chosen ring setup
    - High **soundness** for proof system
    - **Efficiency** of computations
- Ring does not change
    - Proof system works on **same ring**
- Noise analysis
    - Easier to **prove bounds** on ciphertexts

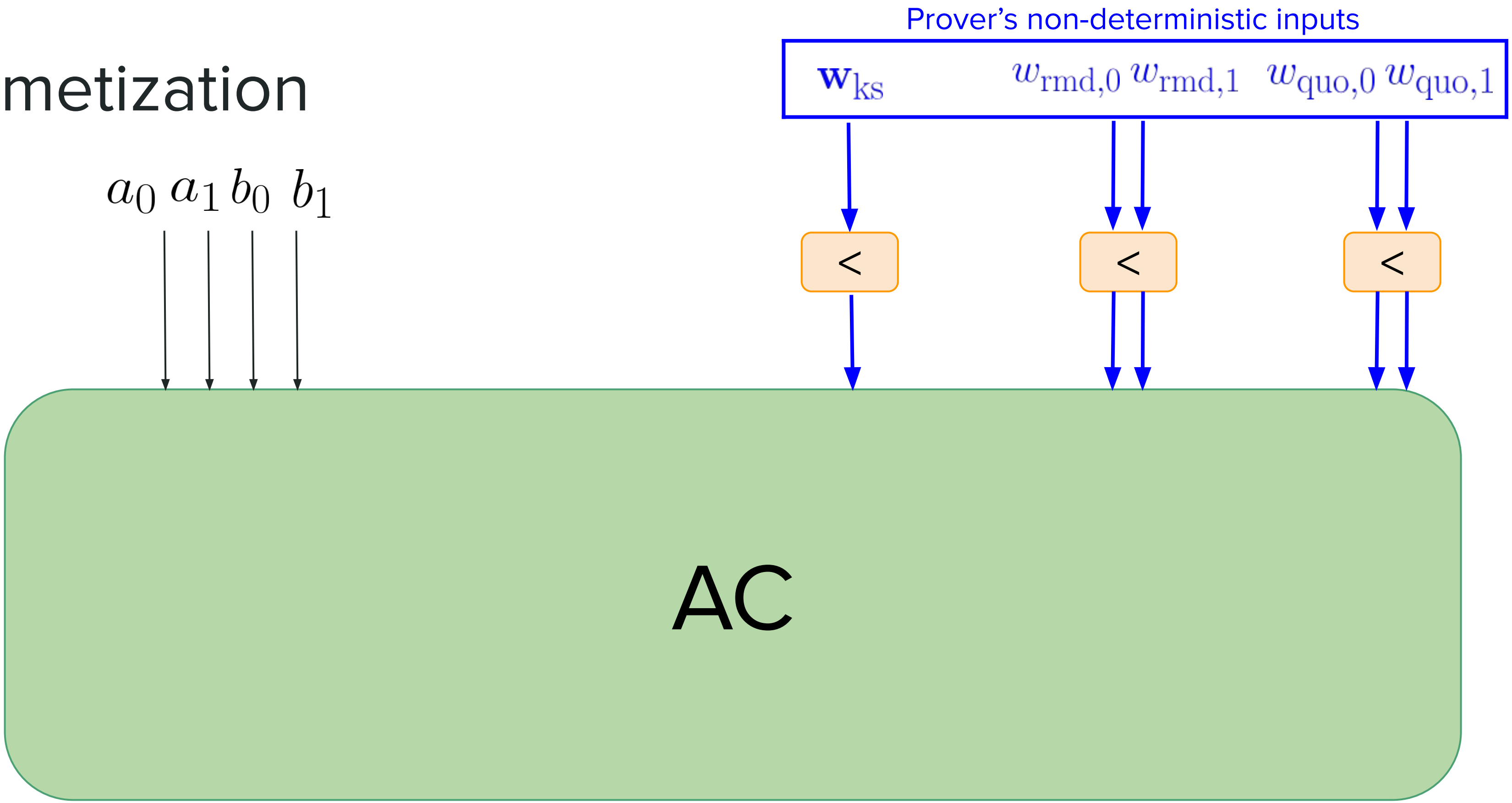
Proof of AC satisfiability



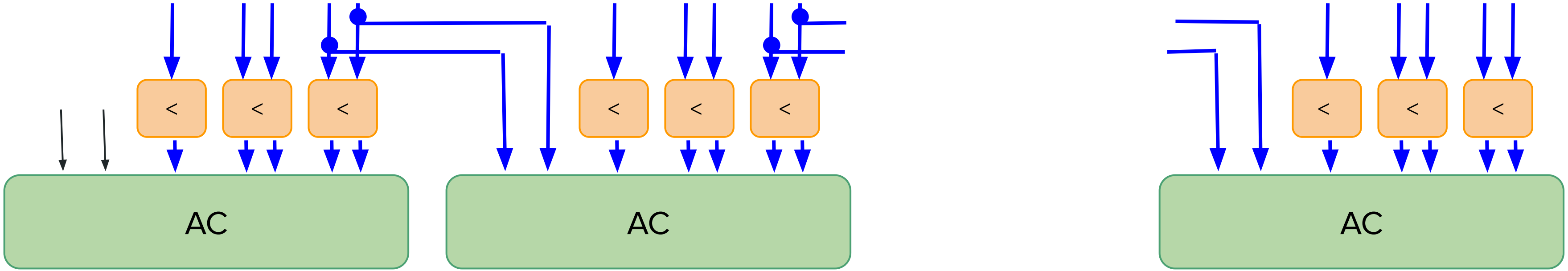
Arithmetization



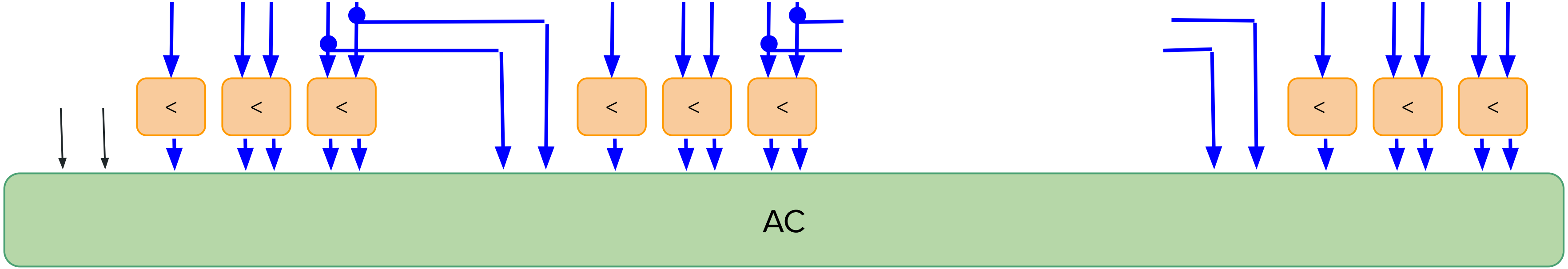
Arithmetization



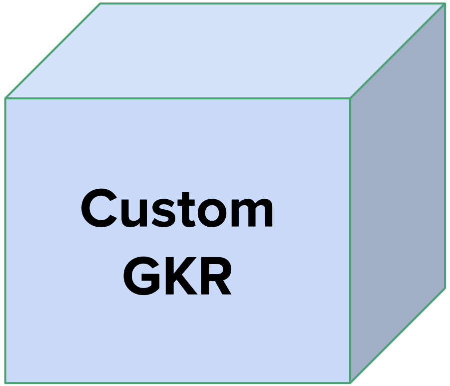
Flattening the circuit



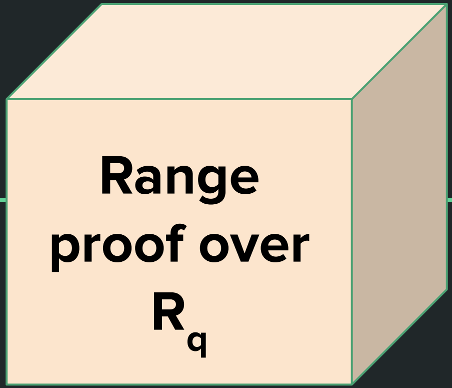
GKR-style proof system for AC



- **Custom gates** (bdcon, rescon, ...)
- Flattened system of relations => **constant depth 4**
- Not affected by recent FS attacks

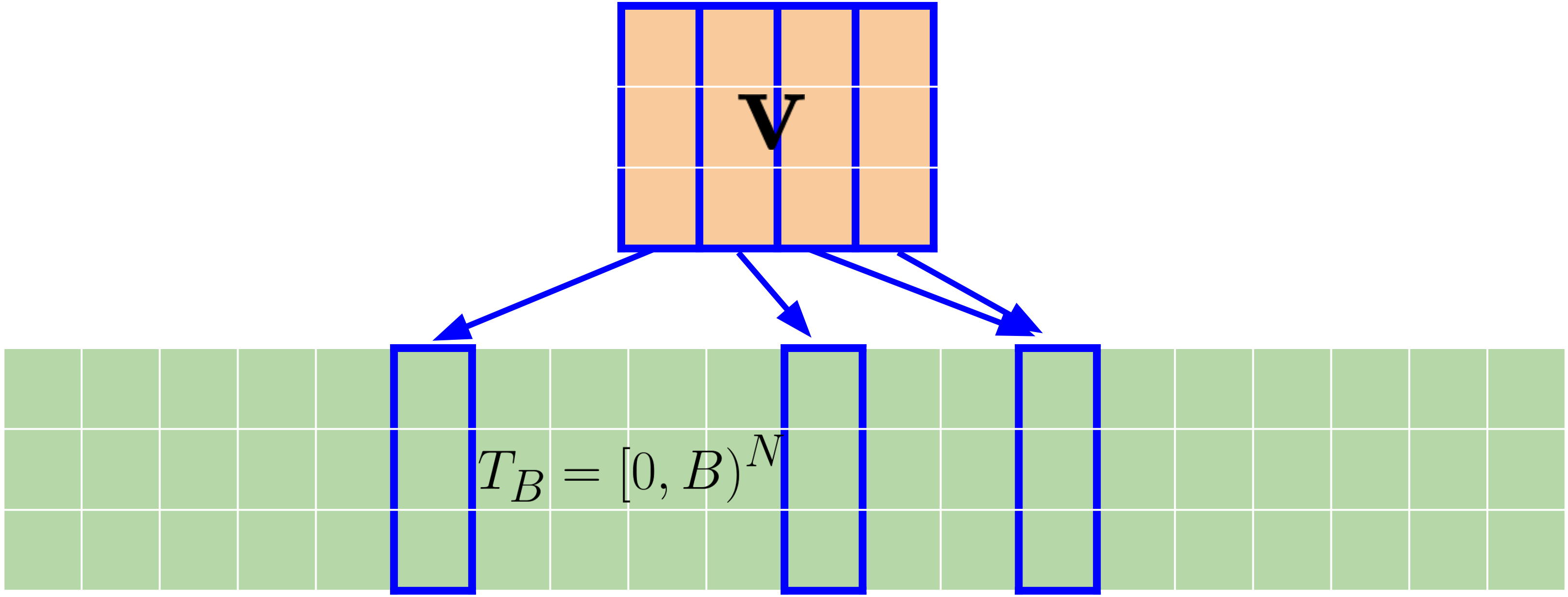


Range checks



Proving ranges

- Prove that vector **v** of *m* elements in R_q has coeffs bounded by B (e.g B = q_l)
- Can be seen as a look-up argument



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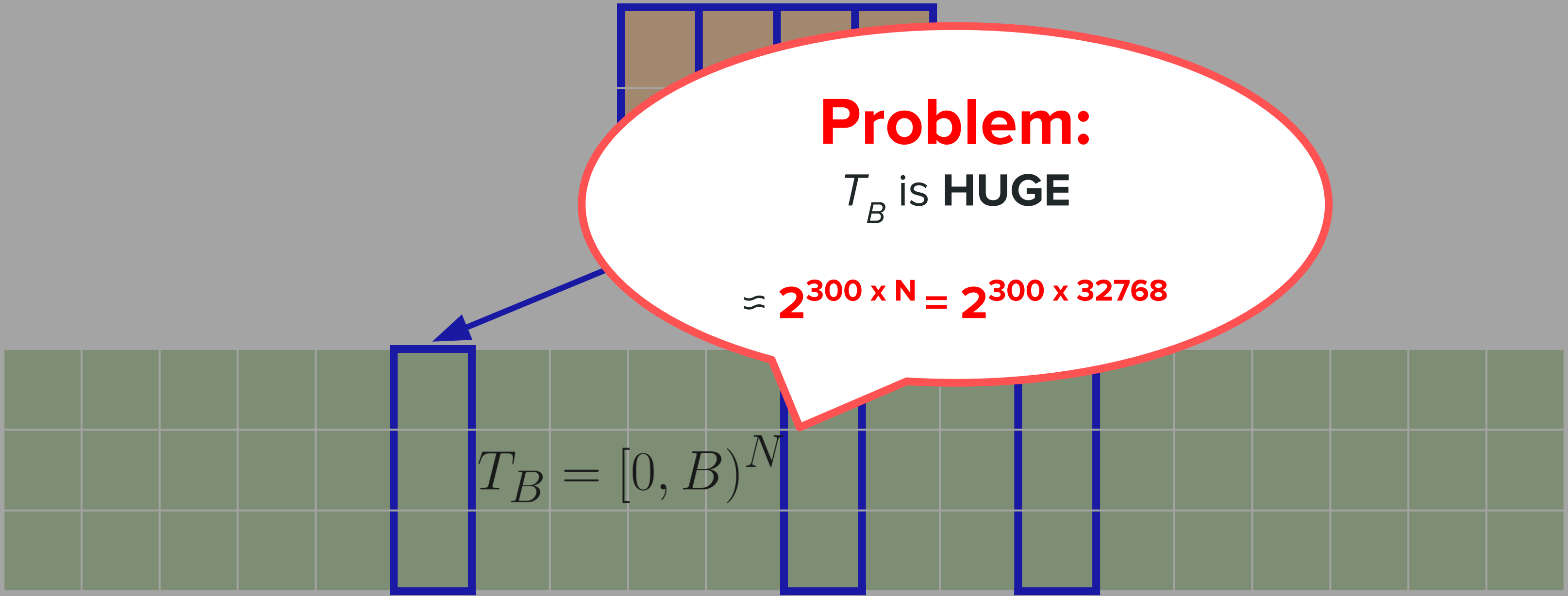


Proving ranges

n R_q has coeffs bounded by B (e.g B = q_l)

nt

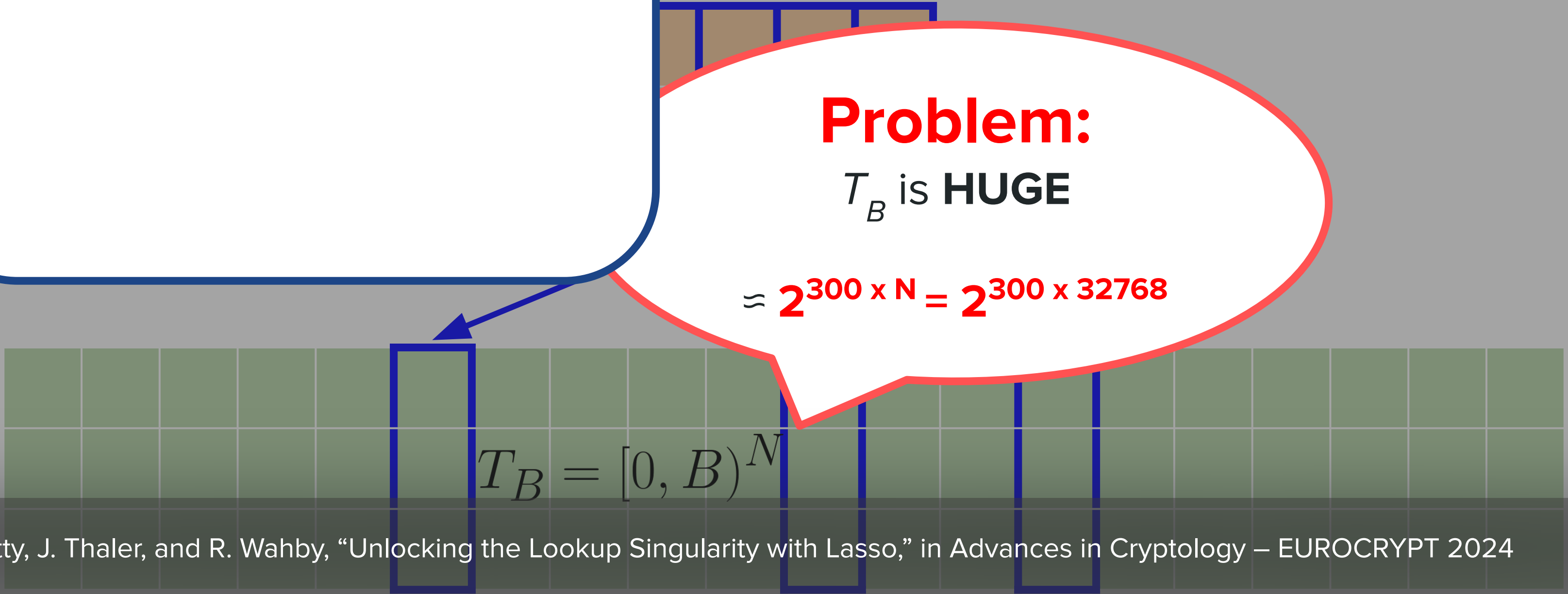
Solution for integers:

Decompose B (e.g. Lasso[1])

Problem:

T_B is **HUGE**

$\approx 2^{300 \times N} = 2^{300 \times 32768}$



$$T_B = [0, B)^N$$

[1] S. Setty, J. Thaler, and R. Wahby, “Unlocking the Lookup Singularity with Lasso,” in Advances in Cryptology – EUROCRYPT 2024

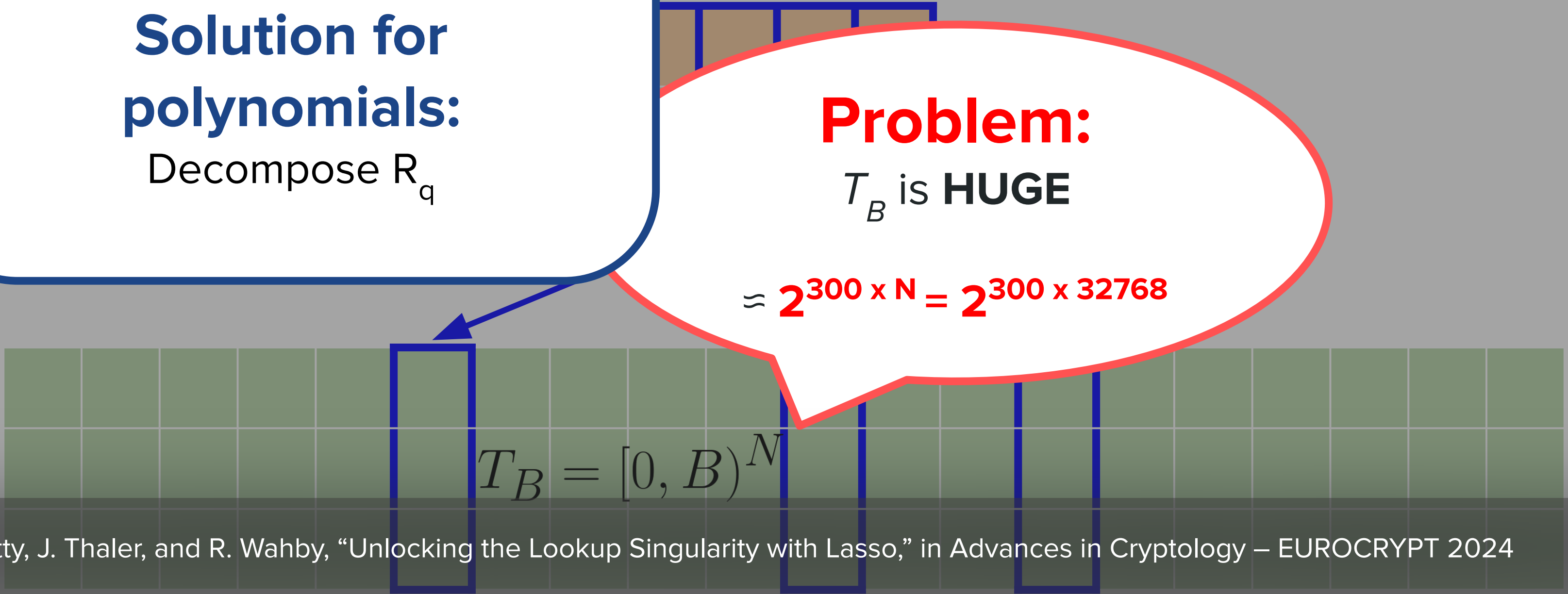
Proving ranges

Solution for integers:
Decompose B (e.g. Lasso[1])

Solution for polynomials:
Decompose R_q

...n R_q has coeffs bounded by B (e.g B = q_l)

...nt



[1] S. Setty, J. Thaler, and R. Wahby, “Unlocking the Lookup Singularity with Lasso,” in Advances in Cryptology – EUROCRYPT 2024

The polynomial commitment

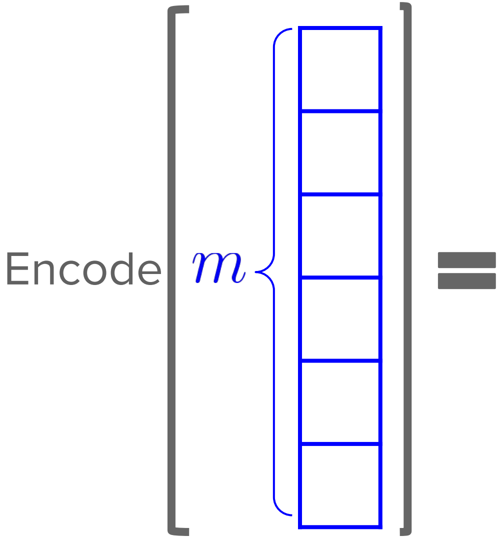
Polynomial Commitment

Need to commit to elements in $R_q[X_1, \ldots, X_\ell]$ where

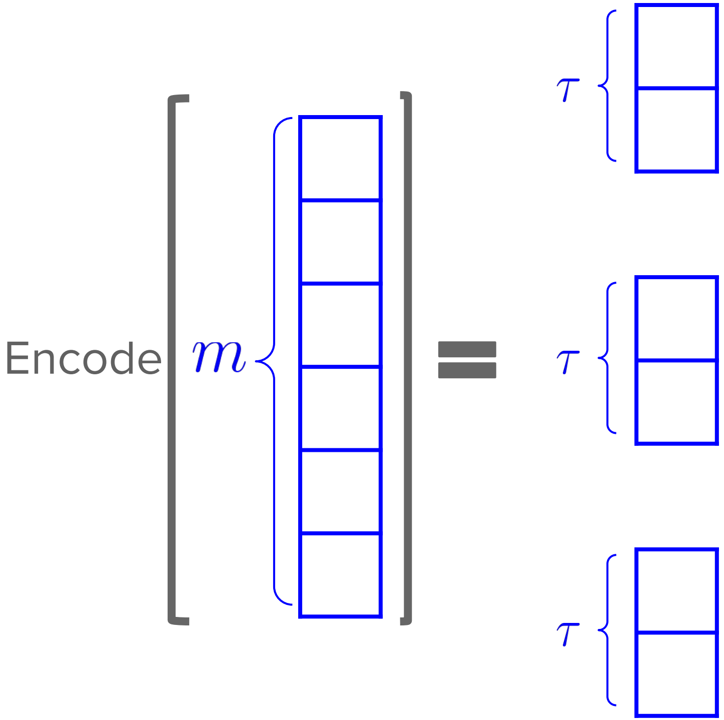
$$R_q \cong \mathbb{F}_{p_0^4} \times \cdots \times \mathbb{F}_{p_L^4}$$

- Reduce MV PC over R_q to MV PC over $\mathbb{F}_{p_i^4}$
- Small-ish fields => **Brakedown** (field-agnostic)
- $\mathbb{F}_{p_i^4}$ has $N/2$ roots of unity. **Can we use them?**

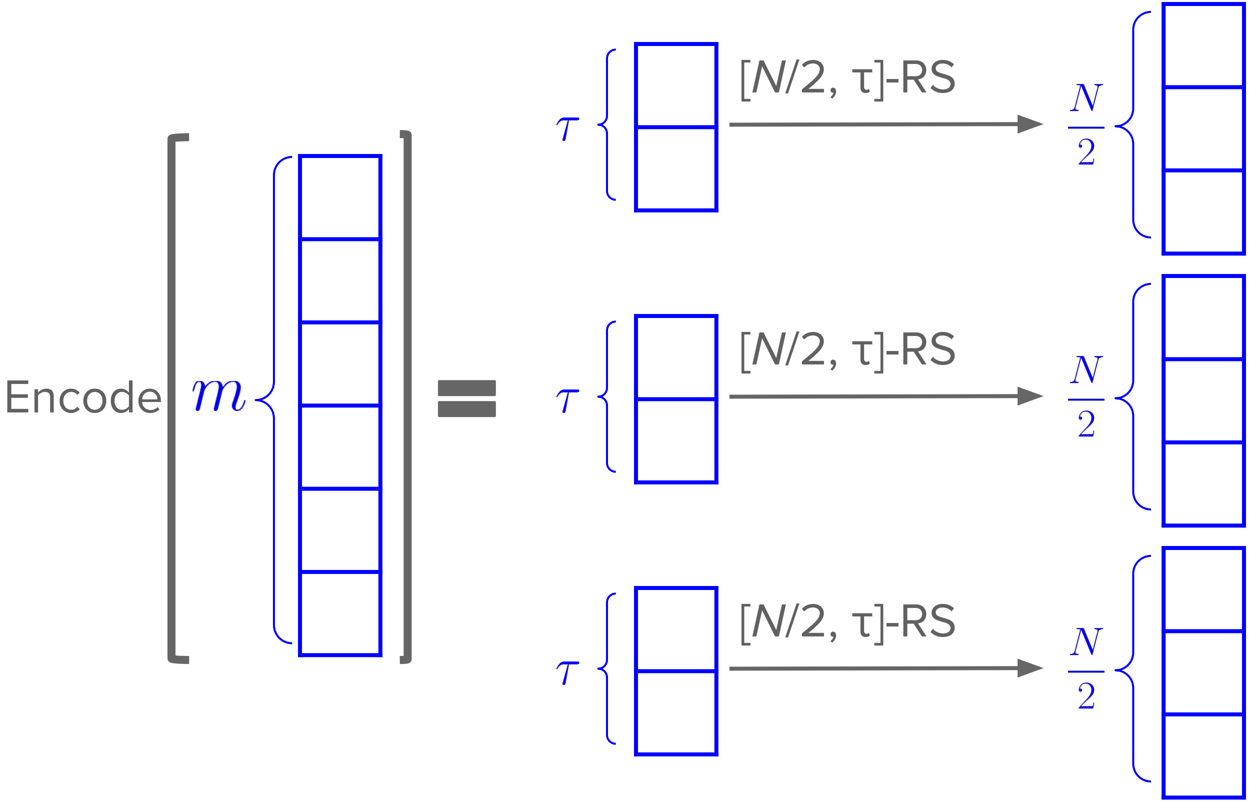
Piecewise Reed-Solomon code



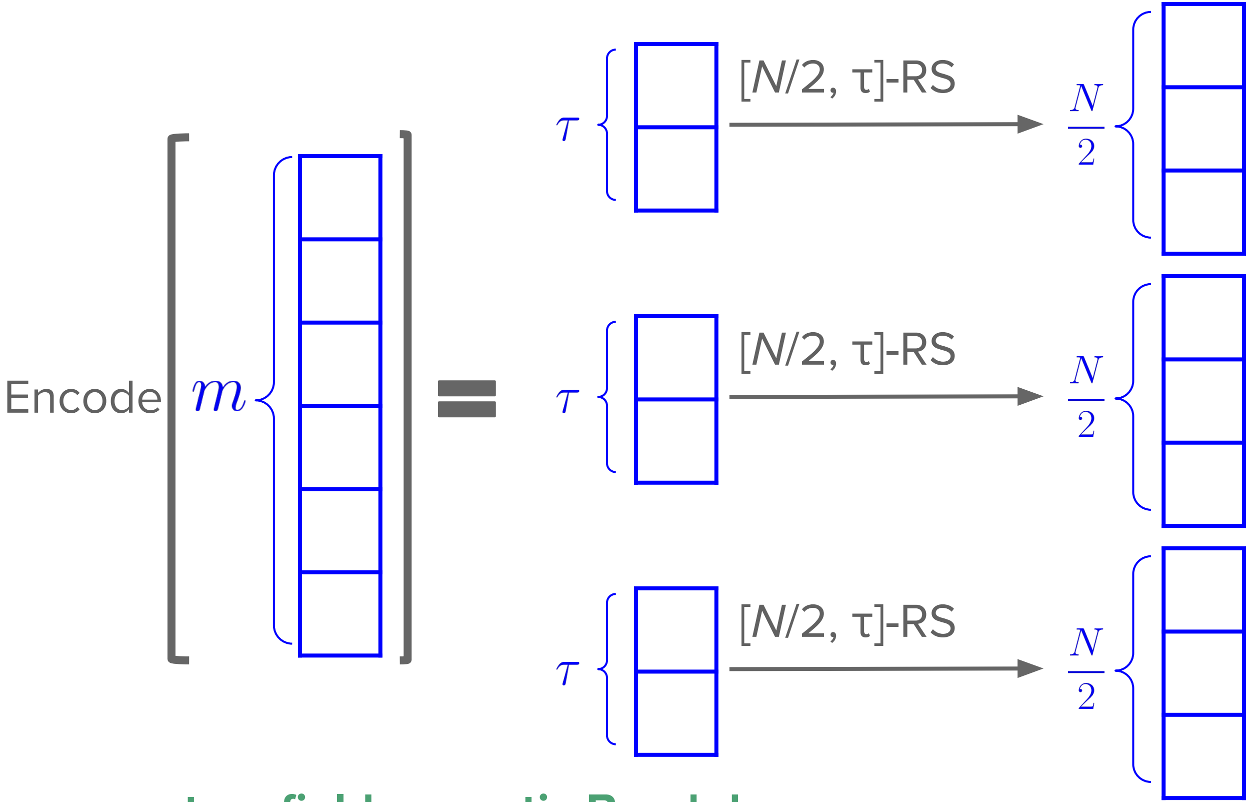
Piecewise Reed-Solomon code



Piecewise Reed-Solomon code



Piecewise Reed-Solomon code



x10 improvement on field-agnostic Breakdown

Conclusions

To summarize

- First practical VC for CKKS
  - Technique extend to FV/BGV
- Description of problem in a modular way (arithmetization)
  - AC satisfiability + range checks
- Design of proof-friendly CKKS
- Design of custom GKR to prove AC over rings
- Design of range proofs for polynomial rings
- Improved Brakedown for medium-sized fields
- Implemented all building blocks

https://eprint.iacr.org/2025/286

Thank you!

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