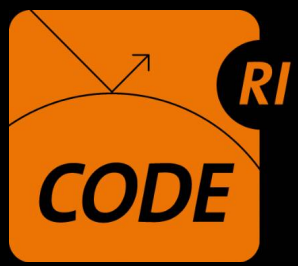
More Efficient Isogeny Proofs of Knowledge via Canonical Modular Polynomials

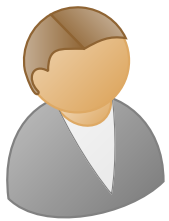
Joint work with Thomas den Hollander,
Sören Kleine, Marzio Mula & Daniel Slamanig

Sebastian A. Spindler • August 18th, 2025



Research Institute
Cyber Defence
Universität der Bundeswehr München

The Scenario



Preston

Veronica

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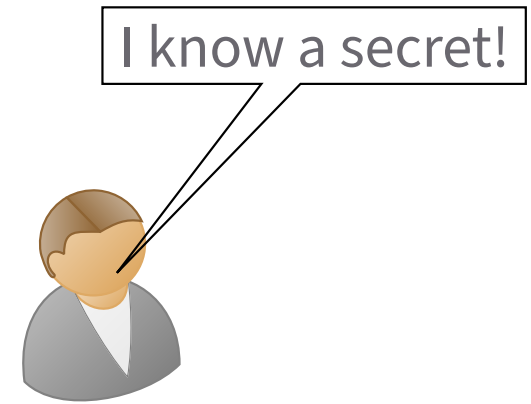

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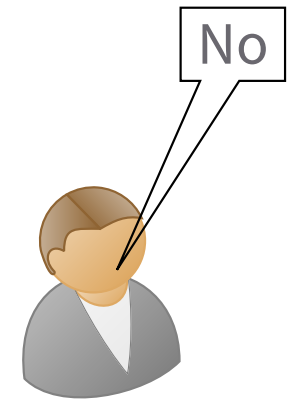

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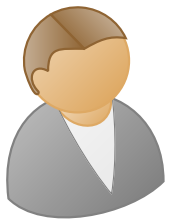

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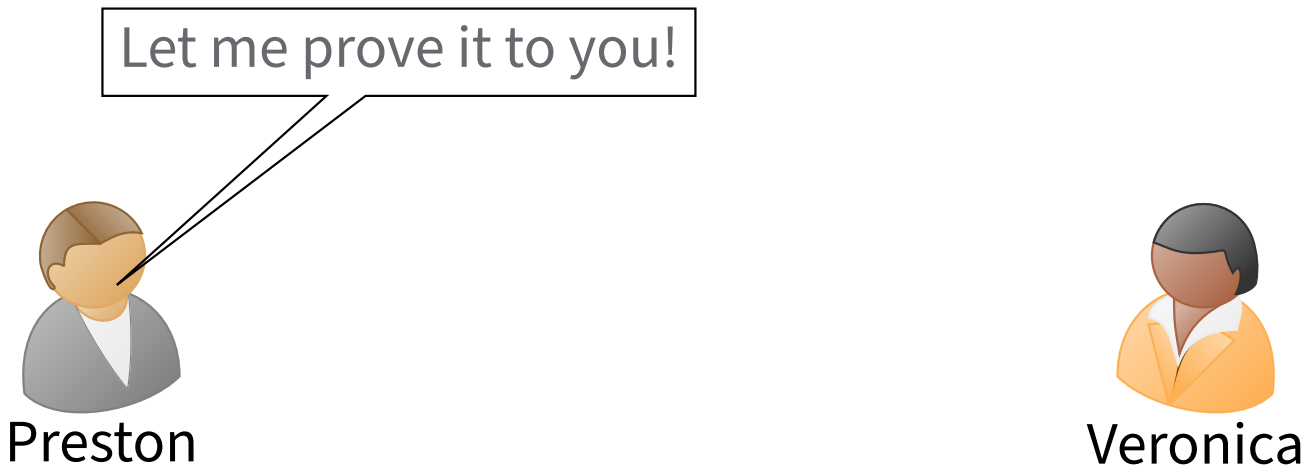

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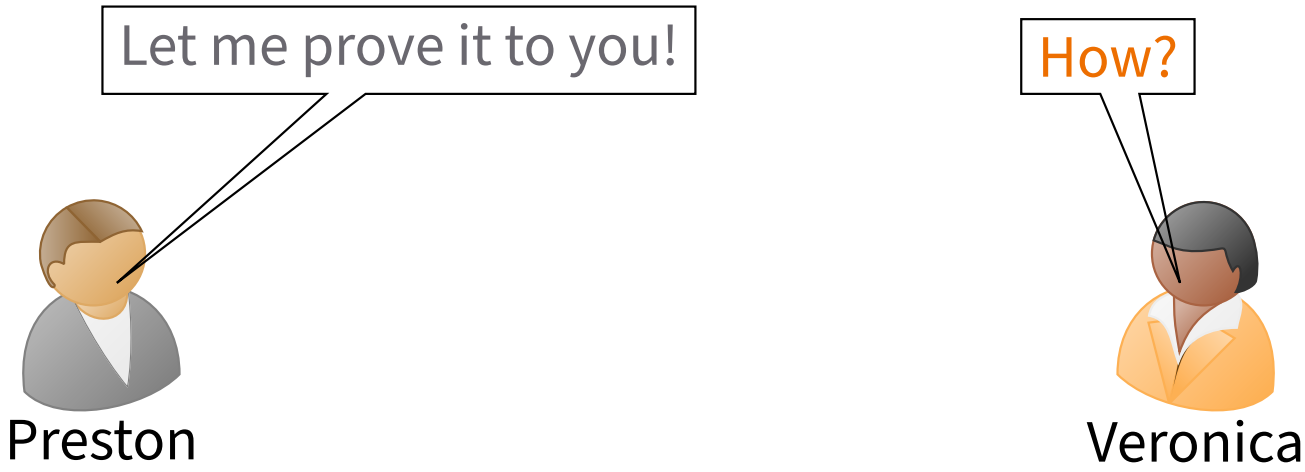

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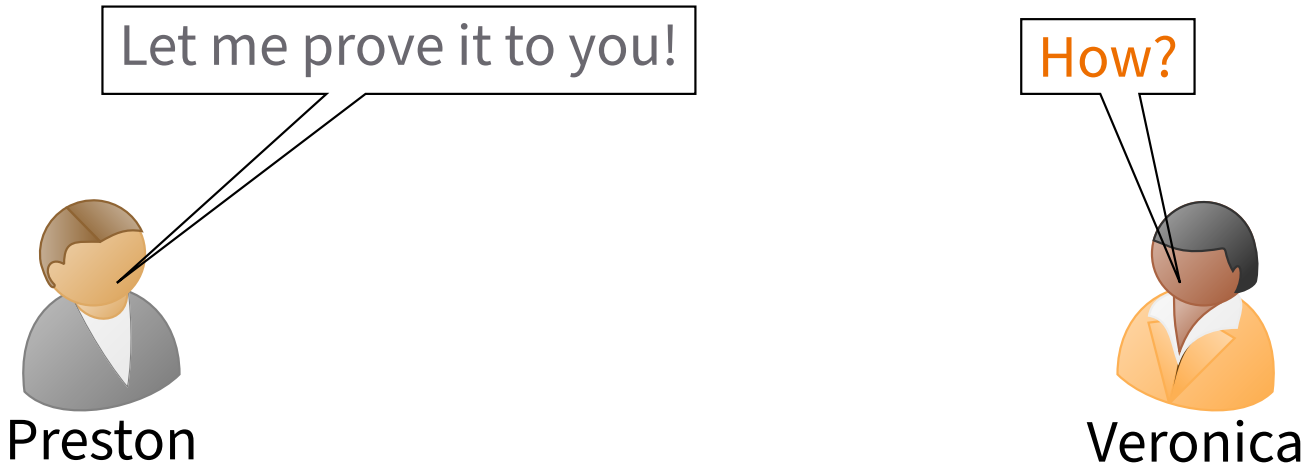

With a **zero-knowledge proof of knowledge**!

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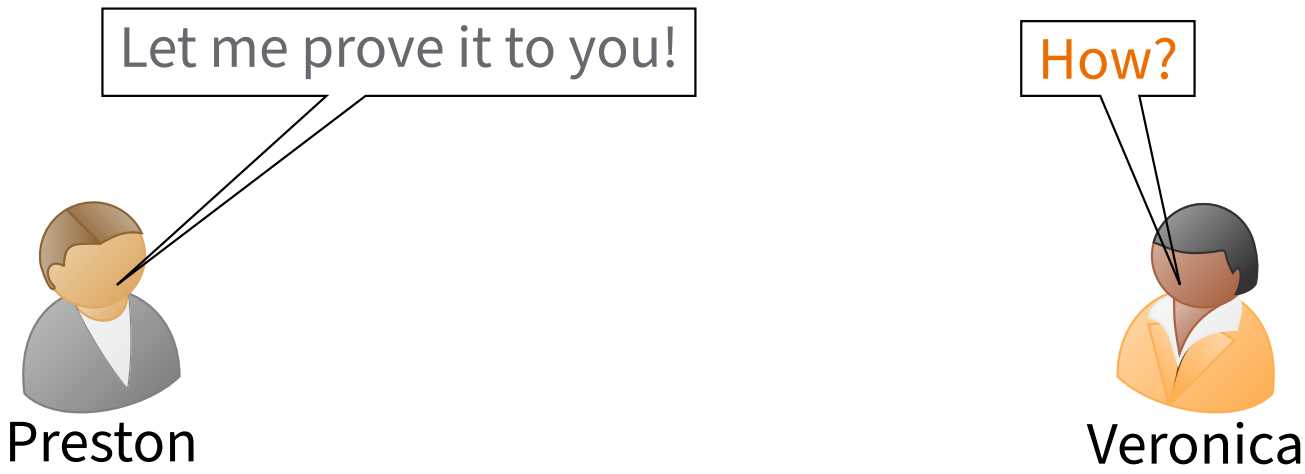

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Multiple applications: Multi-party protocols (honest behavior control, trustless setup), signature schemes, …

General zk-Proofs of Knowledge

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- **Non-Interactivity**: Preston can produce a proof without Veronica, and Veronica can verify it without Preston

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(Parameter sizes: ℓ small, $p \approx 2^{2\lambda}$ for λ bits security)

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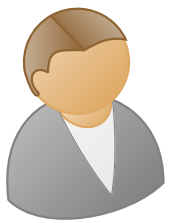

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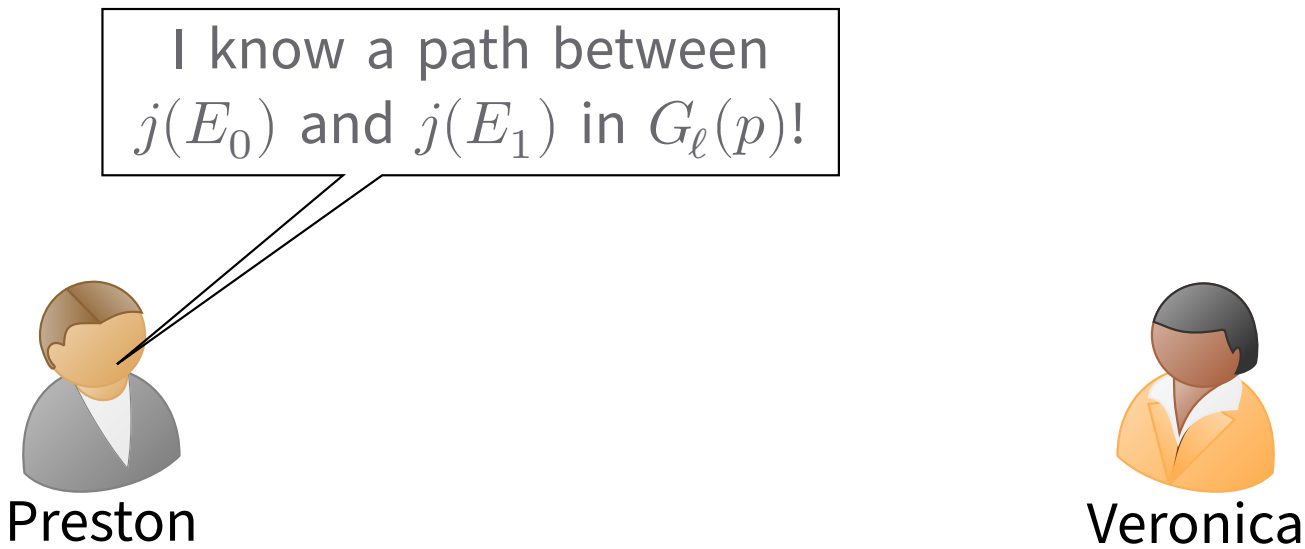

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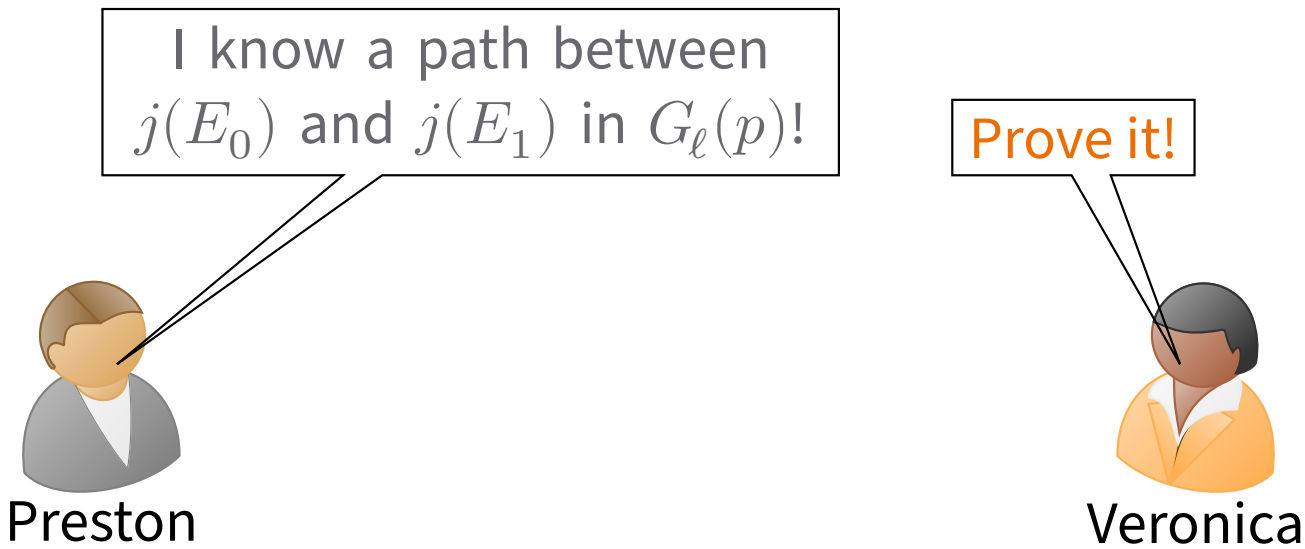

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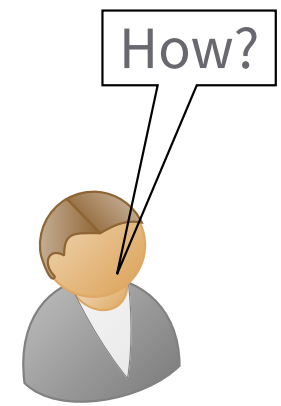

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Generic Machinery: R1CS-based zk-SNARKs

- A rank-1 constraint system is of the form

$$(\mathsf{A}) \begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} \bullet (\mathsf{B}) \begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} = (\mathsf{C}) \begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_n \end{pmatrix}$$

  with statement matrices $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathbb{F}^{m \times (1+n)}$ and witness vector $w = (1, w_1, \ldots, w_n)$

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- This approach was first pursued in [CLL23]

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Why? Isogenies between supersingular curves can be defined over $\mathbb{F}_{p^2}$!

Our Approach: Canonical Modular Polynomial [dH+24]

For each step $j_i \xrightarrow{f_i} j_{i+1}$ rephrase system

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as an R1CS. For $\ell = 2$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ f_i \\ f_i^2 \\ j_i - c_1' \\ j_{i+1} - c_1' \end{pmatrix} \bullet \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & c_2' & c_3' & -1 & 0 \\ 0 & c_3'' & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ f_i \\ f_i^2 \\ j_i - c_1' \\ j_{i+1} - c_1' \end{pmatrix}$$

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Some Stats [dH+24]

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	[CLL23]	Ours	[CLL23]	Ours	[CLL23]	Ours
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3		2.524λ		$2.524\lambda + 1$		11.357λ
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Field		Aurora		Ligero	
		[CLL23]	Ours	[CLL23]	Ours
$\mathbb{F}_{p^2}$	Prover time (ms)	934	669	587	420
	Verifier time (ms)	99	74	847	634
	Proof size (kB)	194	178	1849	1599

Table: Benchmarks for $\ell = 2$

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- Other arithmetizations?

Thank you for your attention!

[dH+24] T. den Hollander, S. Kleine, M. Mula, D. Slamanig, and S. A. Spindler. *More Efficient Isogeny Proofs of Knowledge via Canonical Modular Polynomials*. Cryptology ePrint Archive, Paper 2024/1738. 2024. To appear at CRYPTO 2025.