A quasi-poly. time quantum algorithm for EDCP over power-of-two moduli

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Complexity of EDCP?

This talk

This talk

- A quasi-polynomial time/sample quantum algorithm for EDCP

Learning with Errors (LWE)

(Search) $\mathsf{LWE}_{n,q,\alpha}$

Find a fixed secret $\mathbf{s} \in \mathbb{Z}_q^n$, given $\mathbf{a}_i \leftarrow\!\!\$\ \mathbb{Z}_q^n$ and $e_i \leftarrow\!\!\$\ \mathcal{D}$

$$
\begin{aligned}
\mathbf{a}_1, b_1 &= \langle \mathbf{a}_1, \mathbf{s} \rangle + e_1 \pmod q \\
\mathbf{a}_2, b_2 &= \langle \mathbf{a}_2, \mathbf{s} \rangle + e_2 \pmod q \\
&\vdots \\
\mathbf{a}_m, b_m &= \langle \mathbf{a}_m, \mathbf{s} \rangle + e_m \pmod q
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Typically, $\mathcal{D} = D_{\alpha q}$

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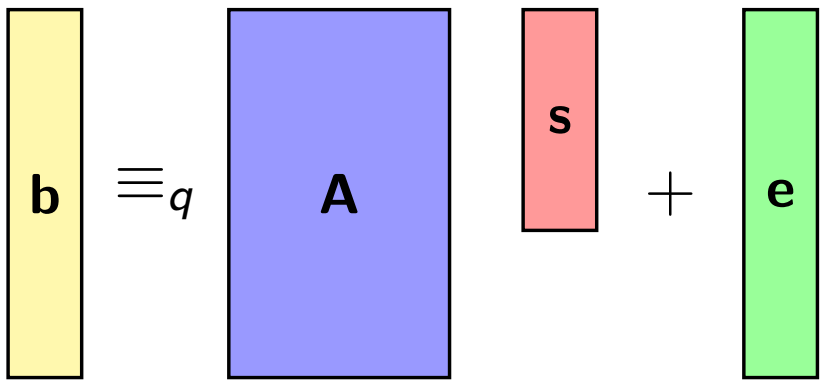

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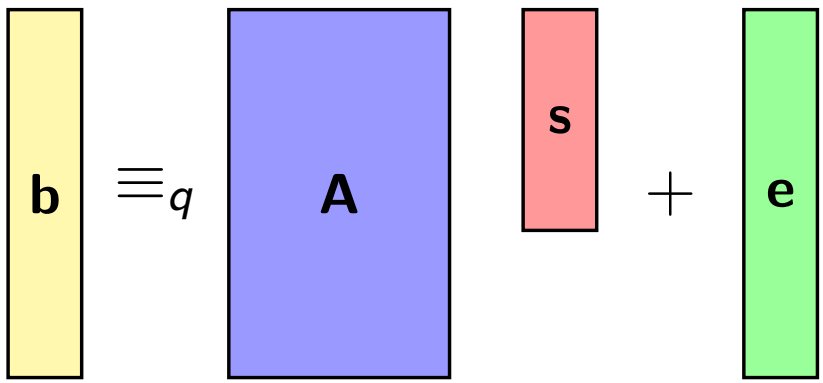

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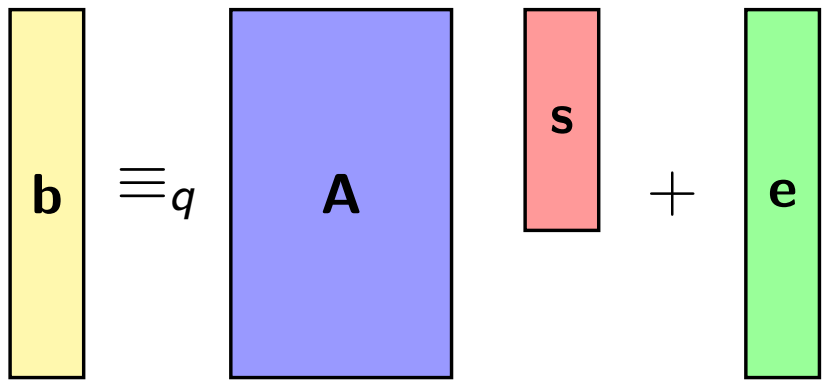

- Hardness of LWE depends on noise-to-modulus ratio $\alpha = \frac{\sigma}{q}$. The smaller α, the easier the LWE problem.
- Standard LWE: $q = \mathrm{poly}(n)$, $m = n \log q$, $\chi = D_{\mathbb{Z},\alpha q}$ and $\alpha q = O(\sqrt{n})$.

Complexity: exponential

DCP vs EDCP

Find a secret s, given ℓ quantum samples

$\mathbf{DCP}_N^\ell$	$\mathbf{EDCP}_{n,q,\chi}^\ell$
$\lvert 0, x_i\rangle + \lvert 1, x_i + s \bmod N\rangle$	$\sum_{j \in \mathrm{Supp}(\mathcal{D})} \mathcal{D}(j) \lvert j\rangle \lvert \mathbf{x}_i + j \cdot \mathbf{s} \bmod q\rangle$

Most common distributions

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- G-EDCP: $\mathcal{D}$ is a Gaussian distribution

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Most common distributions

- G-EDCP: $\mathcal{D}$ is a Gaussian distribution
- U-EDCP$_{n,q,M}$: $\sum_{j=0}^{M-1} |j\rangle\,|\mathbf{x}_i + j \cdot \mathbf{s} \bmod q\rangle$

$\Rightarrow n = 1, M = 2$: DCP instances.

$$\text{G-EDCP} \underset{\text{[BKSW18]}}{\overset{\text{comp}}{\Longleftrightarrow}} \text{U-EDCP}$$

Recall Relation between LWE, DCP and EDCP

$\mathsf{LWE}^{m}_{n,q,1/\mathsf{poly}(n)}$

$\mathsf{DCP}^{\mathsf{poly}(n)}_{q^n}$

$\mathsf{EDCP}^{\mathsf{poly}(n)}_{n,q,M}$

This work

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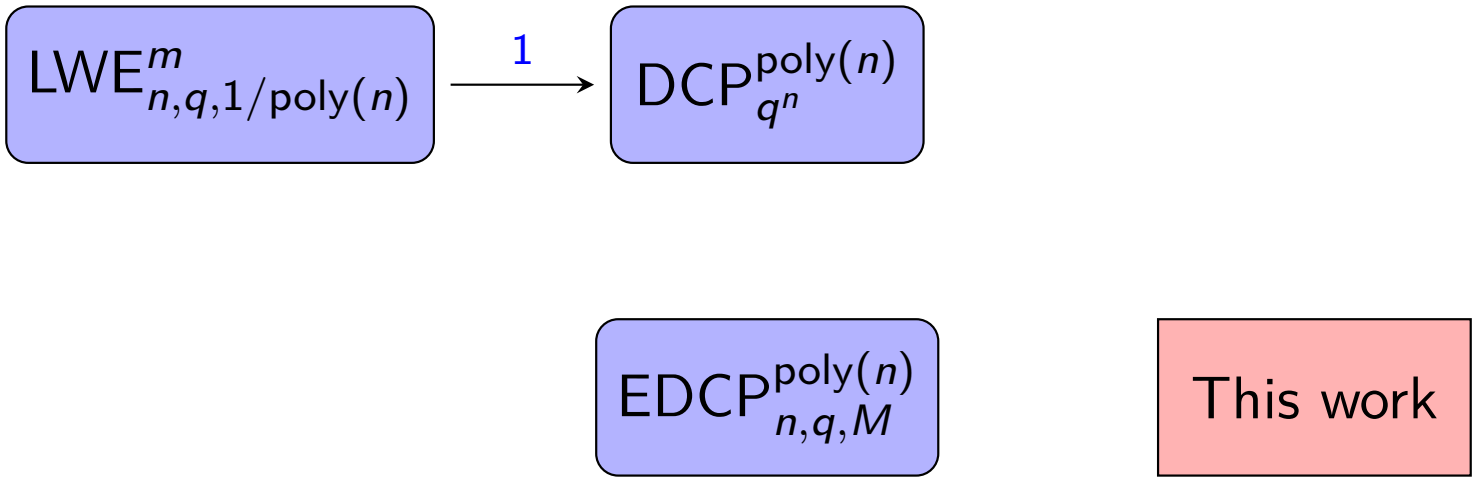

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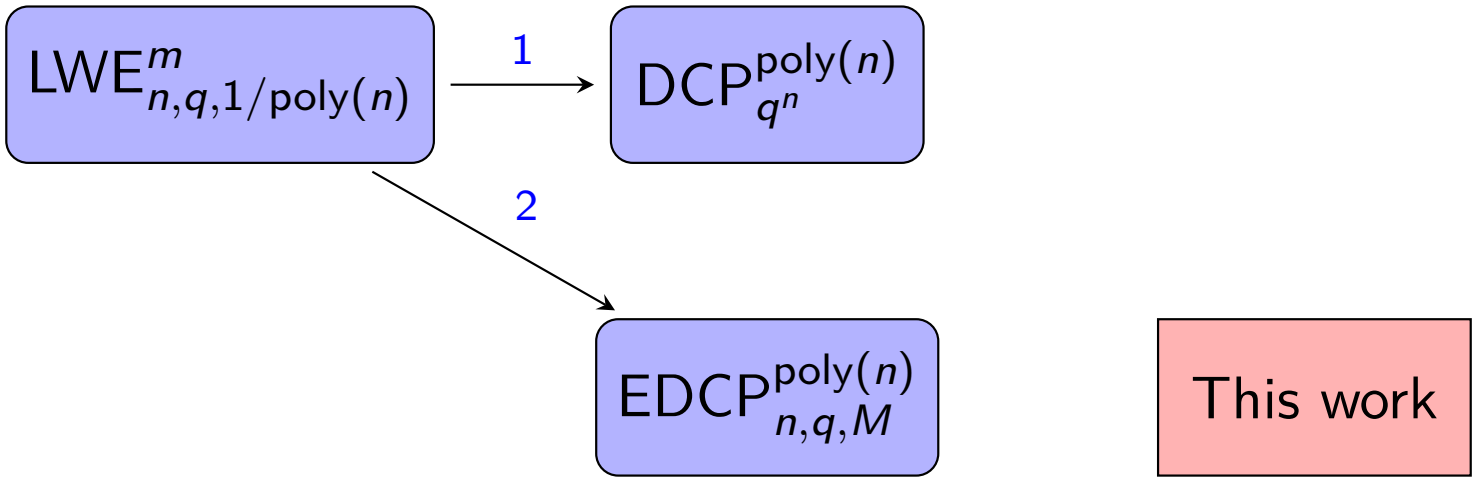

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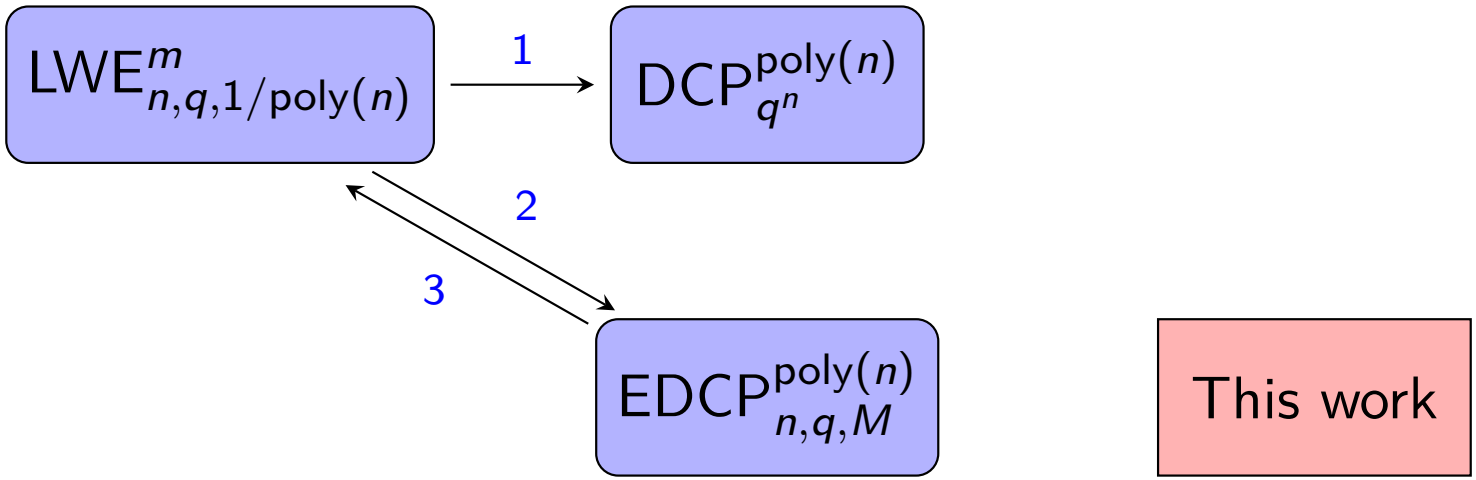

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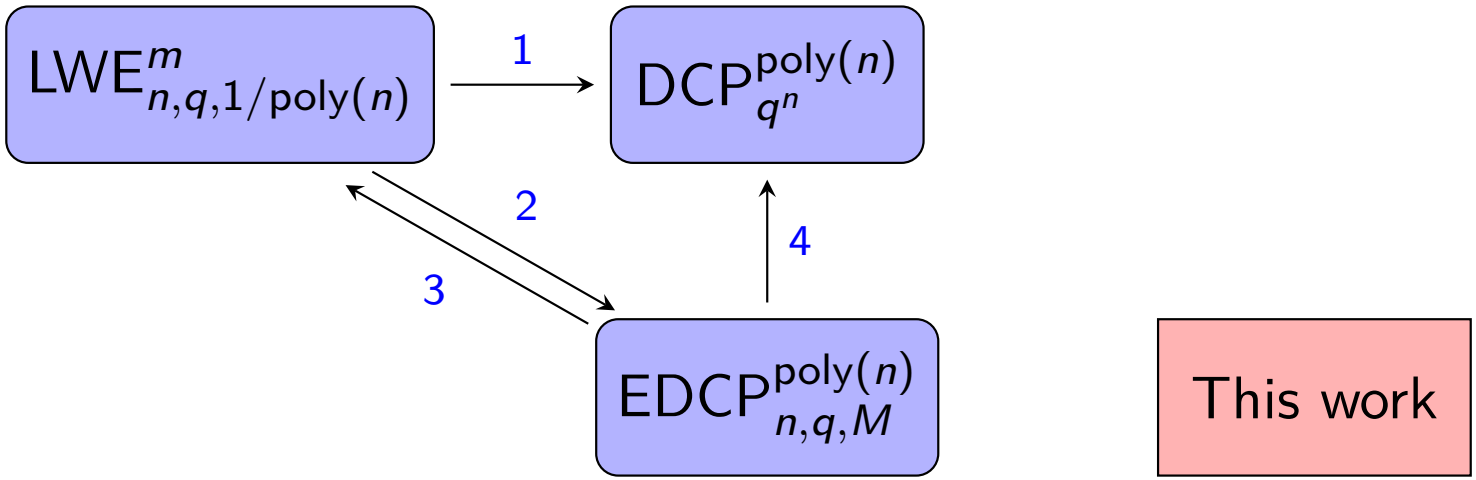

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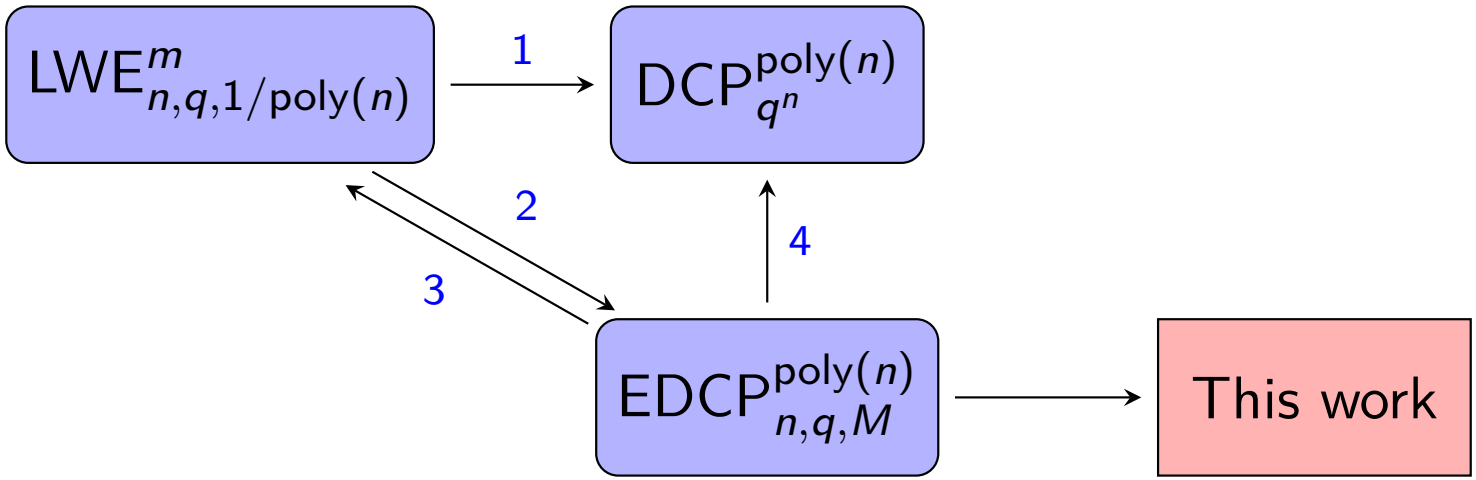

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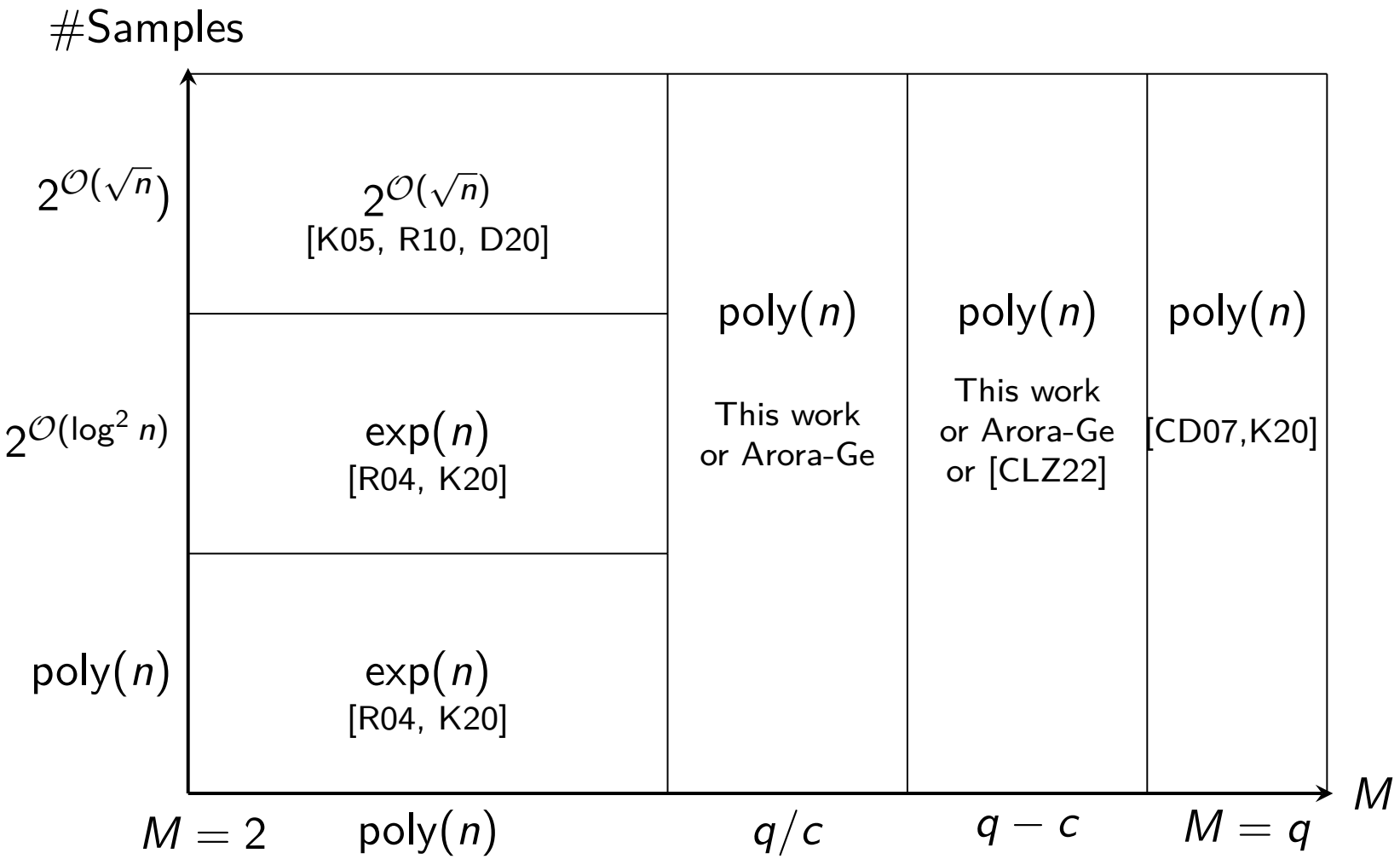
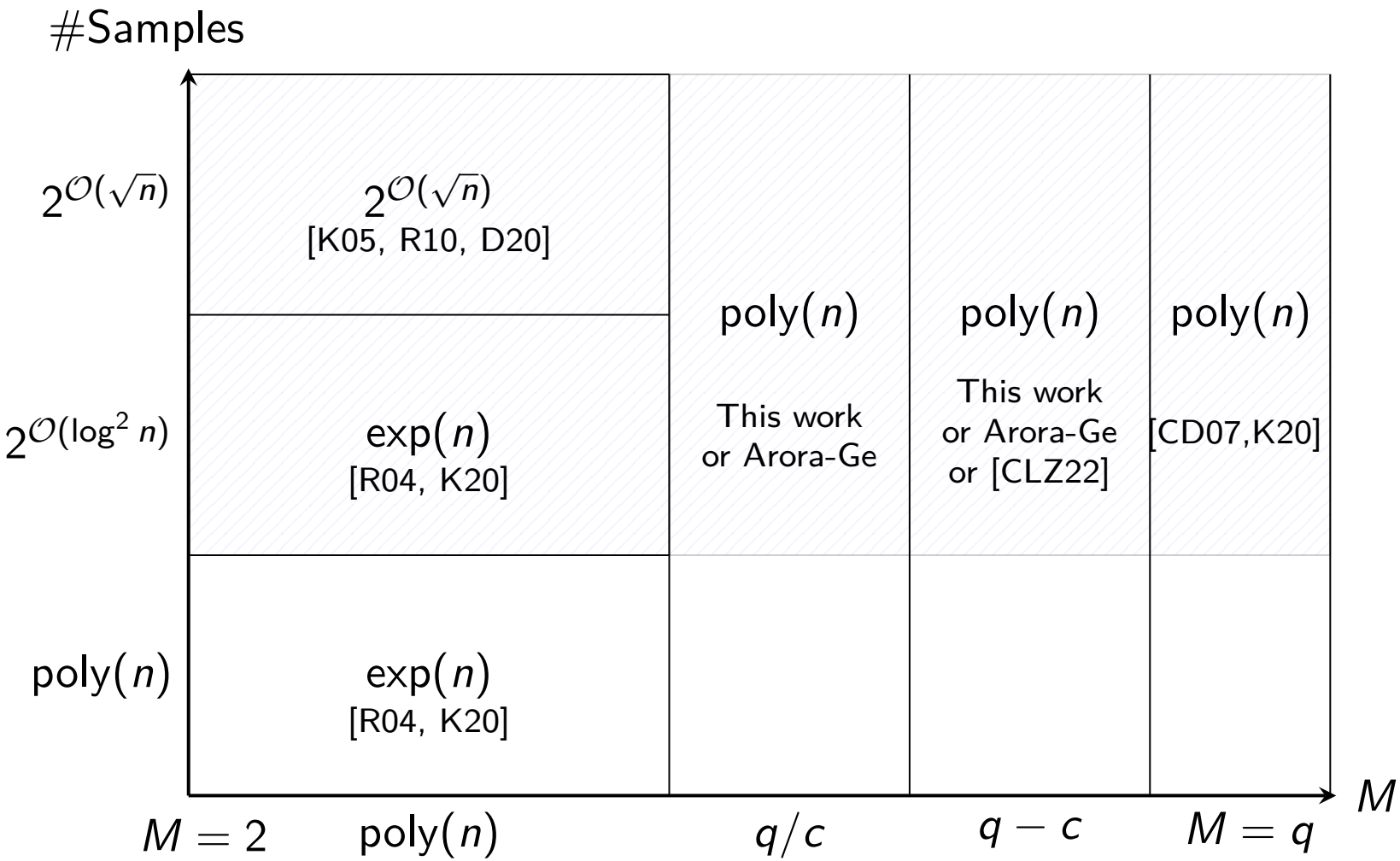
Complexity of U-EDCP$_{n,q,M}$



Our result for U-EDCP$_{n,q,M}$



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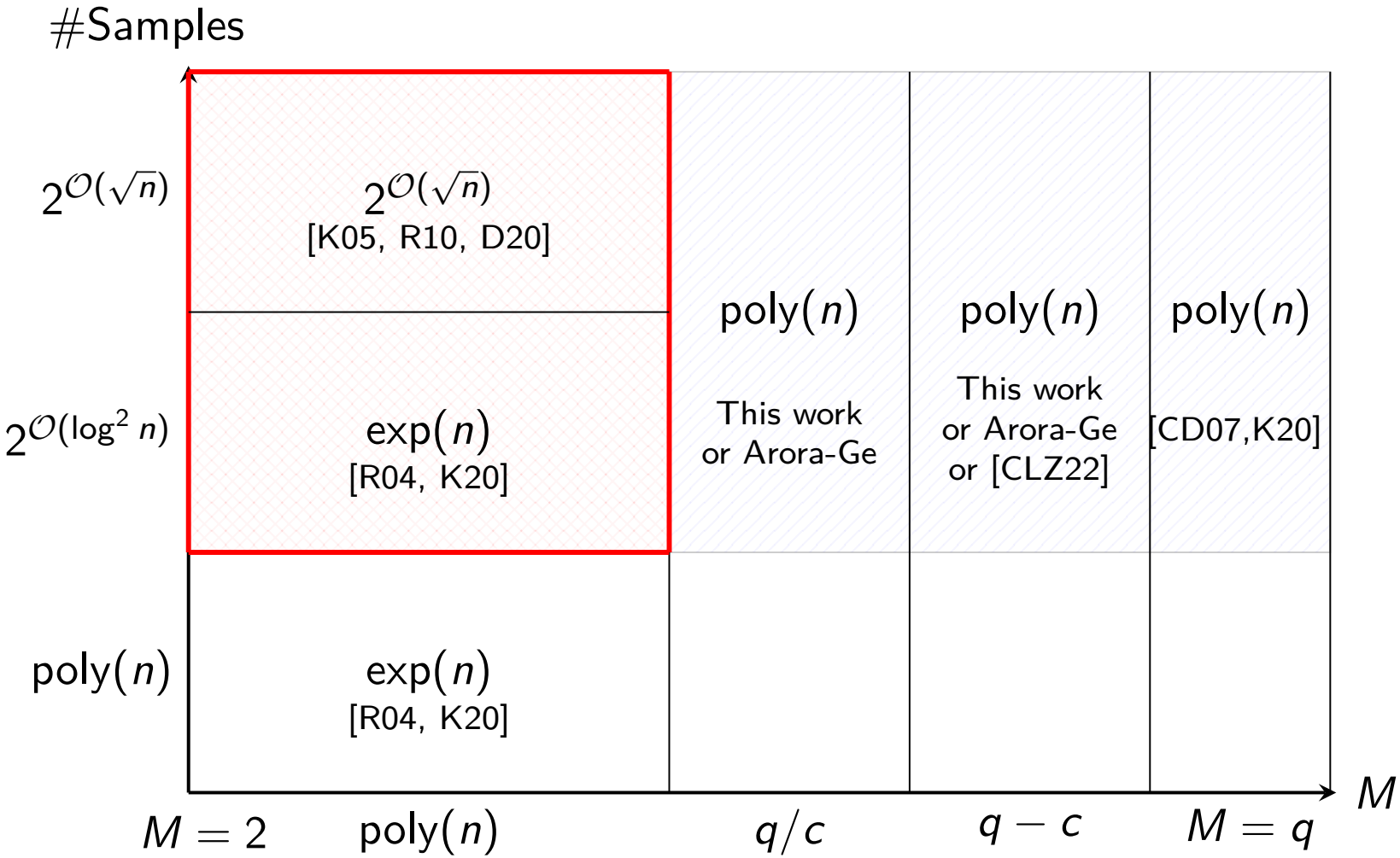

Our algorithm applies

Improve state-of-the-art for power-of-two modulus q: $n^{\mathcal{O}(\log q)}$

Our Algorithm

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Kuperberg's algorithm [K05]

Solves DCP_N^ℓ in time $2^{O(\sqrt{\log_2 N})}$ when $\ell = 2^{O(\sqrt{\log_2 N})}$.

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Kuperberg-like algorithm	This work

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	$\overline{\text{U-EDCP}}$

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1 LSB bit	n LSB bits

This work – Main step

Base case: $M = 2$

Merge $n+1$ $\overline{\text{U-EDCP}}_q$ to get one $\overline{\text{U-EDCP}}_{q/2}$.

$$\overline{\text{U-EDCP}}^{n+1}_{n,q,2} \rightarrow \overline{\text{U-EDCP}}^{1}_{n,q/2,2}$$

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$$\overline{\text{U-EDCP}}^{n+1}_{n,q,2} \to \overline{\text{U-EDCP}}^{1}_{n,q/2,2}$$

1. Tensor input states:

$$\bigotimes_{k=1}^{n+1} |\psi_k\rangle = \bigotimes_{k=1}^{n+1} \left(|0\rangle + \omega_q^{\langle \mathbf{y}_k, \mathbf{s} \rangle} |1\rangle \right) = \sum_{\mathbf{j} \in \mathbb{Z}_2^{n+1}} \omega_q^{\langle \mathbf{Y} \cdot \mathbf{j}, \mathbf{s} \rangle} |\mathbf{j}\rangle$$

where $\mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_{n+1}) \in \mathbb{Z}_q^{n \times (n+1)}$ and known classically.

Merge $n+1$ $\overline{\text{U-EDCP}}_q$ to get one $\overline{\text{U-EDCP}}_{q/2}$.

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2. Compute $\mathbf{Y} \cdot \mathbf{j} \bmod 2$ in a new register and measure to get some $\mathbf{b} \in \mathbb{Z}_2^n$:

$$\sum_{\mathbf{j} \in \mathbb{Z}_2^{n+1} :\, \mathbf{Y} \cdot \mathbf{j} = \mathbf{b} \bmod 2} \omega_q^{\langle \mathbf{Y} \cdot \mathbf{j}, \mathbf{s} \rangle} |\mathbf{j}\rangle \, .$$

3. Recall $\mathbf{y}_k$ is uniform in $\mathbb{Z}_q^n$. So $\mathbf{Y} \pmod 2 \in \mathbb{Z}_2^{n \times (n+1)}$ has full rank with constant probability:

$$\{\mathbf{j} \in \mathbb{Z}_2^{n+1} : \mathbf{Y} \cdot \mathbf{j} = \mathbf{b} \bmod 2\} = \{\mathbf{j}_0, \mathbf{j}_1\}.$$

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$$\omega_q^{\langle \mathbf{Y}\cdot\mathbf{j}_0,\, \mathbf{s}\rangle} |\mathbf{j}_0\rangle + \omega_q^{\langle \mathbf{Y}\cdot\mathbf{j}_1,\, \mathbf{s}\rangle} |\mathbf{j}_1\rangle = |0\rangle + \omega_q^{\langle \mathbf{Y}(\mathbf{j}_1-\mathbf{j}_0),\, \mathbf{s}\rangle} |1\rangle .$$

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$$|0\rangle + \omega_{q/2}^{\langle \mathbf{y}',\, \mathbf{s}\rangle} |1\rangle$$

where $2\mathbf{y}' = \mathbf{Y}(\mathbf{j}_1 - \mathbf{j}_0) \bmod q$.

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- Final Stage: we get n $\overline{\text{U-EDCP}}_{n,2,2}$ samples with known $\{\mathbf{y}_k\}_k$:

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- Measure each in Hadamard basis to learn $\langle \mathbf{y}_k, \mathbf{s}\rangle \bmod 2$.
- Use Gaussian elimination to recover $\bar{\mathbf{s}} := \mathbf{s} \bmod 2$.

The algorithm works for $M = 2$. What about general M?

Lemma (U-EDCP self-reduction) [D20]

Let $M' \leq M$. There is a polynomial time reduction from $\text{U-EDCP}^{\ell}_{n,q,M}$ to $\text{U-EDCP}^{\Theta(\ell)}_{n,q,M'}$ that succeeds with constant probability.

Main theorem

Let $q = \text{poly}(n)$ be a power of two, $2 \leq M \leq q$. There is an algorithm that solves $\text{U-EDCP}^{\ell}_{n,q,M}$ in time $2^{O(\log n \log q)} = 2^{O(\log^2 n)}$ using $\ell = 2^{O(\log^2 n)}$ samples.

Algorithm also applies for $\mathsf{S}\,|\mathrm{LWE}\rangle$

Solve $\mathsf{S}\,|\mathrm{LWE}\rangle^{m}_{n,q,\chi}$ [CLZ22]

Let a function $\chi : \mathbb{Z}_q \to \mathbb{C}$. Given m input states:

$$\Big\{ \sum_{e_i \in \mathbb{Z}_q} \chi(e_i)\, |\langle \mathbf{a}_i, \mathbf{s}\rangle + e_i \bmod q\rangle \Big\}_{i=1}^{m},$$

where $\mathbf{a}_i \sim \mathcal{U}(\mathbb{Z}_q^n)$ and known classically, and find $\mathbf{s} \in \mathbb{Z}_q^n$.

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$$\mathsf{S}\,|\mathsf{LWE}\rangle^{\ell}_{n,q,D_r} \leq \overline{\text{G-EDCP}}^{\ell}_{n,q,\sigma} \leq \overline{\text{U-EDCP}}^{O(\ell)}_{n,q,M}$$

where $\sigma = q/r$ and $M = c \cdot \sigma$.

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Theorem

Let $n, q = \mathrm{poly}(\kappa)$ be integers, where q is a power-of-two. Let $r = \Omega(\sqrt{\kappa})$ and $q/r = \Omega(\sqrt{\kappa})$. There exists a quantum algorithm for $\mathsf{S}\,|\mathrm{LWE}\rangle^{\ell}_{n,q,D_r}$ in time $2^{\mathcal{O}(\log^2 n)}$, when $\ell = 2^{\Omega(\log^2 n)}$.

Impact on LWE

Theorem [BKSW18]

Let $m \geq n \log q$ and $q = \mathsf{poly}(n)$. There is a probabilistic quantum reduction from $\mathsf{LWE}^{m}_{n,q,\alpha} \leq \text{U-EDCP}^{\ell}_{n,q,M}$, where

$$\ell < 1/(M \cdot \alpha \cdot \mathsf{poly}(n)).$$

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- Reduction yields only polynomially many EDCP samples.

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  $\Rightarrow$ Our algorithm does not extend to standard LWE

Future work

1. Can the reduction from LWE to EDCP be improved, or can our algorithm be modified so that it requires only a poly(n) EDCP samples?

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3. EDCP assumption for structured LWE problems?

Thanks for your time

Do you have any questions?

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