Highway to Hull: A new algorithm solving the matrix code equivalence problem

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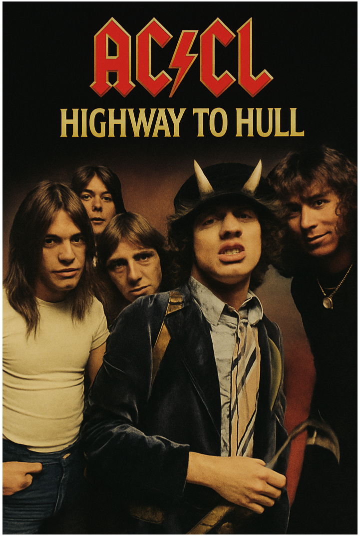
You may have heard of us



Picture created using ChatGPT, inspired by the original cover of the album *Highway to Hell* by AC/DC.

tl;wl

- A particular case of the matrix code equivalence problem is used in recent signature schemes MEDS, ALTEQ.

- Two recent attacks (2024) on some specific instances made them change their parameters.

- We provide an algorithm HtH for a much broader range of parameters, with competitive complexity.

- HtH is based on the following idea: compute one-dimensional *hulls* of various codes, and search for collisions among them.

1. The matrix code equivalence problem

2. Overview of HtH

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2. Overview of HtH

MCE: decisional and search problems

Fix a finite field $\mathbb{F}_q$ and positive integers k, m, n such that $m \leqslant n$ and $k < mn$.

The matrix code equivalence problem (MCE)

Given k-dimensional linear subspaces $\mathcal{C}, \mathcal{D}$ of $\mathbb{F}_q^{m \times n}$:

- MCE Decisional problem: Do there exist matrices $P \in \mathrm{GL}_m(\mathbb{F}_q), Q \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $\mathcal{D} = P\mathcal{C}Q$?
- MCE Search problem: Find, if such matrices exist, $P \in \mathrm{GL}_m(\mathbb{F}_q), Q \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $\mathcal{D} = P\mathcal{C}Q$.
- Particular case $k = m = n$: cubic matrix code equivalence problem (CMCE).

We present a probabilistic algorithm for the MCE search problem.

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$

Related problems

The following problems are equivalent to MCE.

- ◇ 3-Tensor Isomorphism
- ◇ Trilinear Forms Equivalence (TFE)
- ◇ Matrix Space Conjugacy (MCE with $Q = P^{-1}$)

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$

Use of these problems in NIST submissions

Two signature schemes based on these problems were recently put forward.

- ALTEQ: based on TFE
  Parameters (NIST Level I): $q = 2^{32} - 5$, $k = m = n = 13$
  Parameters (NIST Level III): $q = 2^{32} - 5$, $k = m = n = 20$

- MEDS: based on MCE (but actually, the initial parameters were all instances of CMCE)
  Parameters (NIST Level I): $q = 2^{12} - 1$, $k = m = n = 14$
  Parameters (NIST Level III): $q = 2^{12} - 1$, $k = m = n = 22$

Both were thrown out of the NIST competition in October 2024.

- $m \leqslant n$
- $k < mn$
- $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- $k = \dim(\mathcal{C})$

Recent algorithms for MCE

[Narayanan, Qiao, Tang (2024)]

Method: Find matching points of corank 1 in both codes and construct equivalence from them

Strength: Best theoretical complexity to date $\widetilde{\mathcal{O}}(q^{n/2})$

Weakness: Only applies to CMCE problem ($k = m = n$)

[Ran, Samardjiska (2024)]

Method: Construct graphs associated with tensors, find triangles in them and construct equivalence from these triangles (low-degree polynomial system solving)

Strength: Best practical complexity to date

Weakness: Only works in $1/q$ of all cases (when graphs contain triangles)

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$

How HtH compares to the state of the art

- ◇ Parameter range: $\{k, k^{\perp} < \min(m,n)^2 - 1\}$ where $k^{\perp} = mn - k$

$\rightarrow$ much broader than both previous algorithms.

- ◇ Time complexity when $k = m = n$: $\widetilde{\mathcal{O}}(q^{n/2})$

$\rightarrow$ similar to Narayanan et al.

$\rightarrow$ in practice, not better than Ran and Samardjiska.

- ◇ Space complexity when $k = m = n$: $\mathcal{O}\left(nq^{\frac{n}{2}-1}\right)$

$\rightarrow$ $\mathcal{O}(n^2)$ times smaller than Narayanan et al.

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$

General structure

Data k-dimensional matrix codes $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$.

Goal Find P, Q such that $\mathcal{D} = P\mathcal{C}Q$.

1. Reduce to conjugacy problem (i.e. $Q = P^{-1}$) for codes $\mathcal{C}_A, \mathcal{D}_B$ computed from $\mathcal{C}, \mathcal{D}$
   ... for which we know a pair of conjugate 1-dimensional subspaces in $\mathcal{C}$ and $\mathcal{D}$ (collision search).

2. Solve this conjugacy problem.

3. Deduce solution to the original problem.

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$

Reducing to conjugacy problem (1/2)

If $\mathcal{D} = P\mathcal{C}Q$, then for each $A \in \mathcal{C}^\perp$, $B = (P^{-1})^\top A(Q^{-1})^\top \in \mathcal{D}^\perp$.

We produce codes $\mathcal{C}_A = \mathcal{C}A^\top, \mathcal{D}_B = \mathcal{D}B^\top$ where $A \in \mathcal{C}^\perp, B \in \mathcal{D}^\perp$, hoping to find a matching pair (A, B). For such a pair, $\mathcal{D}_B = P\mathcal{C}_A P^{-1}$.

- ◇ Checking whether two k-dimensional codes are conjugate is hard.
- ◇ Checking whether two 1-dimensional codes are conjugate is easy.

→ We try to find A, B such that $\mathcal{C}_A$ and $\mathcal{D}_B$ contain specific conjugate 1-dimensional subspaces: their respective *hulls*.

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$
- ☐ $A \in \mathcal{C}^\perp$
$B \in \mathcal{D}^\perp$
- ☐ $\mathcal{C}_A = \mathcal{C}A^\top$
$\mathcal{D}_B = \mathcal{D}B^\top$

Reducing to conjugacy problem (2/2)

Definition

In this talk, the hull of a matrix code $\mathcal{C} \subset \mathbb{F}_q^{m \times m}$ is

$$h(\mathcal{C}) = \{A \in \mathcal{C} \mid \forall B \in \mathcal{C}, \mathrm{Tr}(AB) = 0\}.$$

- ◇ Two conjugate codes have conjugate hulls.
- ◇ Roughly $1/q$ of all codes have 1-dimensional hull.
- ◇ Given a uniformly random code $\mathcal{C}$ and a uniformly random matrix A such that $h(\mathcal{C}_A)$ has dimension 1, the characteristic polynomial of a generator of $h(\mathcal{C}_A)$ is uniformly random.

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$
- ☐ $A \in \mathcal{C}^\perp$
  $B \in \mathcal{D}^\perp$
- ☐ $\mathcal{C}_A = \mathcal{C}A^\top$
  $\mathcal{D}_B = \mathcal{D}B^\top$

Overview of the collision search

- Construct a dictionary (polynomial:matrix):

  1. Pick matrix $A \in \mathcal{C}^\perp$
  2. If $\dim h(\mathcal{C}_A) = 1$, compute the characteristic polynomial χ of a normalized generator of $h(\mathcal{C}_A)$, add $(\chi : A)$ to the dictionary
  3. Continue until the dictionary contains $\mathcal{O}(q^{(m-3)/2})$ entries

- Find matching pairs:

  1. Pick matrix $B \in \mathcal{D}^\perp$
  2. If $\dim h(\mathcal{D}_B) = 1$, compute the characteristic polynomial of a normalized generator of $h(\mathcal{D}_B)$ and look for collision in the dictionary

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$
- ☐ $A \in \mathcal{C}^\perp$
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- ☐ $\mathcal{C}_A = \mathcal{C}A^\top$
$\mathcal{D}_B = \mathcal{D}B^\top$

Solving conjugacy problem

Data Codes $\mathcal{C}_A, \mathcal{D}_B \subset \mathbb{F}_q^{m\times m}$ containing two conjugate vectors $U_A \in h(\mathcal{C}_A)$, $V_B \in h(\mathcal{D}_B)$:

$$V_B = P_0 U_A P_0^{-1}.$$

Goal Find P such that $\mathcal{D}_B = P\mathcal{C}_A P^{-1}$.

Solution If such a P exists, then there exist $f, g \in \mathbb{F}_q[t]_{\leqslant m}$ such that

$$P = P_0 f(U_A) \text{ and } P^{-1} = g(U_A)P_0^{-1}.$$

We compute the coefficients of f and g by linearizing the bilinear system:

$$P_0 f(U_A)\mathcal{C}_A g(U_A)P_0^{-1} \subseteq \mathcal{D}_B.$$

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- ☐ $A \in \mathcal{C}^\perp$
  $B \in \mathcal{D}^\perp$
- ☐ $\mathcal{C}_A = \mathcal{C}A^\top$
  $\mathcal{D}_B = \mathcal{D}B^\top$

Solving the initial problem

Data Codes $\mathcal{C}, \mathcal{D}$ and matrices $A \in \mathcal{C}^\perp, B \in \mathcal{D}^\perp$, $P \in \mathrm{GL}_m(\mathbb{F}_q)$ such that $\mathcal{D}_B = P\mathcal{C}_A P^{-1}$.

Goal Find a matrix Q such that $\mathcal{D} = P\mathcal{C}Q$.

Solution Let $(C_1, \ldots, C_k)$ be a basis of $\mathcal{C}$. Define the linear map:

$$\begin{aligned} \psi_{\mathcal{C}} \colon \mathbb{F}_q^{n \times n} &\longrightarrow (\mathbb{F}_q^{m \times n})^k \\ Q &\longmapsto (PC_1Q, \ldots, PC_kQ). \end{aligned}$$

The suitable matrices Q are exactly the elements of $\psi_{\mathcal{C}}^{-1}(\mathcal{D}^k) \cap \mathrm{GL}_n(\mathbb{F}_q)$.

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
- ☐ $k = \dim(\mathcal{C})$
- ☐ $A \in \mathcal{C}^\perp$, $B \in \mathcal{D}^\perp$
- ☐ $\mathcal{C}_A = \mathcal{C}A^\top$, $\mathcal{D}_B = \mathcal{D}B^\top$

Total complexity

Theorem

For parameters k, m, n such that $m(n-m)+2 < k \leqslant mn/2$, the average number of operations in $\mathbb{F}_q$ required by algorithm HtH *to solve* $\mathsf{MCE}_{k,m,n}$ *is*

$$\widetilde{\mathcal{O}}\left(q^{\max\left(\frac{k^\perp}{2}, k^\perp - m + 2\right)}\right)$$

where $k^\perp = nm - k$.

Remark

For $k = m = n$, the above complexity is

$$\widetilde{\mathcal{O}}\left(q^{n/2}\right).$$

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
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- ☐ $k^\perp = mn - k$
- ☐ $A \in \mathcal{C}^\perp$
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tl;dl

- We provide an algorithm HtH which solves the MCE problem with time complexity $\widetilde{\mathcal{O}}\left(q^{\max\left(\frac{k^\perp}{2}, k^\perp - m + 2\right)}\right)$ for a broad range of parameters.
- HtH is based on the following idea: compute one-dimensional hulls of various codes, and search for collisions among them.
- In the cubic case $k = m = n$: matches the state of the art in terms of time complexity and beats it in terms of space complexity.
- In the general case: first algorithm with this complexity.

- ☐ $m \leqslant n$
- ☐ $k < mn$
- ☐ $\mathcal{C}, \mathcal{D} \subset \mathbb{F}_q^{m \times n}$
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