Schnorr Signatures are Tightly Secure in the ROM under a Non-Interactive Assumption

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https://ia.cr/2024/1528

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- Hardness of problem P then determines parameters (e.g. key length) for instantiating scheme Π in real world
- Unfortunately, for many schemes we only have loose reductions (i.e., adversary B needs to spend much more effort than adversary A)

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- Algebraic properties of Schnorr signatures have been instrumental in achieving advanced functionalities, such as threshold, blind, adaptor signatures…
- Existentially unforgeable (EUF-CMA-secure) in the ROM under DL

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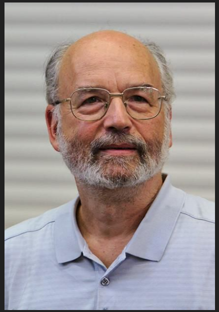
We should use a group order of 256 bits!



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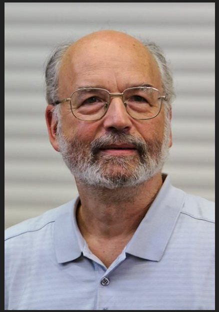

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Best known attack is breaking DL, which on twisted Edwards curves takes time $O(\sqrt{|\mathbb{G}|})$

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[PS96] in **ROM**:

$$\mathsf{Adv}^{\text{euf-cma}}_{\mathsf{Sch}[\mathbb{G}]} \leq q_h \cdot \sqrt{\mathsf{Adv}^{\text{dl}}_{\mathbb{G}}} + \ldots$$

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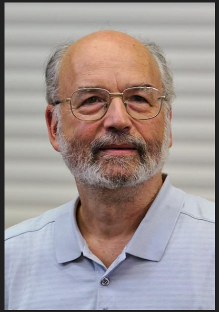

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Our Question

Is there a ***tight security proof*** *for Schnorr signatures?*
If so, under what ***assumption****?*

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⟹ Getting even a *semi*-tight reduction requires interactive, non-falsifiable assumptions or additional idealized models!

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Is there such a representation-dependent assumption or non-generic reduction that gets around the above?

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Verify$(vk, m, (R, s))$:

1. $c := H(R, m)$
2. $g^s = R \cdot vk^c$?

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Circular Discrete-Logarithm (CDL) Assumption

- CDL solution: $(R, z) \in \mathbb{G} \times \mathbb{Z}_p$ such that $f(R) \neq 0 \wedge g^z = R \cdot h^{f(R)}$

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  - In fact, we don't even need to know what f is!

Main Result

Theorem (in ROM):

$$\mathsf{Adv}^{\text{euf-cma}}_{\mathsf{Sch}[\mathbb{G}]} \leq \mathsf{Adv}^{\text{cdl}}_{\mathbb{G},f} + \frac{q_s(q_s + q_h) + q_h \cdot |f^{-1}(0)|}{p}$$

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- Arbitrary efficiently computable function $f : \mathbb{G} \to \mathbb{Z}_p$!
- Take f that minimizes advantage

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- Sparkle+ has a loose reduction from static security to DL (in the ROM)
- We give a tight proof of static security under CDL (in the ROM)

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2. Idealized function:
   - We show that for the ECDSA conversion function

$$f : (x, y) \mapsto x \bmod p$$

   CDL reduces to DL in the algebraic bijective ROM [FKP16, QCY21]

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Summary

- We introduce the circular discrete-logarithm problem, a new non-interactive and falsifiable variant of DL which uses a function $f : \mathbb{G} \to \mathbb{Z}_p$

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- We conjecture that the ECDSA conversion function works as f for a suitable elliptic-curve group and give evidence by proving it in suitable idealized models
- We give a tight proof of (static) security of the Sparkle+ threshold signature scheme [CKM23] under CDL

Future Directions

- Is there a function for which CDL reduces to a standard assumption, maybe even DL?
- Is CDL applicable to:
  - Additional threshold Schnorr schemes?
  - Additional advanced primitives based on Schnorr signatures like adaptor signatures, multisignatures, or blind signatures?
- Could CDL be useful for instantiating Schnorr signatures under EUF-CMA in the standard model?

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Thanks!

Questions?

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Proof Intuition

- On CDL instance h, we run the forger with public key h
- We simulate signing queries as in [PS96]
- For hash queries, we want to embed outputs of f in responses such that:
  1. Responses are independent and uniform
  2. The forgery can be used to extract a CDL solution
- On the i-th hash query (R, m), we set $R' := R \cdot h^{a_i} \cdot g^{b_i}$ for random $a_i, b_i \in \mathbb{Z}_p$ and return

$$f(R') + a_i \bmod p$$

Proof Intuition

- Now adversary's forgery $m, (R, s)$ will correspond to a hash query, so:

$$g^s = R \cdot h^c = R \cdot h^{f(R \cdot h^a \cdot g^b)+a}$$

- Multiplying both sides by g^b gives:

$$g^{s+b} = R \cdot h^a \cdot g^b \cdot h^{f(R \cdot h^a \cdot g^b)}$$

- So, we can return the CDL solution:

$$(R \cdot h^a \cdot g^b,\ s + b \bmod p)$$

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