On Weak NIZKs, One-way Functions and Amplification

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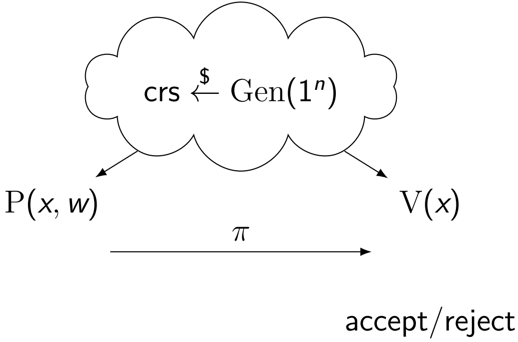
Non-Interactive Zero-Knowledge (for NP)



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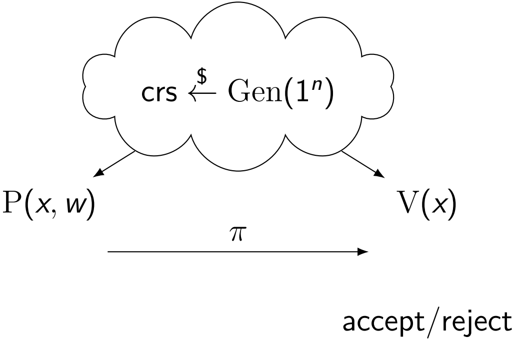

Completeness: If $x \in \mathcal{L}$, $\Pr[\mathrm{V}\text{ accepts}] \geq 1 - \epsilon_{\mathsf{c}}$.

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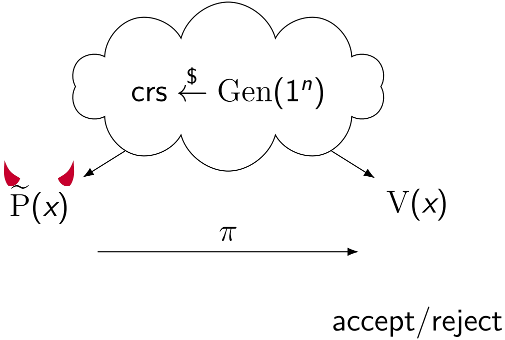

Soundness: If $x \notin \mathcal{L}$, then for any nuPPT cheating prover $\widetilde{\mathrm{P}}$, $\Pr[\mathrm{V} \text{ accepts}] \leq \epsilon_{\mathsf{s}}$.

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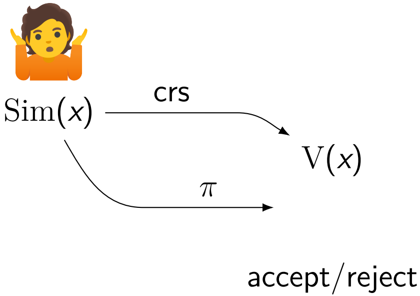

Zero-Knowledge: There is a simulator Sim that for every $x \in \mathcal{L}$ is ϵ_{zk} computationally indistiguishable from V's view.

NIZK and OWF

Key question: how “hard” are NIZKs to construct?
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Our goal: understand what happens if the error parameters are allowed to be large, even constant.

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Recent work [GJS19,BKP+24,BG24,AK25] has shown that we can amplify "weak" NIZKs to get negligible errors.

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Viewpoint 1: understand if the hardness of constructing NIZKs is "inherent" or only comes from needing the errors to be small.

Viewpoint 2: if weak NIZKs give one-way functions, we can amplify "for free"!

Main Results

Suppose NP $\not\subseteq$ ioP/poly and we have a weak NIZK for NP. Then one-way functions exist if for any polynomial p, any of the following hold:

- $\epsilon_c(n) + \epsilon_s(n) + 2\epsilon_{zk}(n) < 1 - \frac{1}{p(n)}$
- $\epsilon_c(n) + \epsilon_{zk}(n) + 2\sqrt{\epsilon_s(n)} < 1 - \frac{1}{p(n)}$
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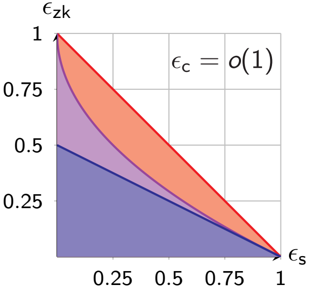

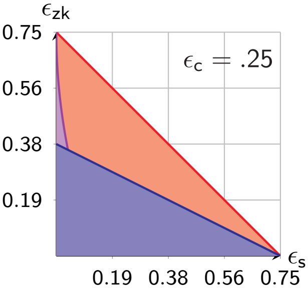

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(* as long as *either* NP $\not\subseteq$ ioP/poly or NP $\subseteq$ BPP)

Overview of Techniques

Each parameter regime uses different techniques to show that if one-way functions don't exist but NP has a weak NIZK, NP $\subseteq$ ioP/poly:

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This replaces an *additive* ϵ_{zk} loss with a *multiplicative* $\frac{1}{\sqrt{\epsilon_s}}$ loss!

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- Upcoming work: some progress, but seems stuck at constant rounds.

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Thanks!