Sometimes-Decryptable Homomorphic Encryption from Sub-exponential DDH

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NTT and Johns Hopkins University

Zhengzhong Jin
Northeastern University

Homomorphic Encryption 🔒 (HE)

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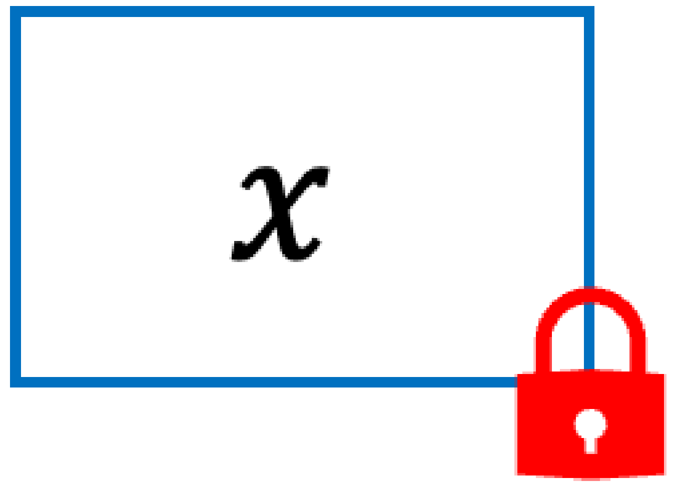

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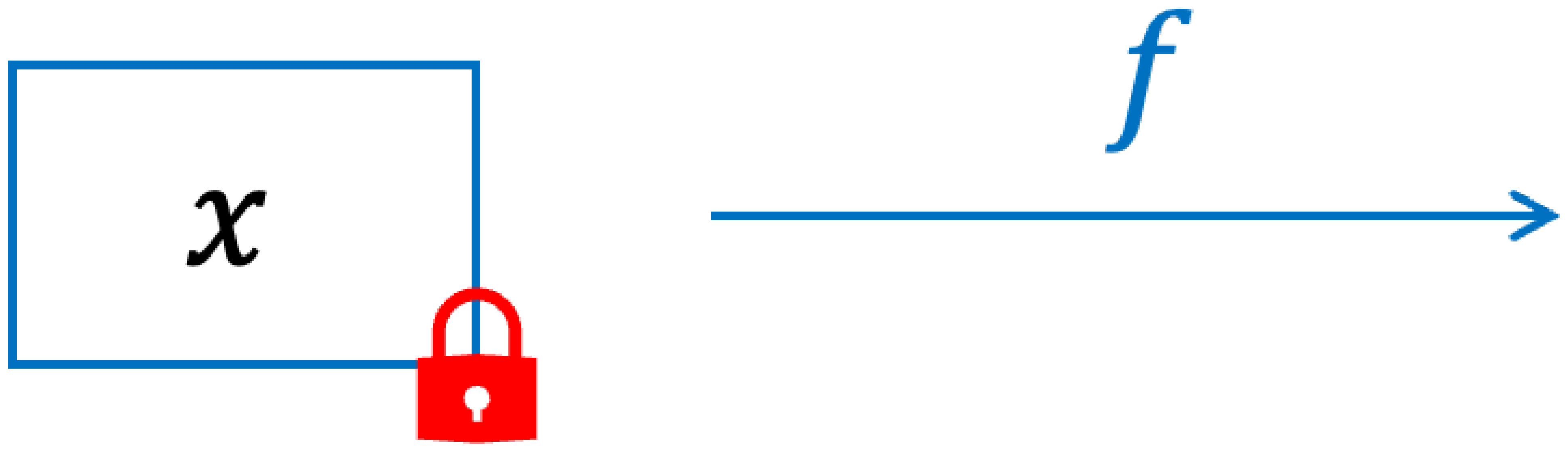

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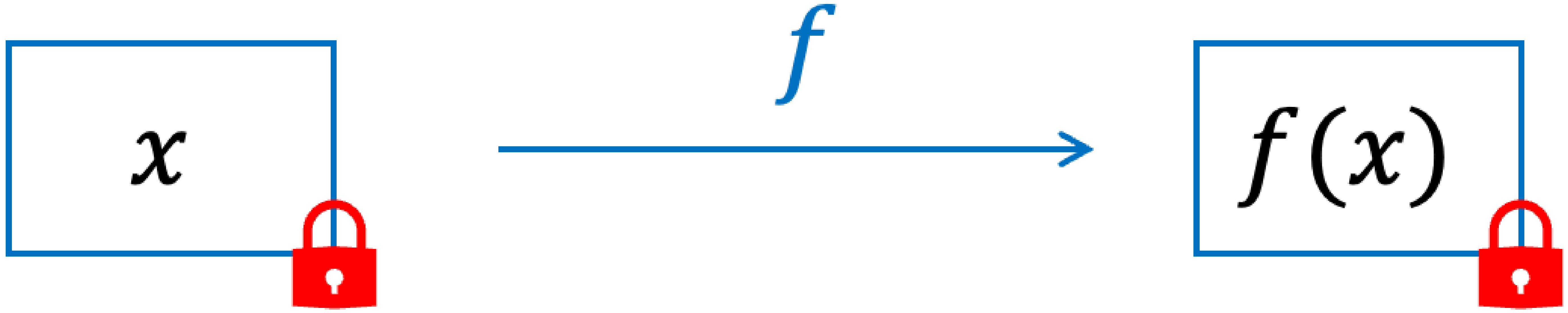

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Many Applications: computing over encrypted data

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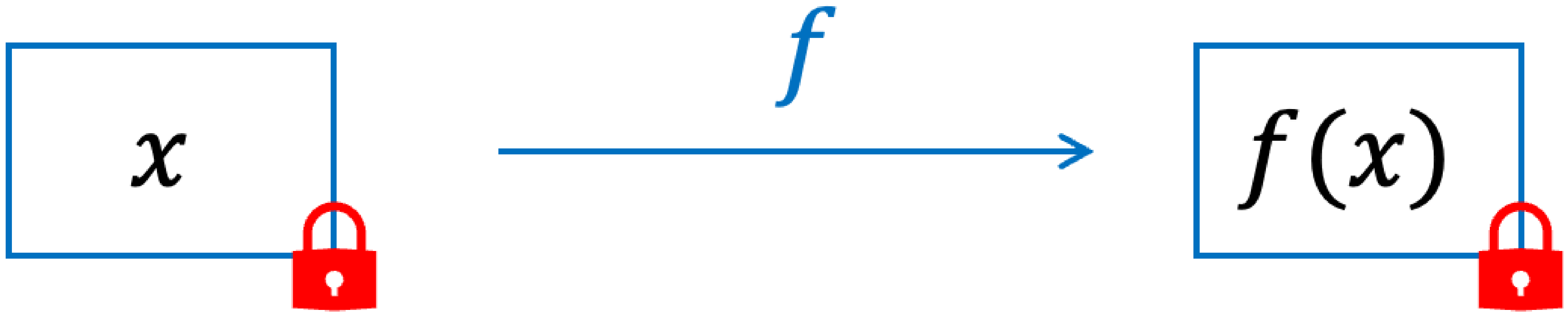

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Prior Work

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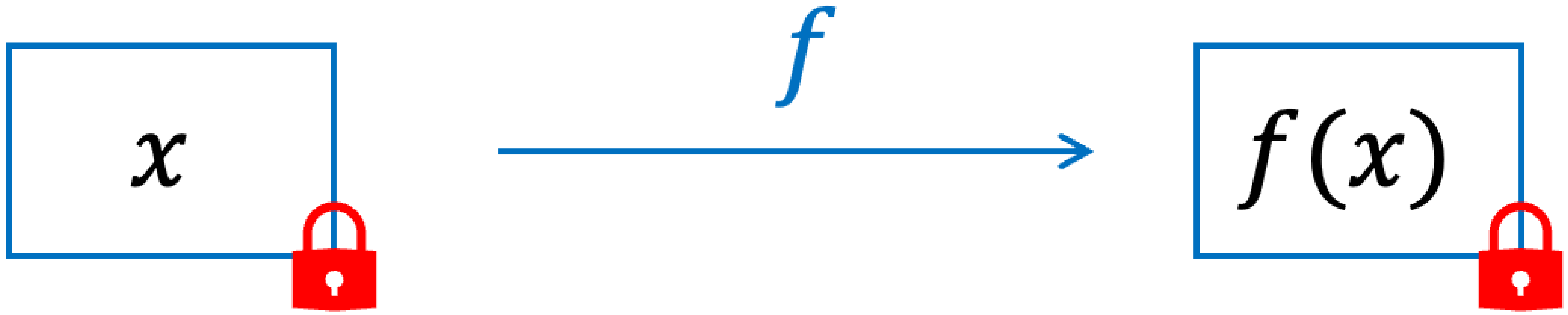

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- Fully homomorphism:

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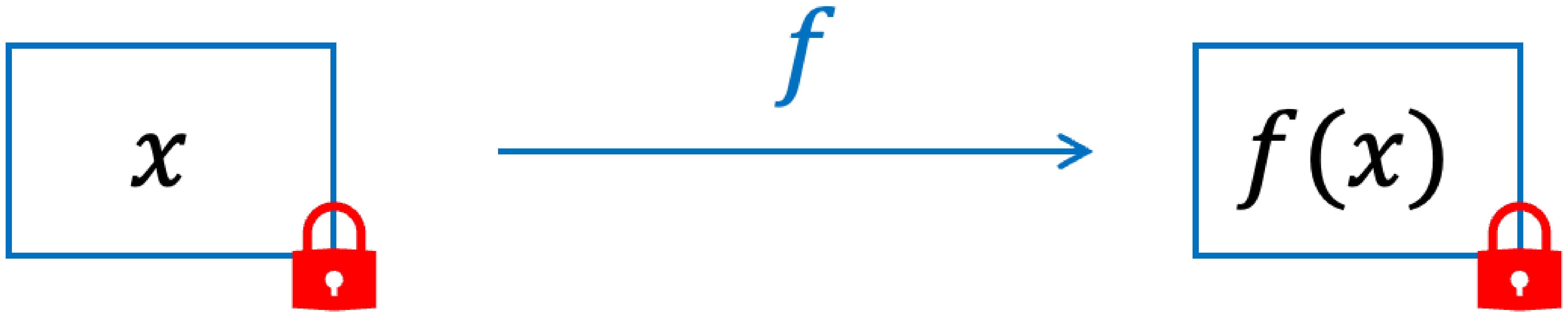

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- 2-DNF: bilinear maps [Boneh-Goh-Nissim'05]

Can we build HE from group-based assumptions, for a larger class of functionality?

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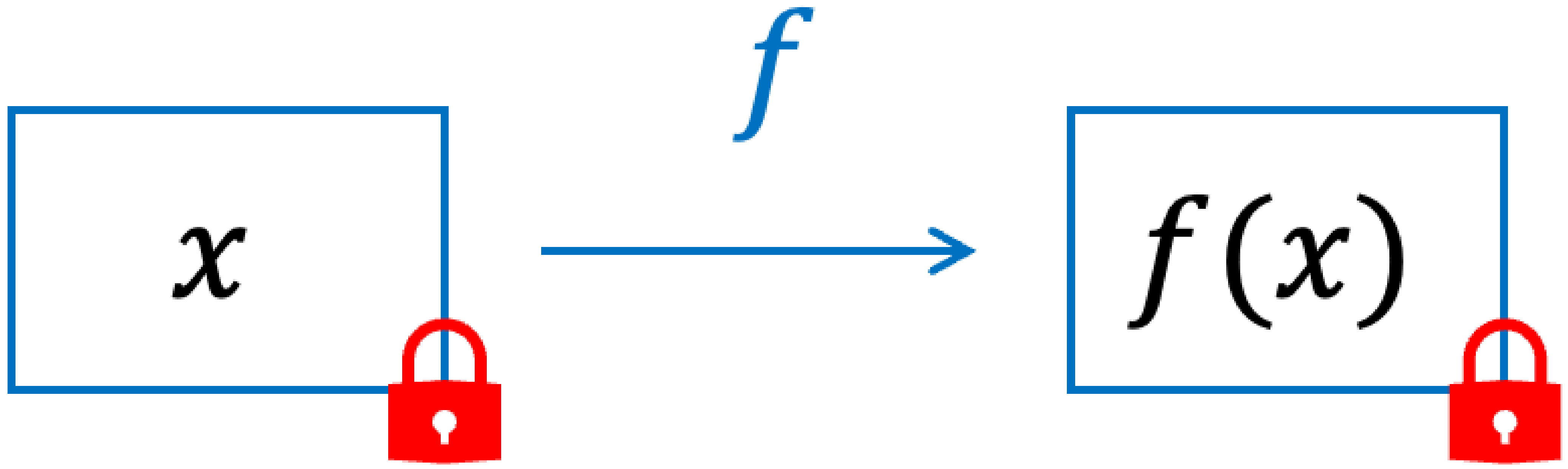

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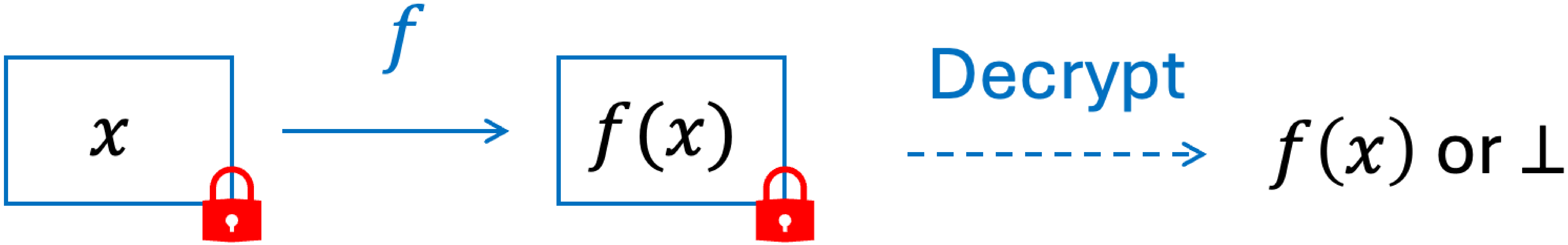

x

f

$f(x)$

Decrypt

$f(x)$ or $\perp$

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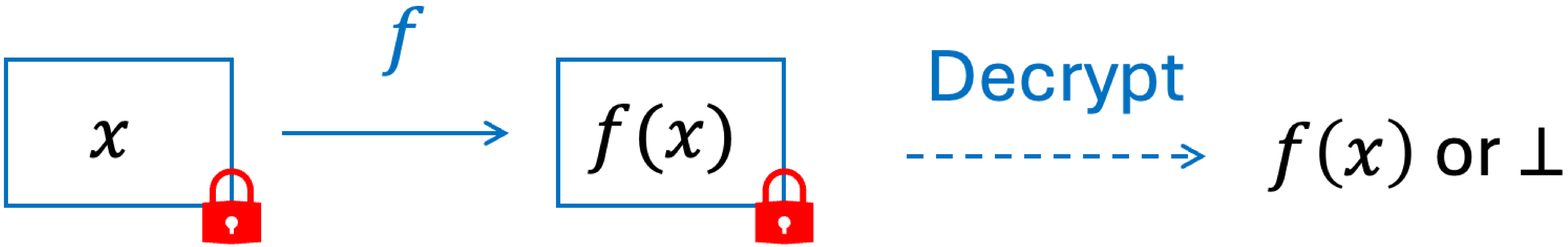

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Formal Definition: later

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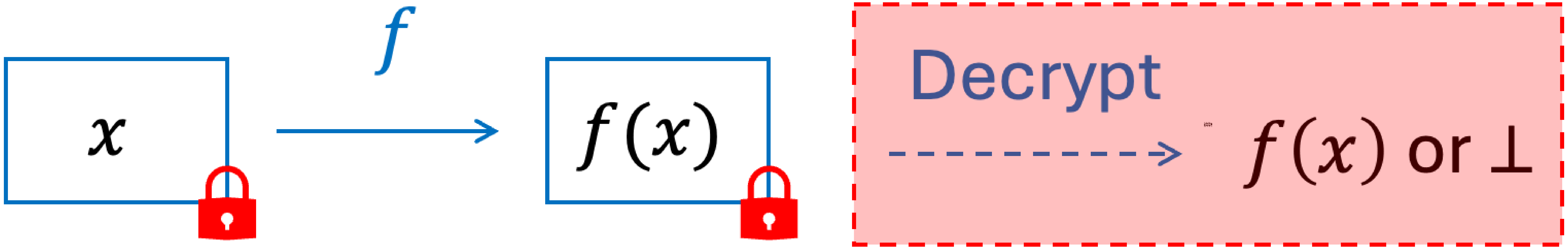

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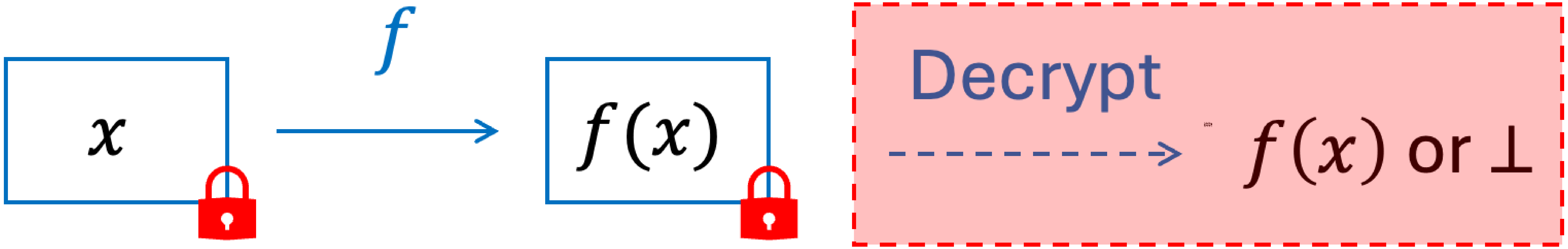

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Useful When: **Decryption is only needed in security proof**
e.g. proof systems

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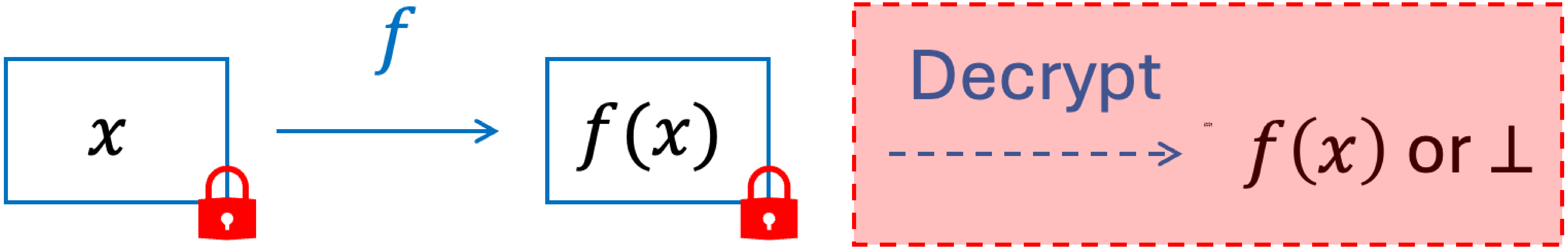

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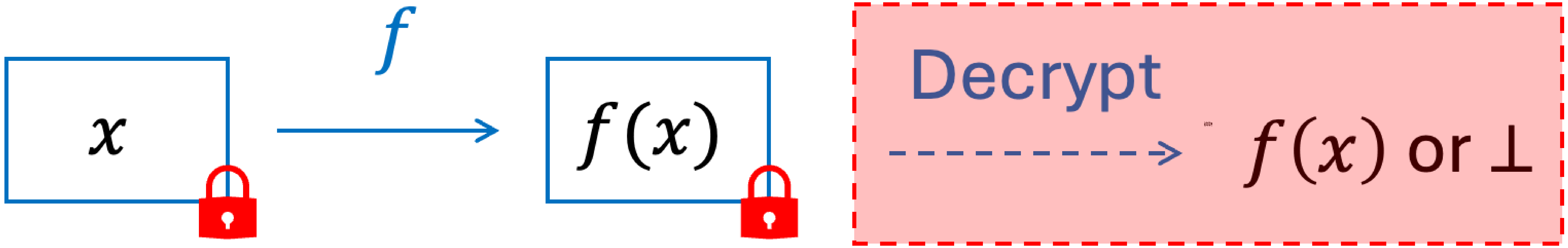

x $\xrightarrow{f}$ $f(x)$

Decrypt $\dashrightarrow$ $f(x)$ or $\perp$

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Formal Definition: later

Decryption/Extraction only in the Security Proof:

- Sometimes extractable commitment → *statistical Zaps* [Kalai-Khurana-Sahai'18]
- Somewhere extractable commitment [Hubacek-Wichs'15], predicate-extractable commitment [Brakerski-Brodsky-Kalai-Lombardi-Paneth'23] → *SNARGs*
- Correlation intractable hash [Canetti-Chen-Holmgren-Lombardi-Rothblum-Rothblum-Wichs, Peikert-Shiehian'19] → *NIZKs/SNARGs*

Our Result

Assuming sub-exponential hardness of Decisional Diffie-Hellman (DDH), there exists a sometimes-decryptable homomorphic encryption for TC^0.

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(TC^0: constant-depth threshold circuits)

Application: Succinct Non-interactive ARGuments (SNARGs)

CRS: Common Reference String

P

V

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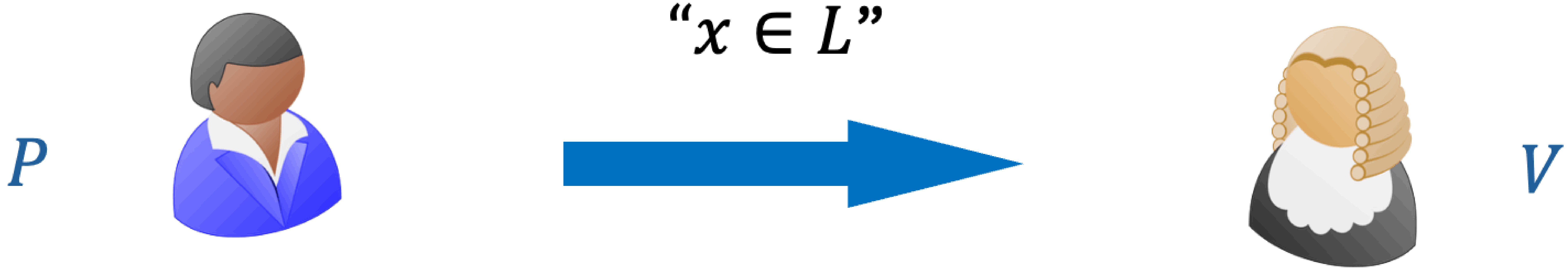

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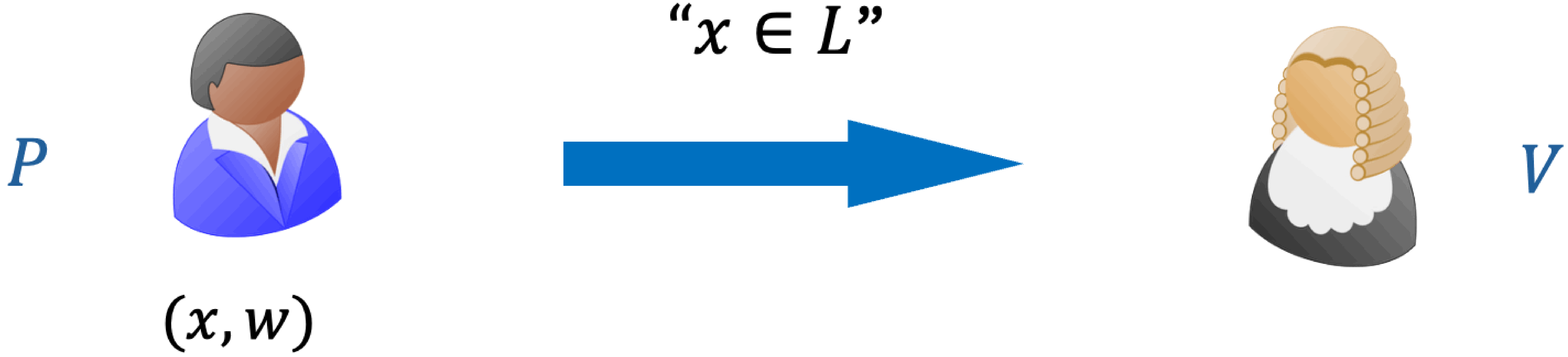

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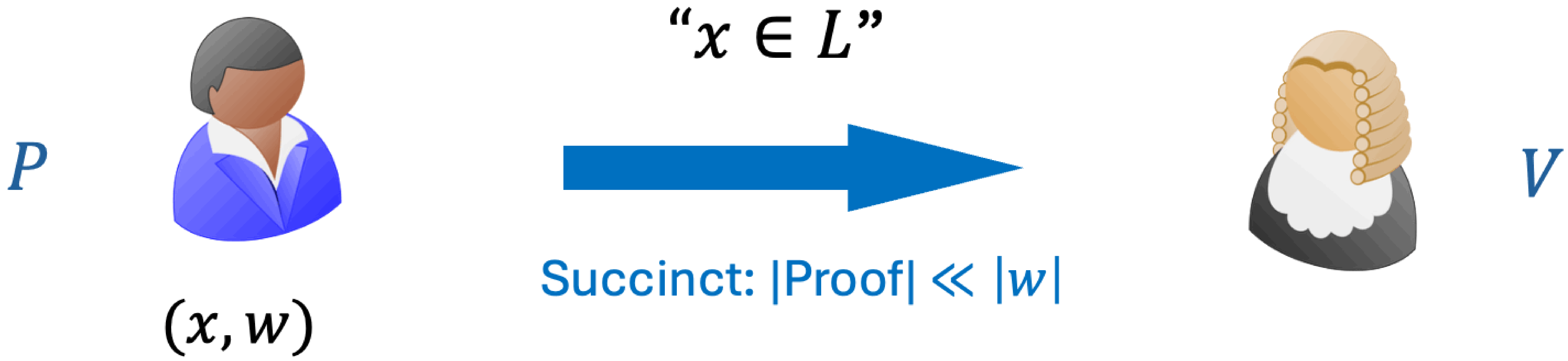

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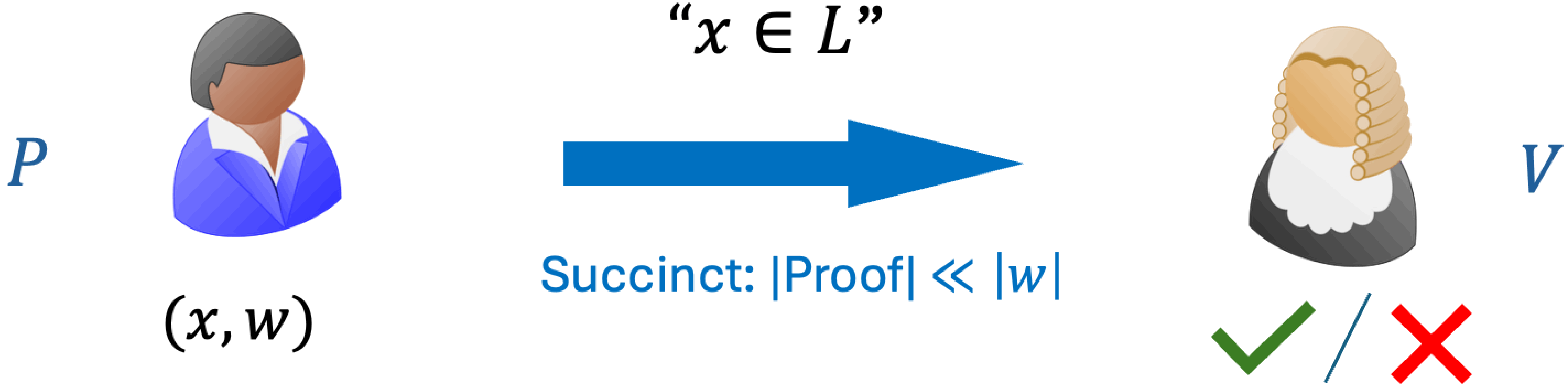

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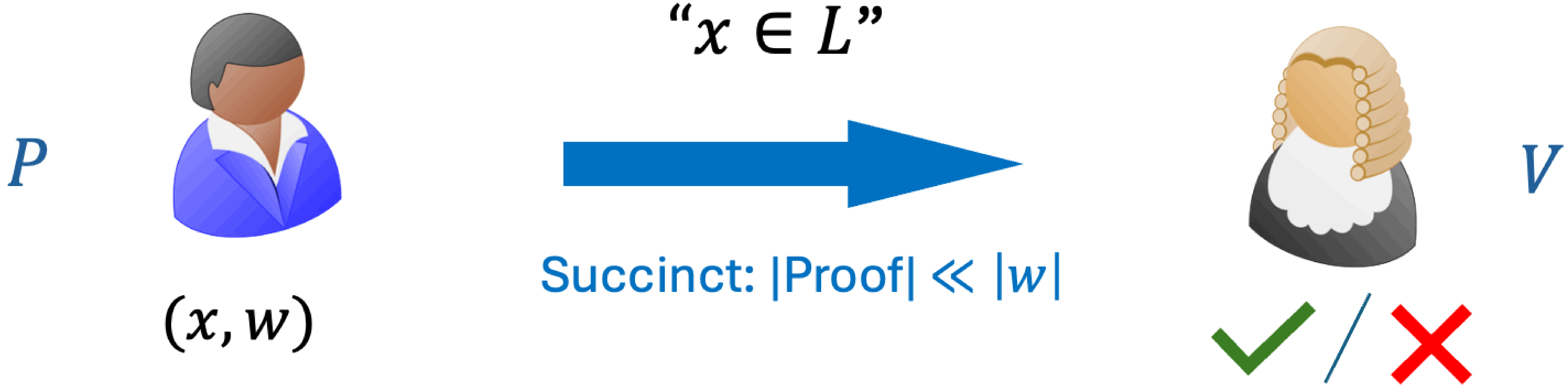

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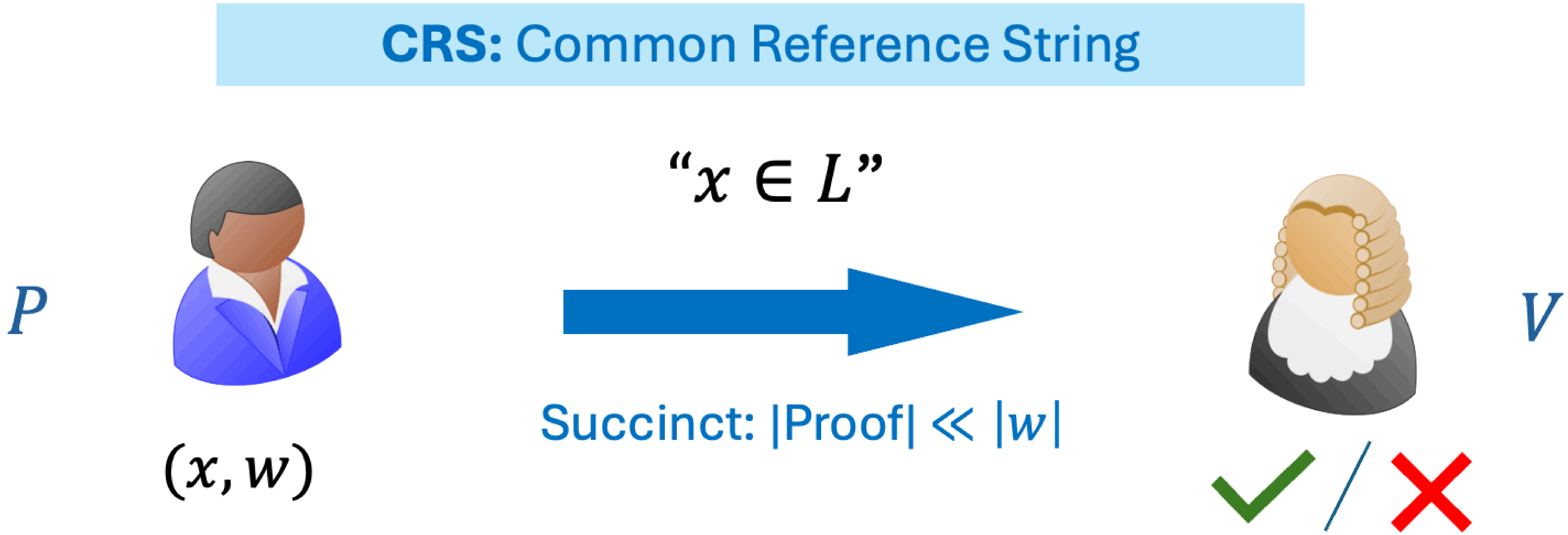

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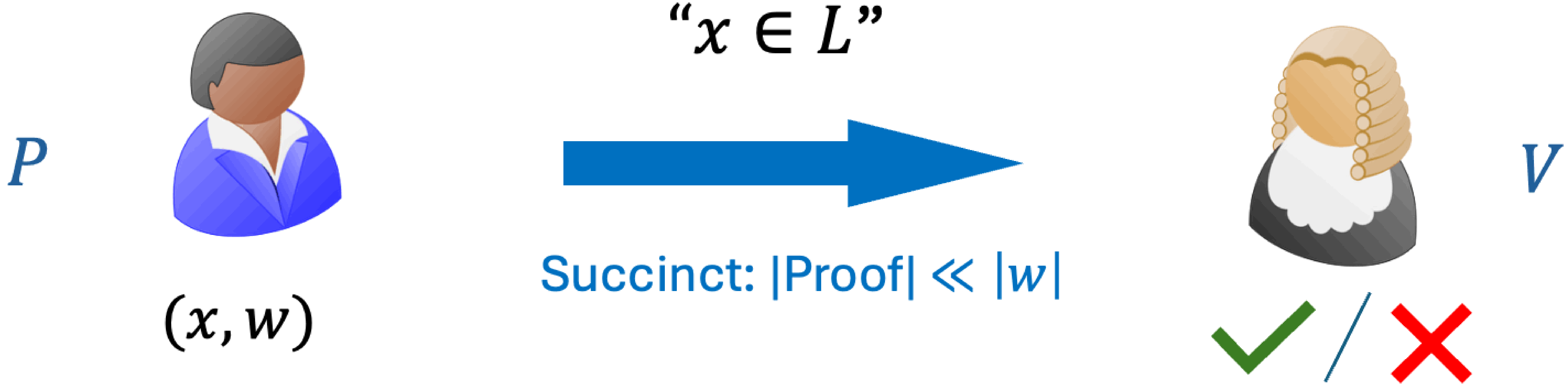

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Many applications: delegation of computation, blockchain and cryptocurrency, etc.

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Example: DDH Language

$$\{(g, h, g^s, h^s) | s \in \mathbb{Z}, g, h \in \mathbb{G}\}$$

Implication: Monotone-Policy Batch Arguments

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$x_1 \ldots x_k, w_1 \ldots w_k$

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“$f\left(1_{x_1 \in L}, \dots, 1_{x_k \in L}\right) = 1$”, f: a monotone circuit

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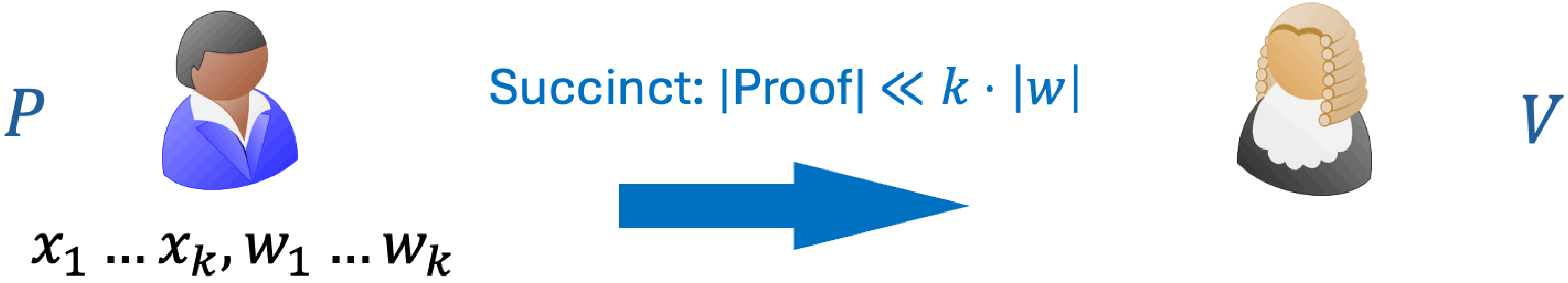

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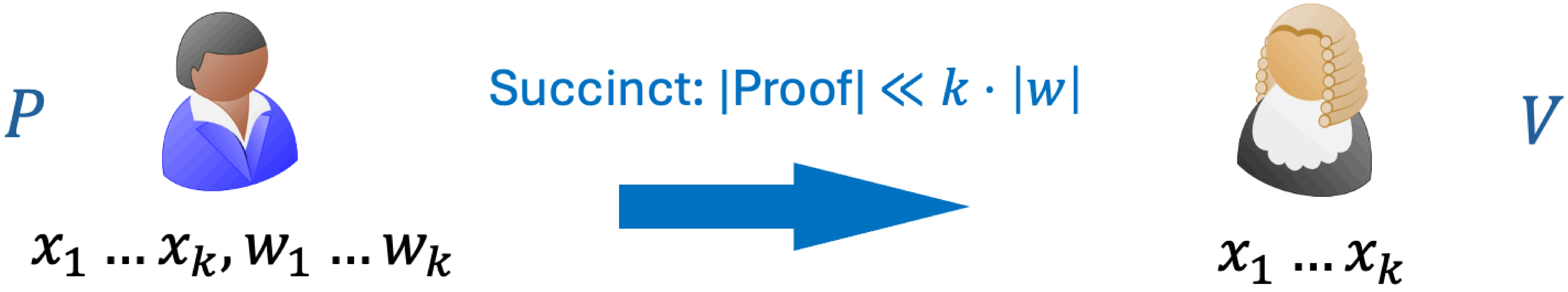

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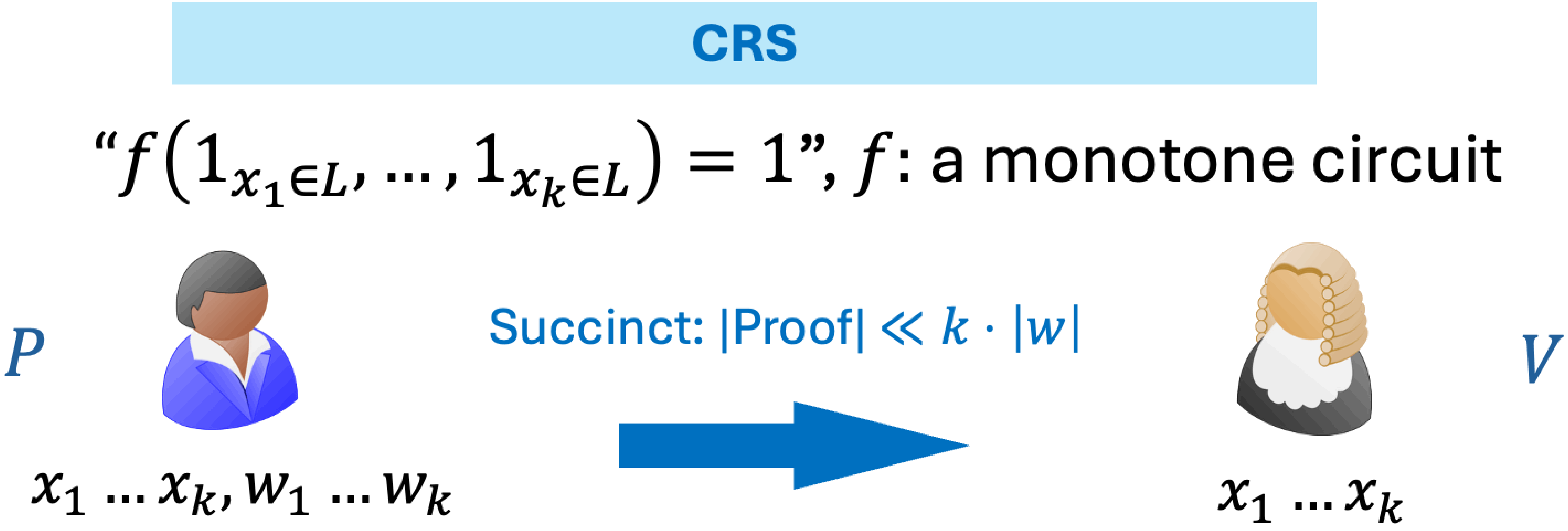

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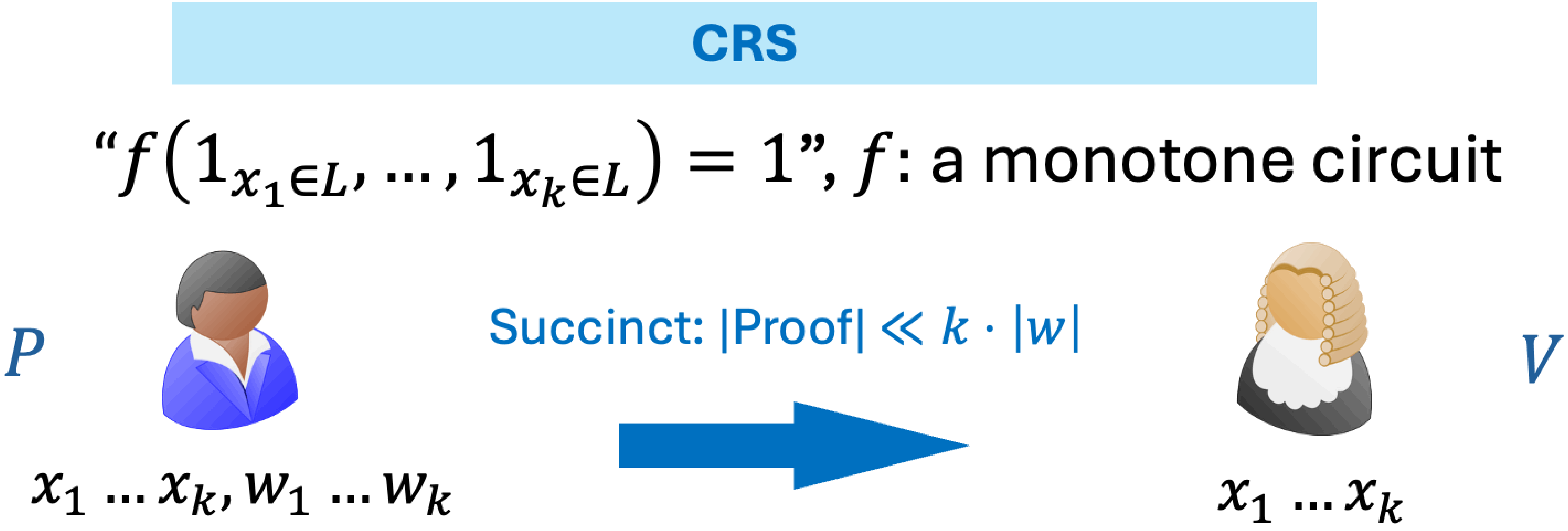

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- Concurrent: [Nassar-Waters-Wu'24] from sub-exp DDH (different approach), or poly-hard k-Lin in pairing groups

Application (2): Monotone-Policy BARGs

Assuming sub-exponential hardness of DDH, there exists a monotone-policy BARGs for all polynomial-size monotone circuits.

More in the paper: Predicate-Extractable hash and
Correlation-Intractable hash from sub-exp DDH.

Rest of the Talk

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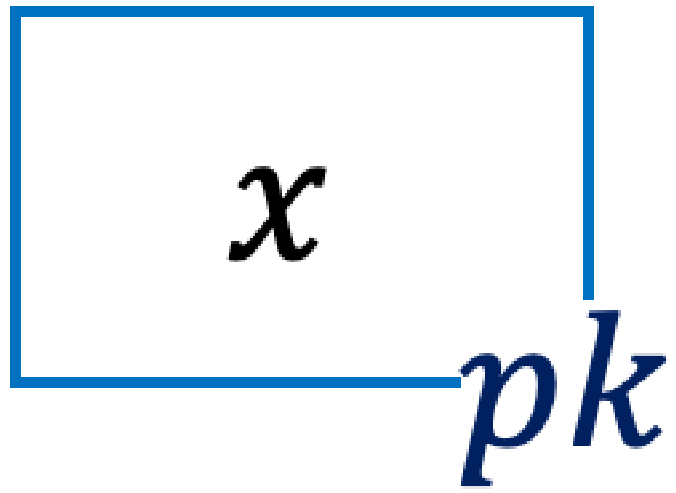

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PPT.

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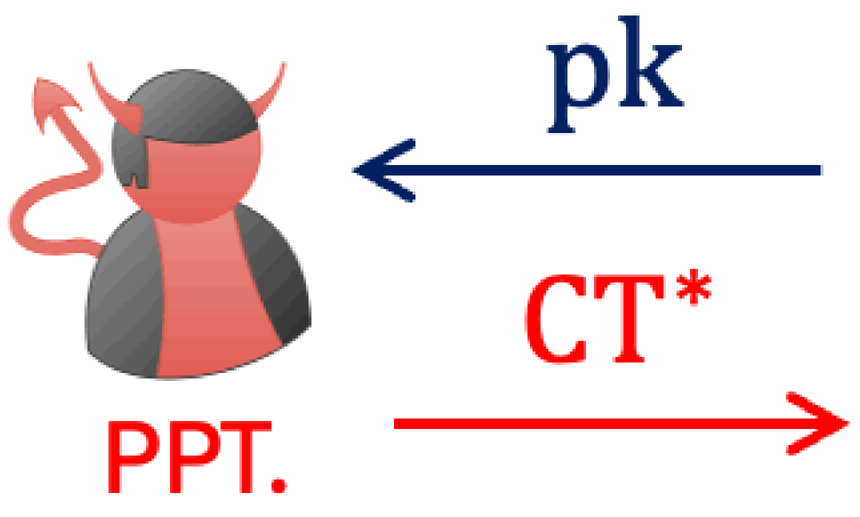

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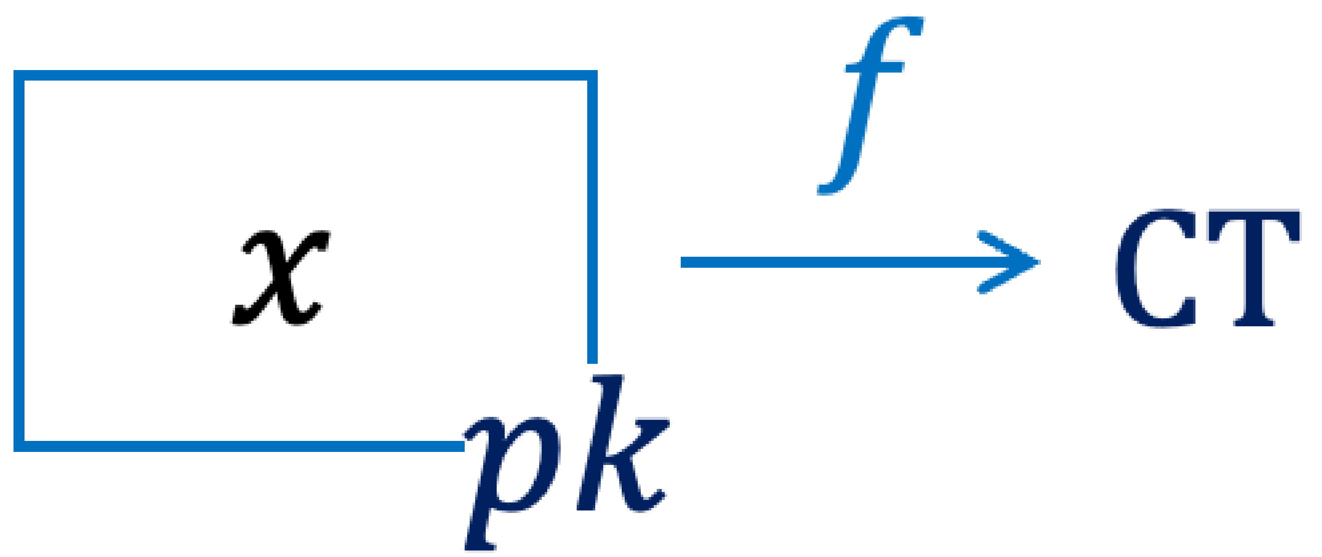

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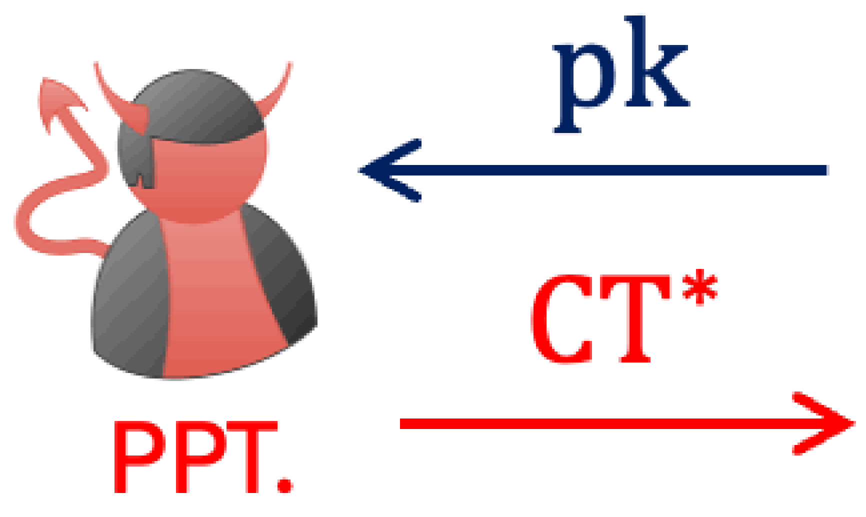

$\Pr[\mathrm{Pred}(\mathrm{CT}^*) = 1] > 2^{-\lambda^c}$

Issue: Probability Can’t Compose

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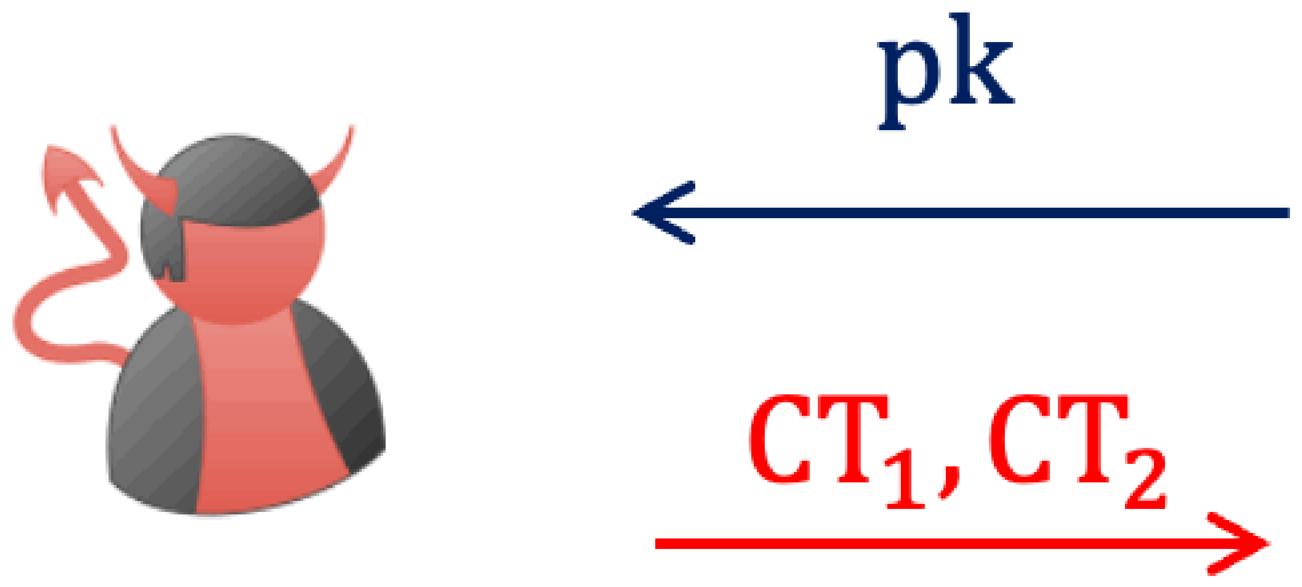

pk

$\mathrm{CT}_1, \mathrm{CT}_2$

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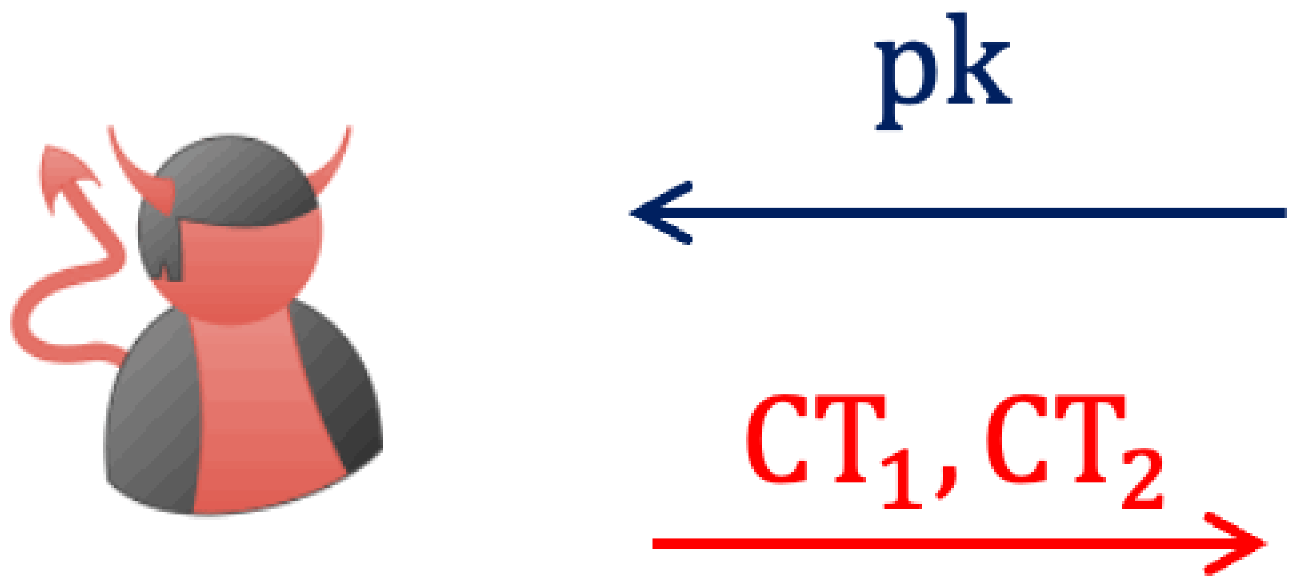

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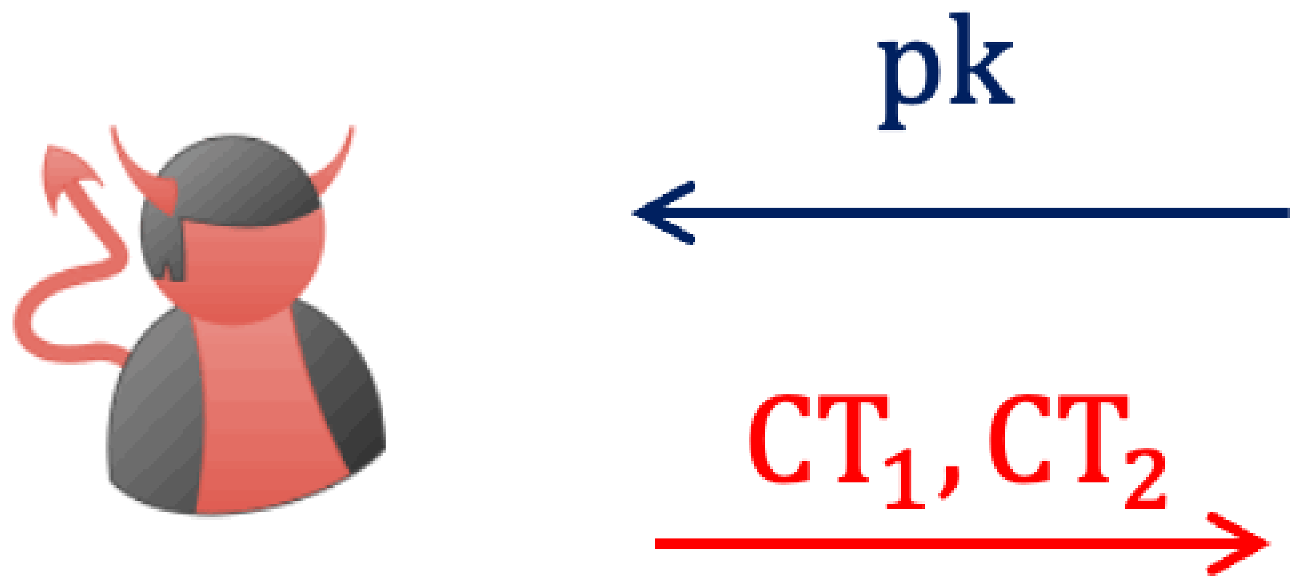

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We can’t conclude $\Pr[\mathrm{Pred}(\mathrm{CT}_1) \wedge \mathrm{Pred}(\mathrm{CT}_2)] \geq \mu^2$

Avoid Composition via Multi-bit Evaluation

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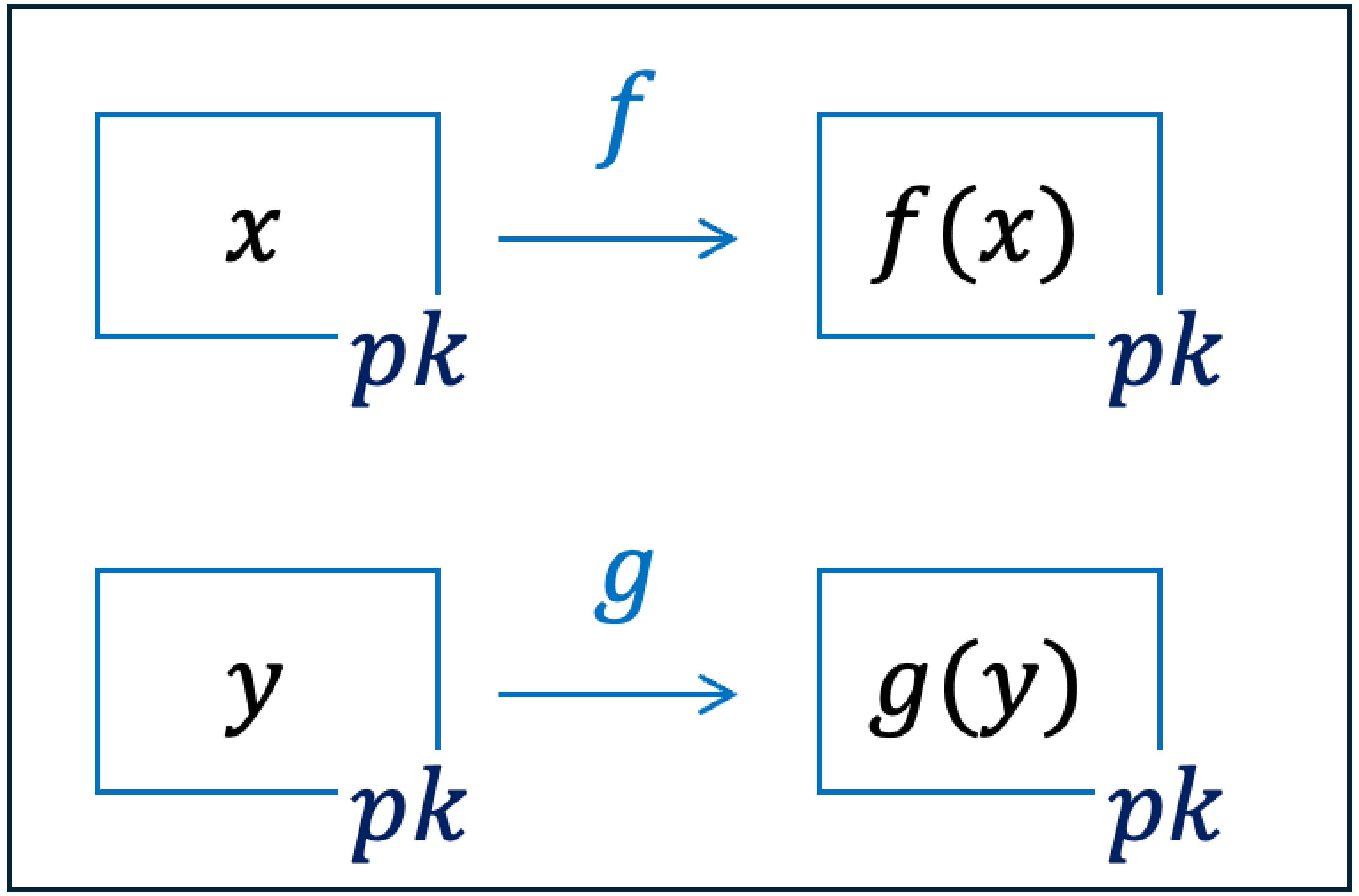

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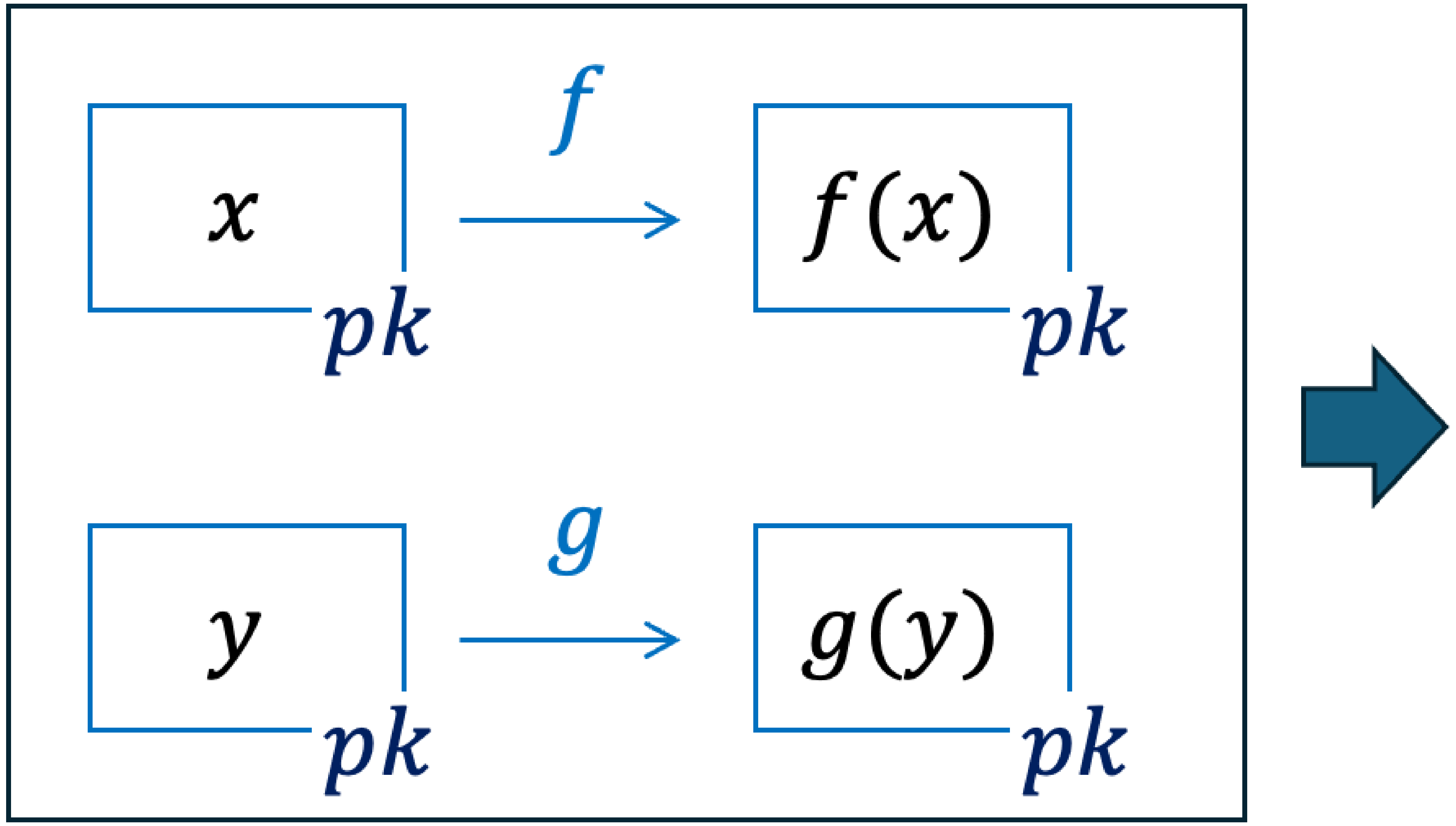

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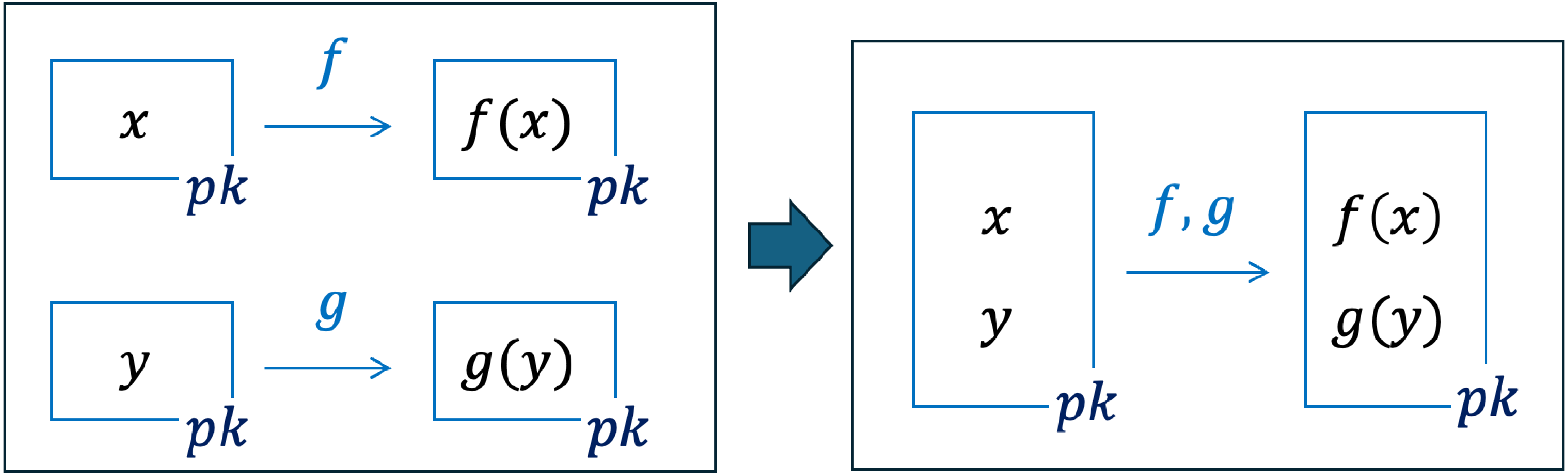
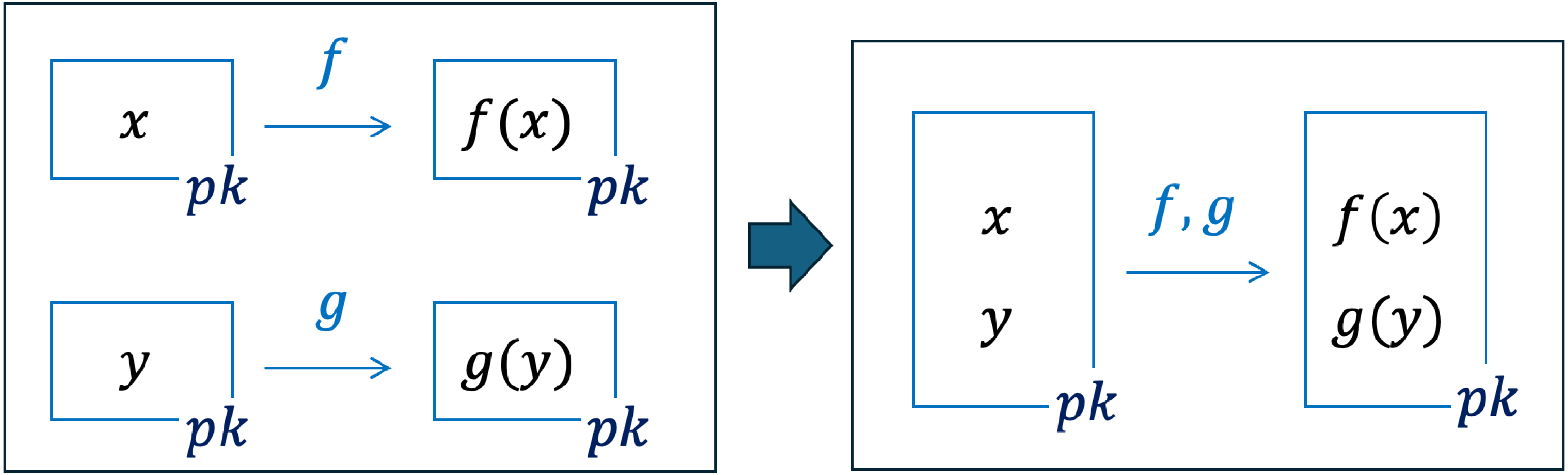


Avoid Composition via Multi-bit Evaluation



New Issue: we lost ‘**gate-by-gate**’ structure in HE evaluation——
Can’t talk about ‘intermediate ciphertext’ for a gate in f, g.

Homomorphic Eval Provides Intermediate CT

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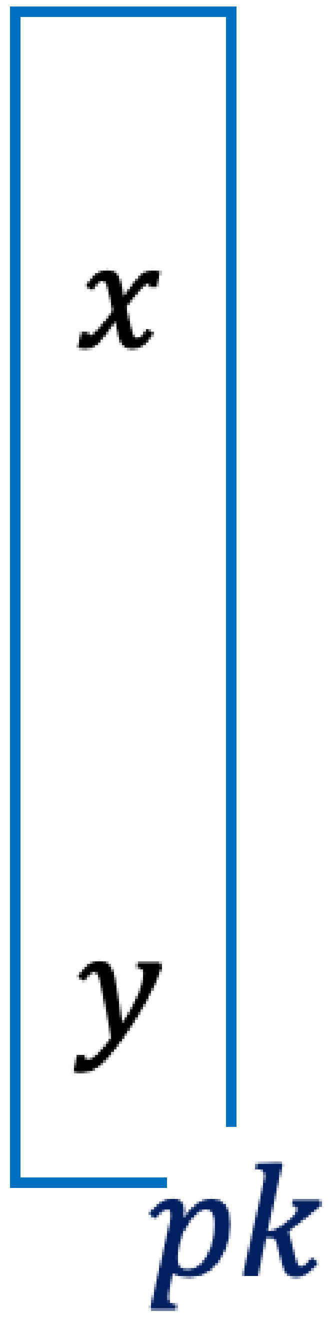
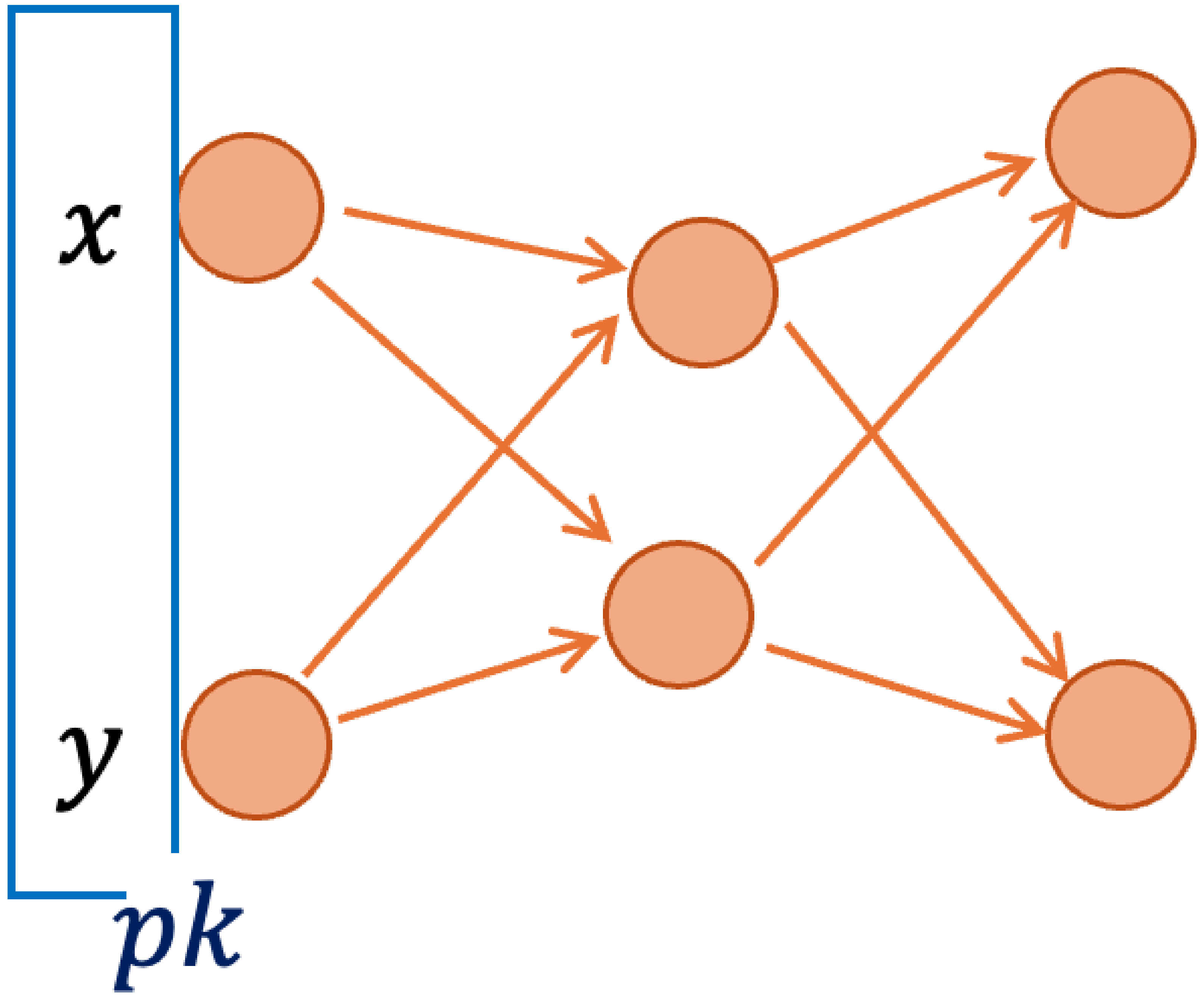


x

y

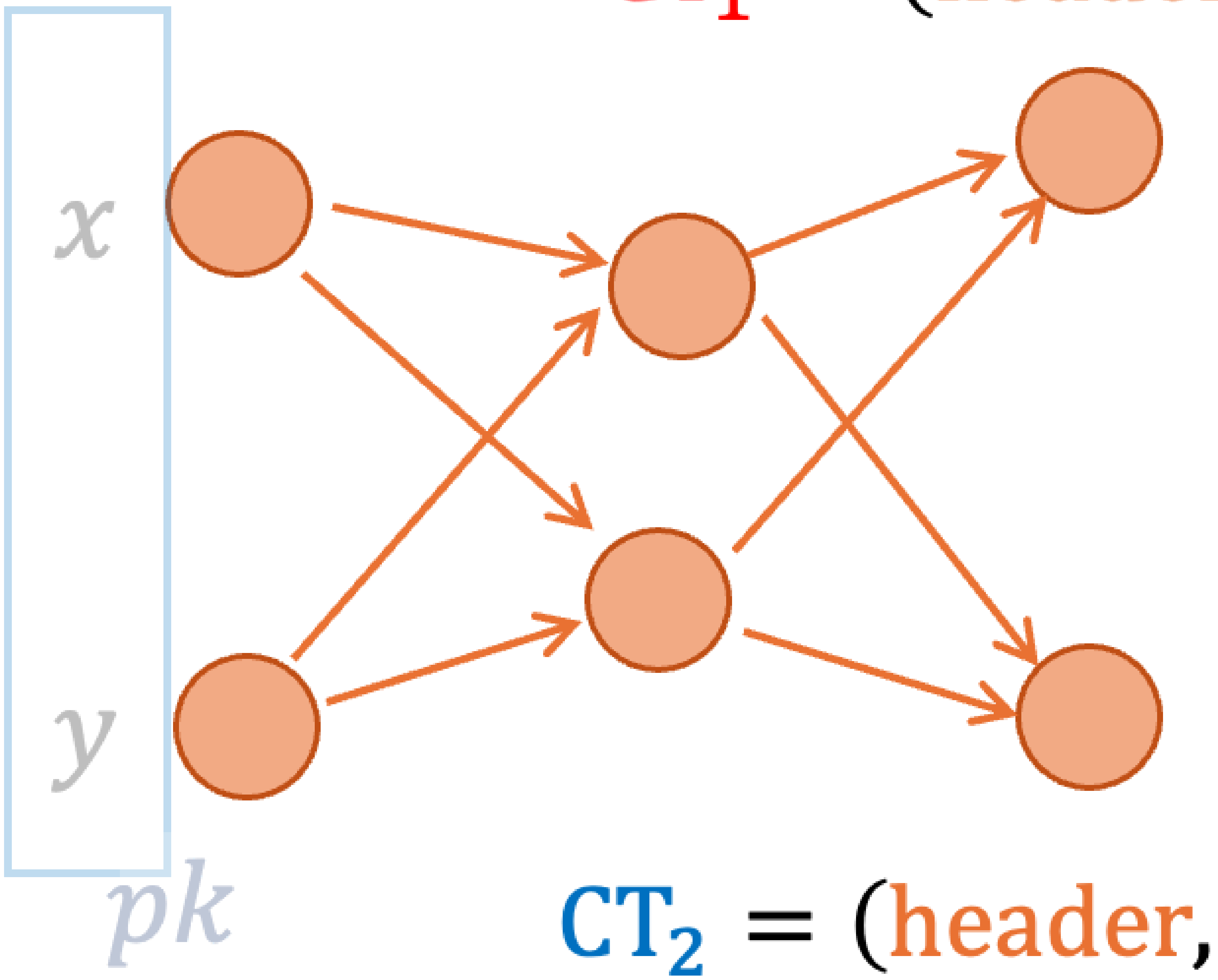
pk

Homomorphic Eval Provides Intermediate CT



Homomorphic Eval Provides Intermediate CT

$\mathrm{CT}_1 = (\mathrm{header}, \mathrm{payload}_1)$



Eval also outputs **intermediate CT** for each gate

$\mathrm{CT}_2 = (\mathrm{header}, \mathrm{payload}_2)$

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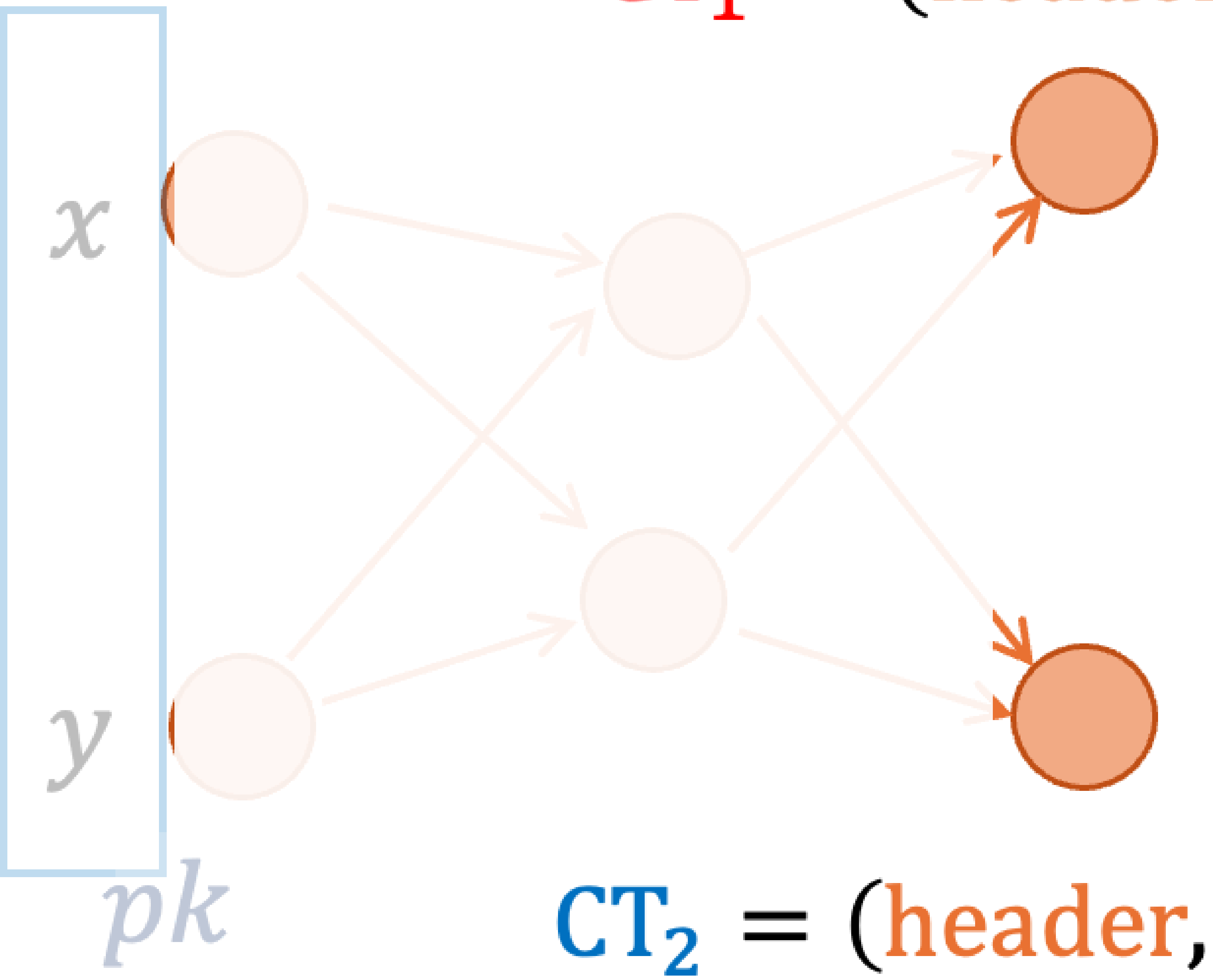

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Header-Payload Structure

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headers are the same for all gates

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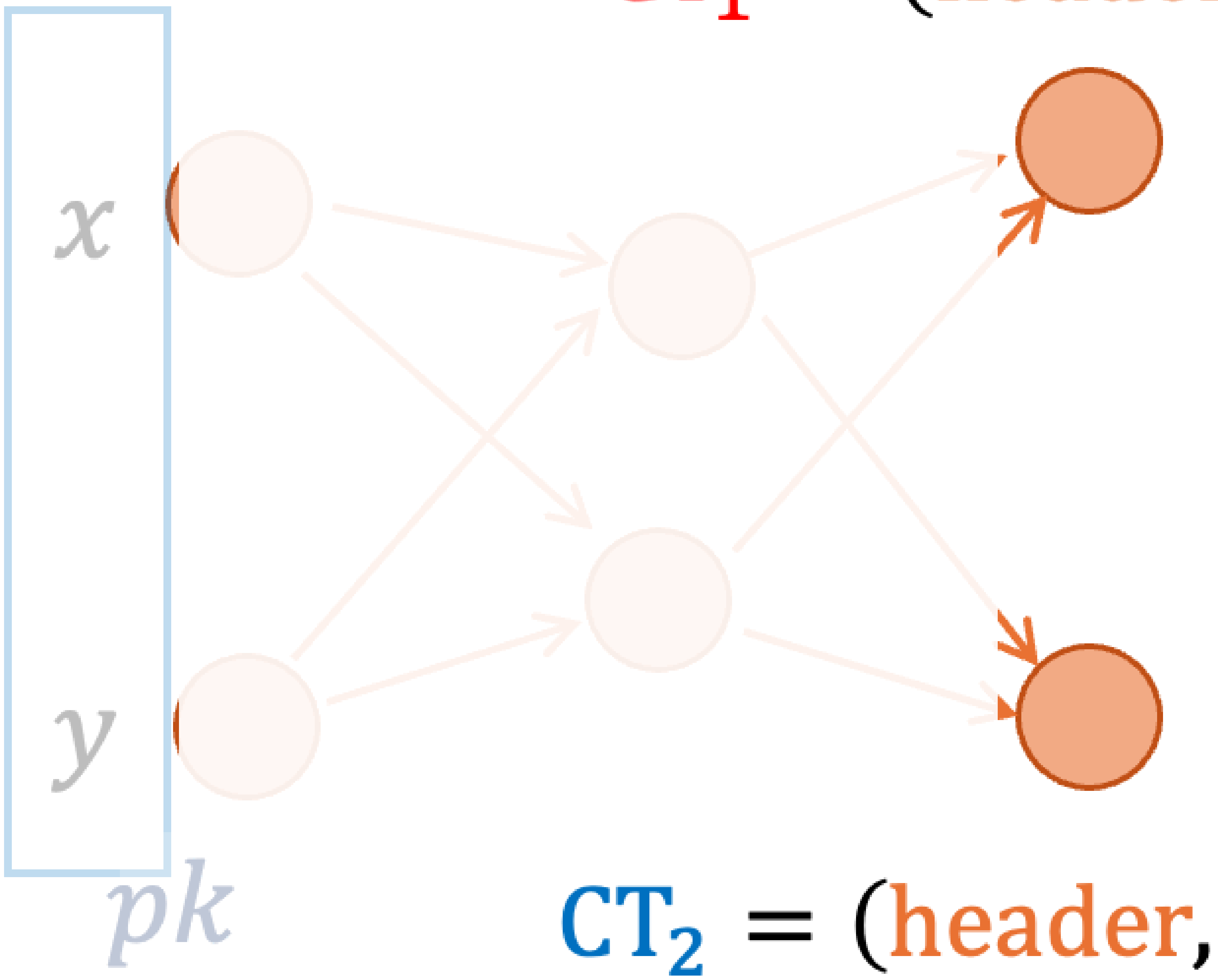

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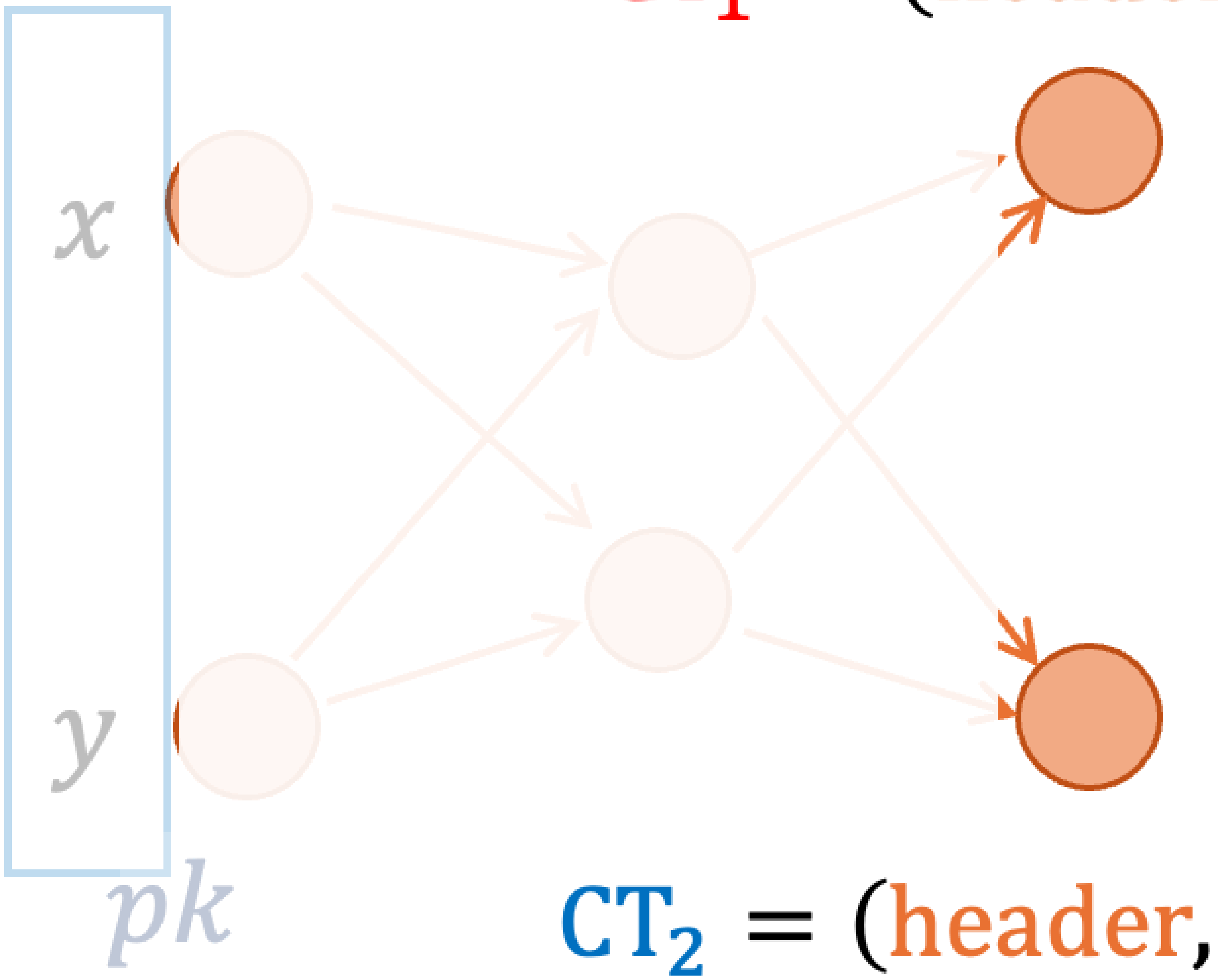

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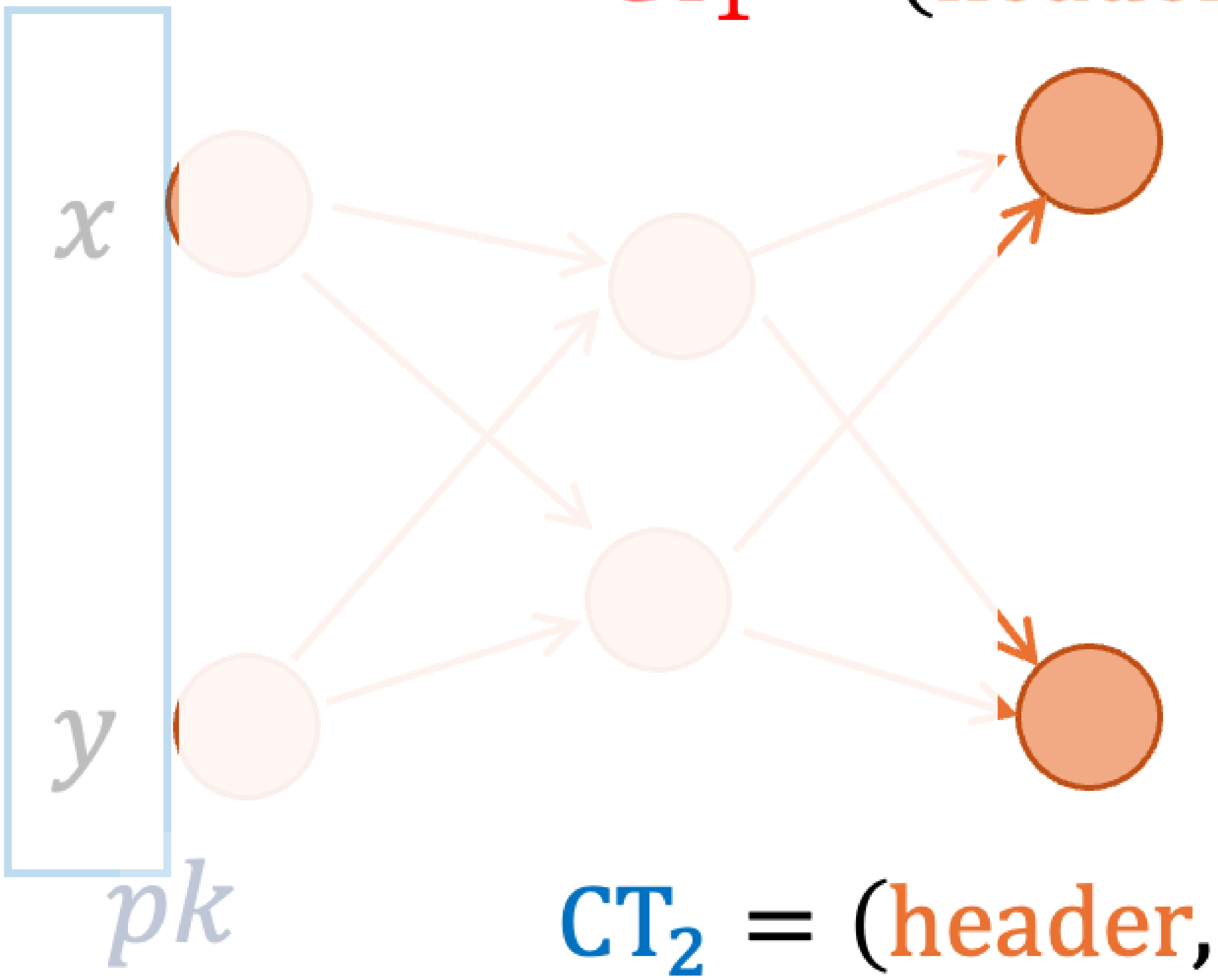

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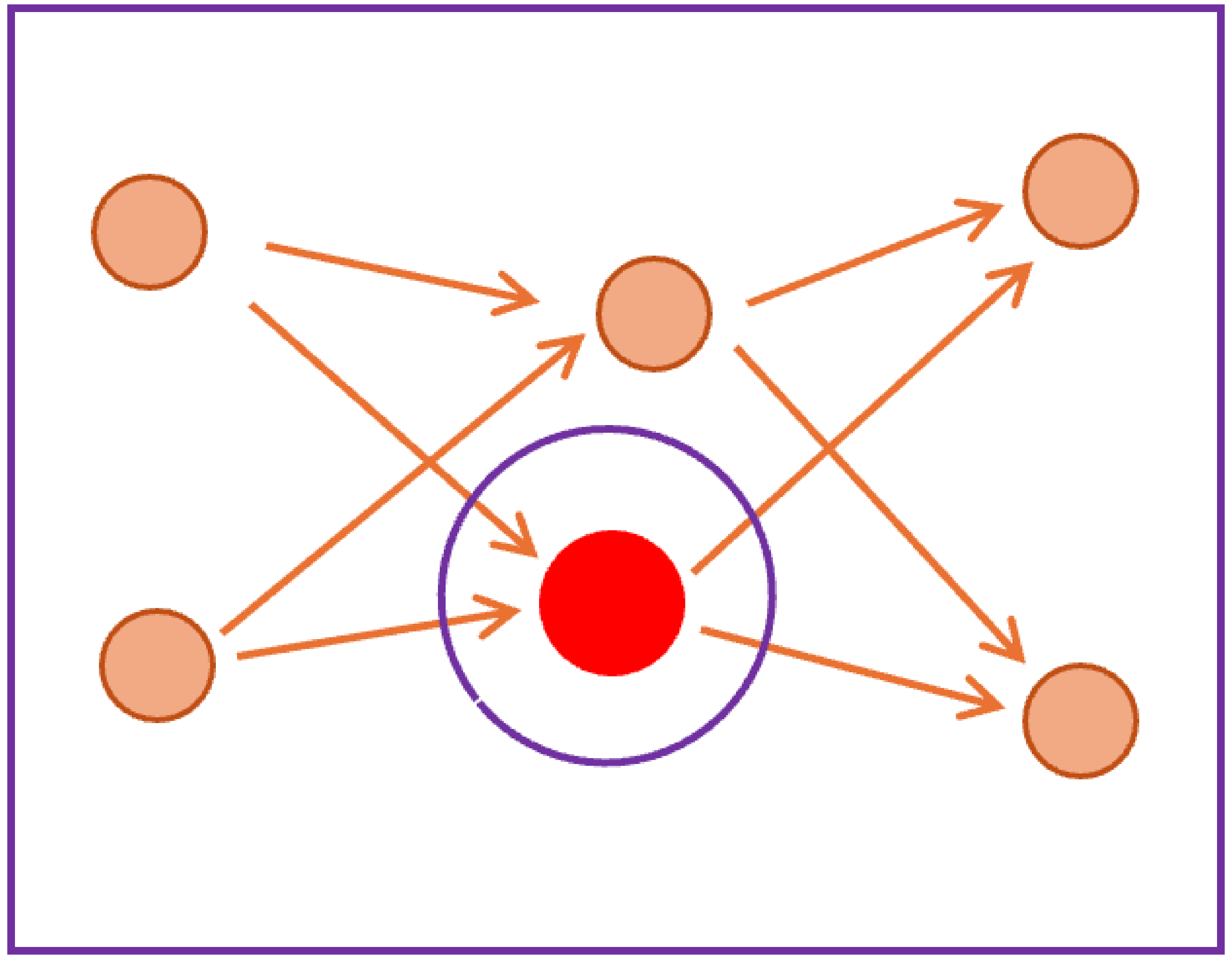
How to locally certify the correctness of intermediate ciphertexts?

SNARG for Local Correctness



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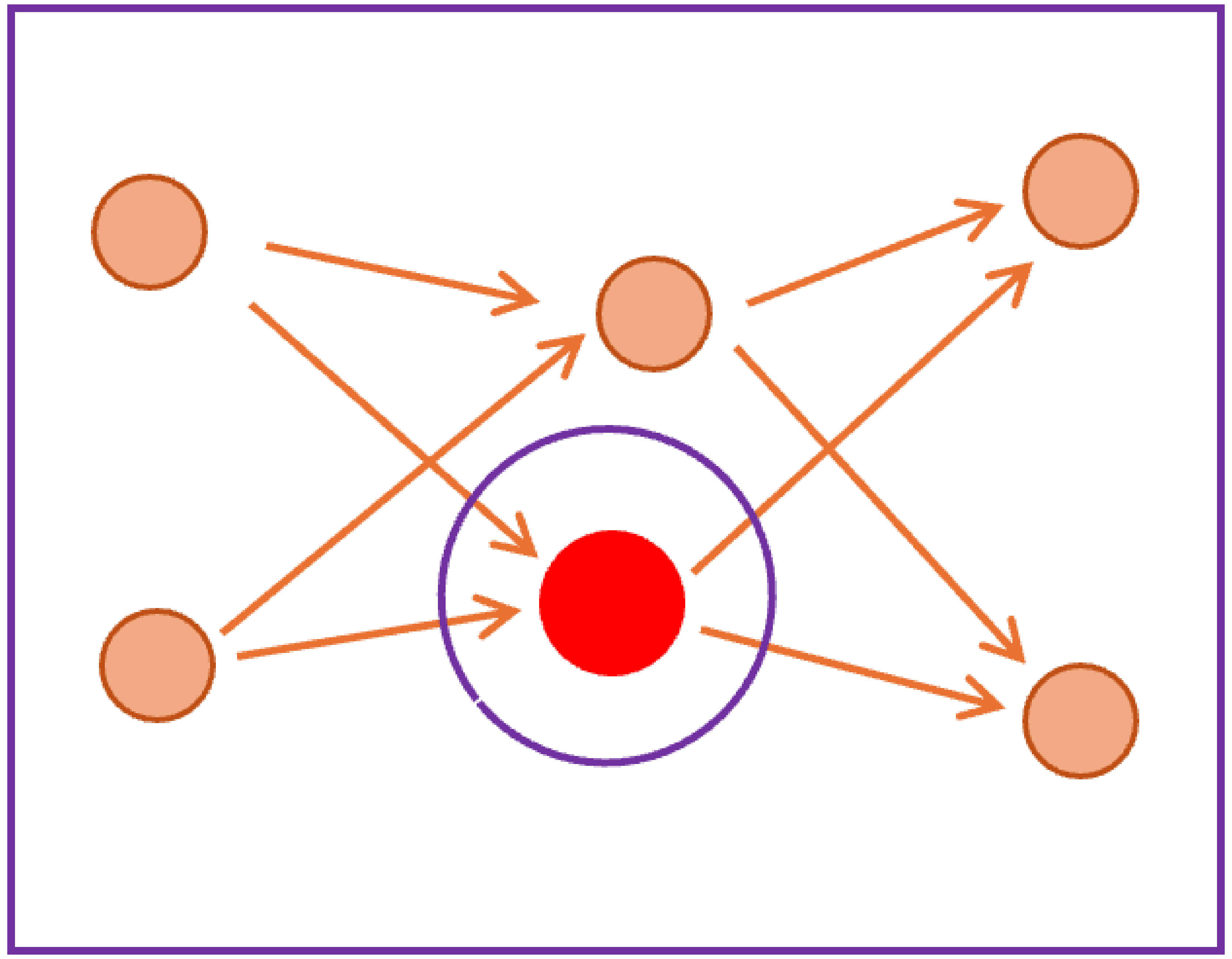


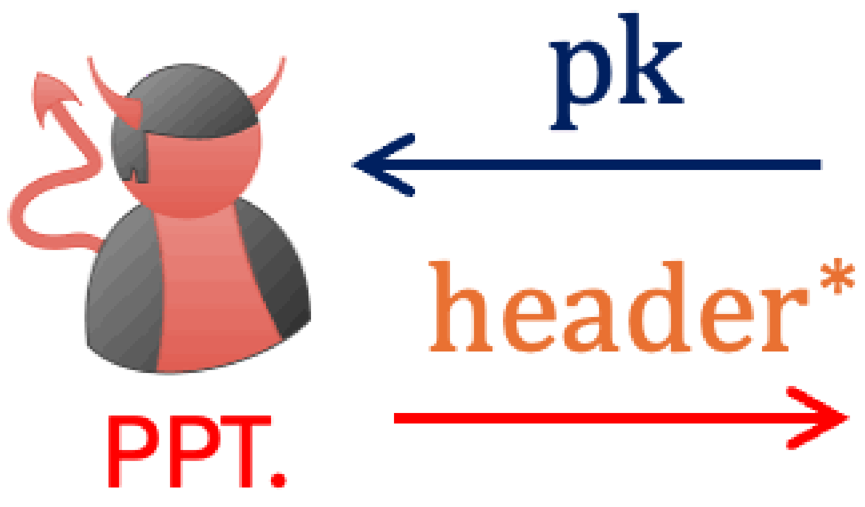
We generate a SNARG proof to certify the correctness for each intermediate ciphertext.

Summary of Definition for s-HE

$\mathrm{Gen}(1^{\lambda}) \to (pk, sk, \mathrm{Pred})$ Pred : privately computable

Homomorphic Evaluation

- Header-payload structure: CT = (header, payload)
- If $\mathrm{Pred}(\mathrm{header}) = 1$, then decryption is correct.
- Sometimes Decryptable for Malicious CT:



$$\Pr[\mathrm{Pred}(\mathrm{header}^{*}) = 1] > 2^{-\lambda^{c}}$$

- SNARGs for local correctness of intermediate CT

Rest of the Talk

Formal Definition of Sometimes-Decryptable HE

Construction of Sometimes-Decryptable HE

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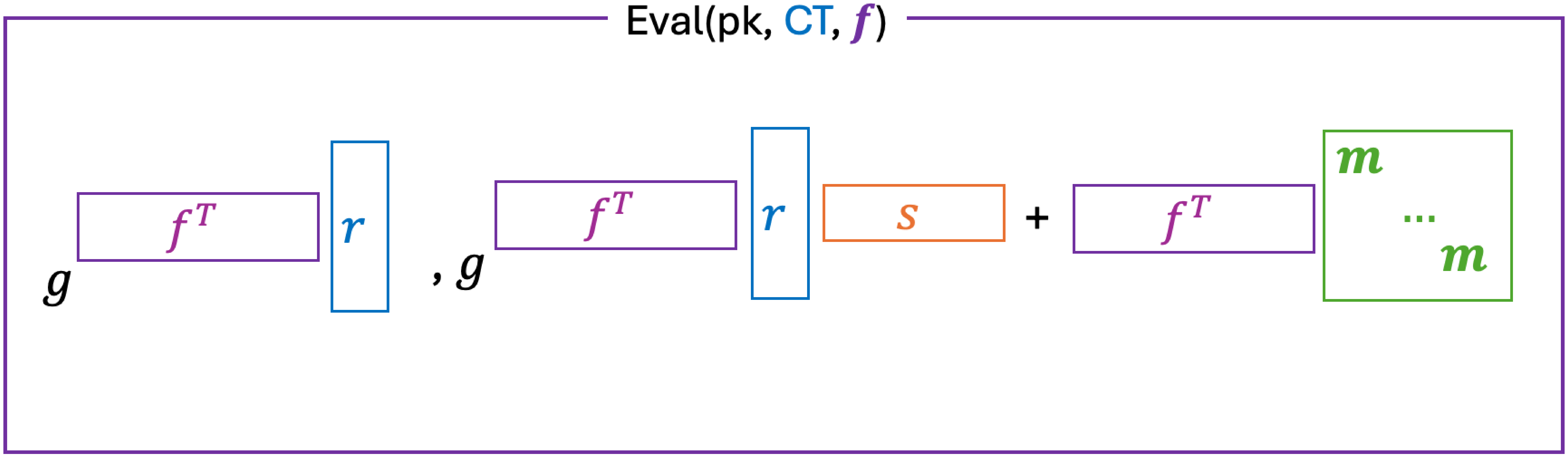
$$g^{\boxed{f_1^T, \ldots, f_\ell^T}\boxed{r}}, g^{\boxed{f_1^T, \ldots, f_\ell^T}\boxed{r}\boxed{\quad s \quad} + \boxed{f_1^T, \ldots, f_\ell^T}\boxed{\begin{matrix} \boldsymbol{m} & & \\ & \ldots & \\ & & \boldsymbol{m} \end{matrix}}}$$

HE for Linear Functions: Decryption



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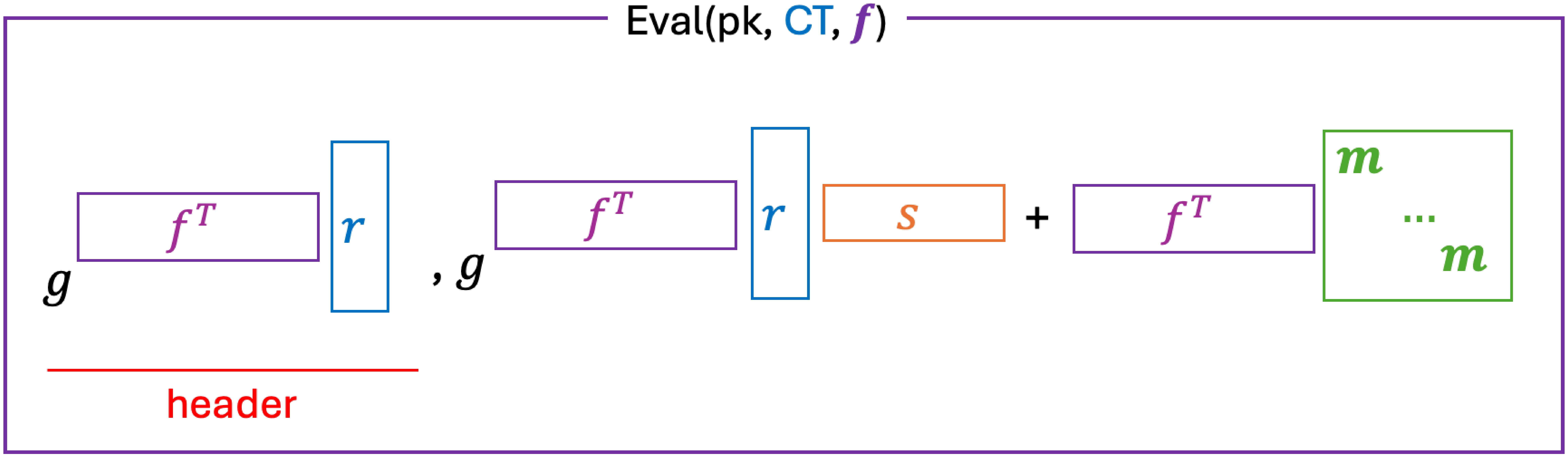
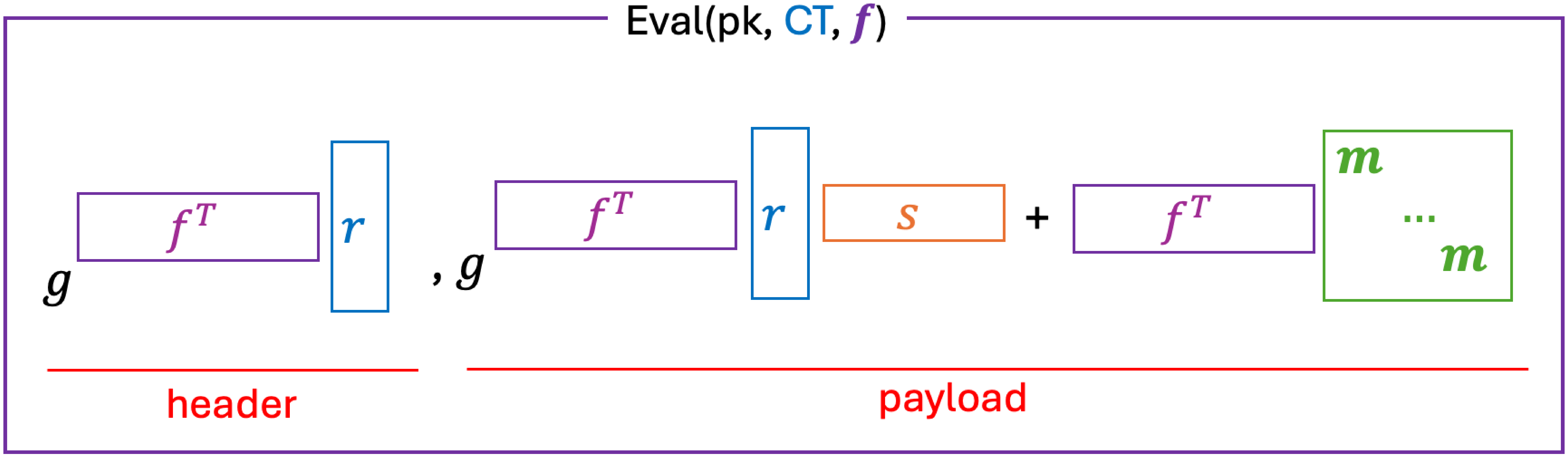


Eval(pk, CT, f)

header

HE for Linear Functions: Decryption



HE for Linear Functions: Decryption

Eval(pk, CT, f)

$$\underbrace{g^{\boxed{f^T}\boxed{r}}}_{\text{header}},\ \underbrace{g^{\boxed{f^T}\boxed{r}\boxed{s} + \boxed{f^T}\boxed{\begin{matrix} m & & \\ & \cdots & \\ & & m \end{matrix}}}}_{\text{payload}}$$

- Decryption: divide payload by header$^{\boxed{s}}$, get $g^{\boxed{f_1 \cdot m, \ldots, f_\ell \cdot m}}$

SNARGs for Local Correctness

$$\mathsf{CT} = \left(g^{\boxed{r}},\ g^{\boxed{r}\,\boxed{\ s\ } + \boxed{\begin{matrix} m & & \\ & \dots & \\ & & m \end{matrix}}}\right)$$

Eval(pk, CT, f)

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$$\boxed{f_1^T, \dots, f_\ell^T} = \boxed{\boldsymbol{f_1^T \dots 0 \dots f_\ell^T}} + \boxed{\boldsymbol{0 \dots f_i^T \dots 0}}$$

SNARGs for Local Correctness

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i-th Output of Eval$(f,\cdot)$

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SNARGs for Local Correctness

Prove this Part via SNARGs for Linear relations from sub-exp DDH
[Choudhuri-Garg-Jain-J-Zhang'23]

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The verifier can compute by itself in time poly(input arity of i-th output)

Bootstrap from Linear Functions to TC^0 (high-level)

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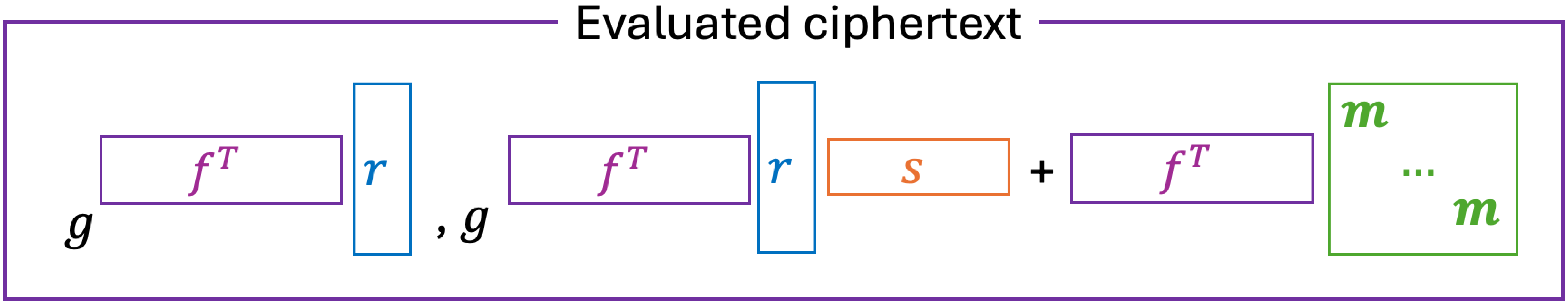
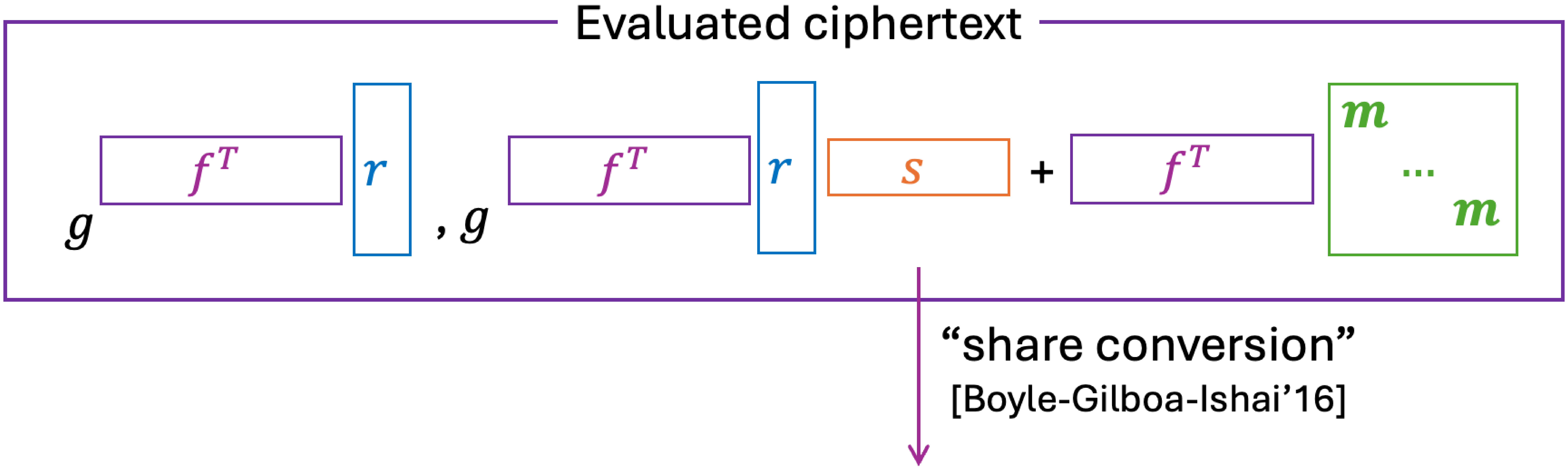


Evaluated ciphertext

$$g^{\boxed{f^T}\boxed{r}},\ g^{\boxed{f^T}\boxed{r}\boxed{s} + \boxed{f^T}\boxed{\begin{matrix} m & & \\ & \cdots & \\ & & m \end{matrix}}}$$

Bootstrap from Linear Functions to TC^0 (high-level)

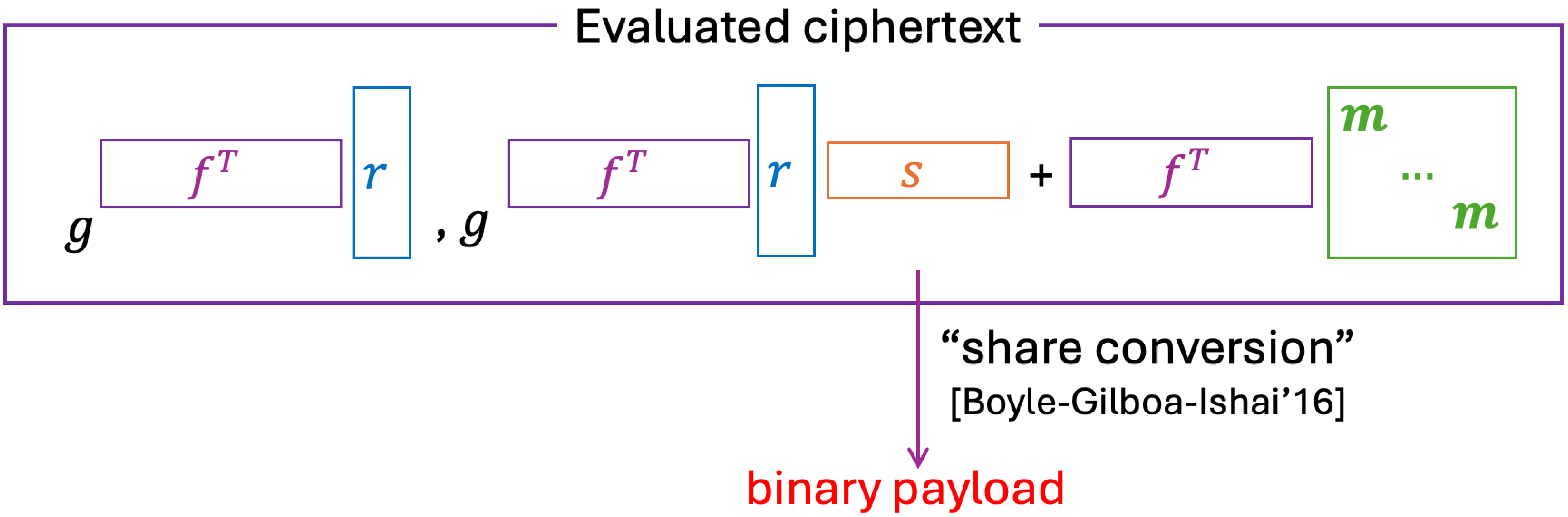


Evaluated ciphertext

“share conversion”

[Boyle-Gilboa-Ishai’16]

Bootstrap from Linear Functions to TC^0 (high-level)



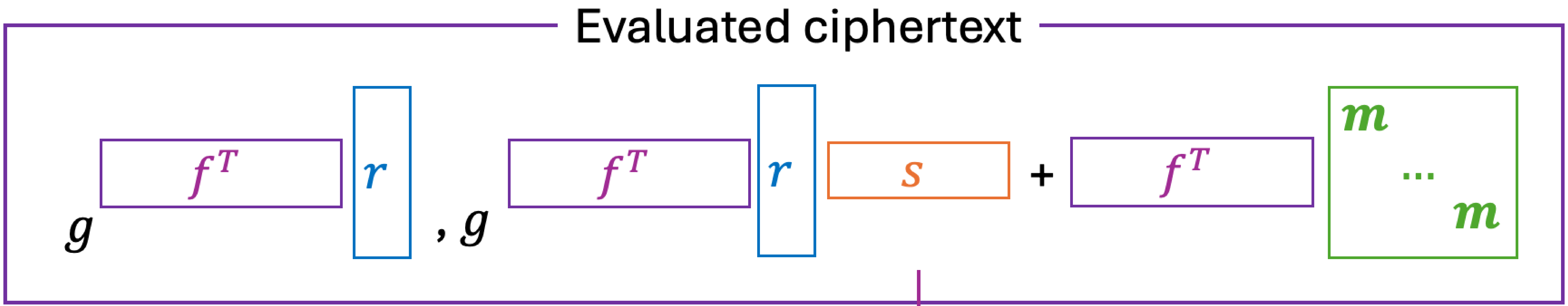
Evaluated ciphertext

“share conversion”

[Boyle-Gilboa-Ishai’16]

binary payload

Bootstrap from Linear Functions to TC^0 (high-level)



Evaluated ciphertext

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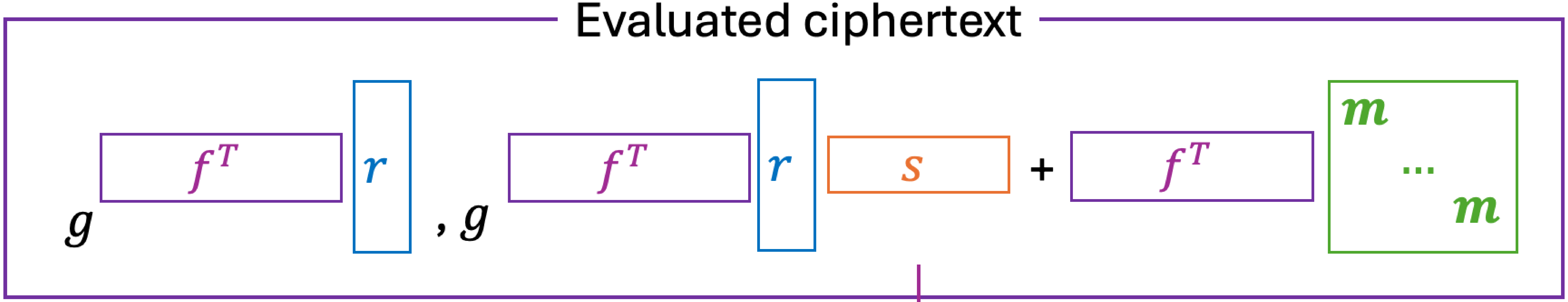
[Boyle-Gilboa-Ishai'16]

Additive decryption: Decryption = binary payload ⊕ BGI(header^s)

Bootstrap from Linear Functions to TC^0 (high-level)

Evaluated ciphertext



“share conversion”

[Boyle-Gilboa-Ishai’16]

Additive decryption: Decryption = binary payload ⊕ BGI(header^s)

→ s-HE for a layer of Threshold Gates → s-HE for full TC^0

[Jain-J’21] techniques

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Take away

Can replace FHE in “proof-system applications” (e.g. NIZK/SNARG) to achieve constructions from DDH in pairing-free groups!

Thank you!

Q & A