Stationary Syndrome Decoding for Improved PCGs

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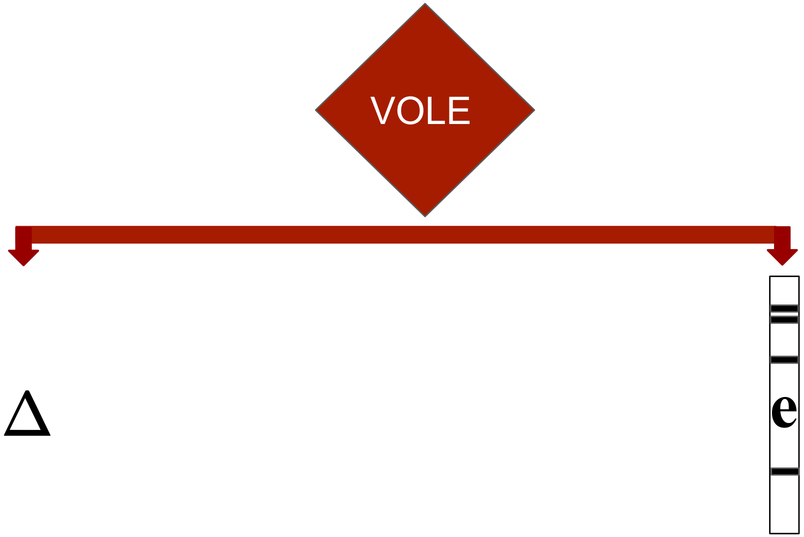
Joint work with:
Vlad Kolesnikov, Srini Raghuraman, Peter Rindal

Georgia Institute of Technology

VISA

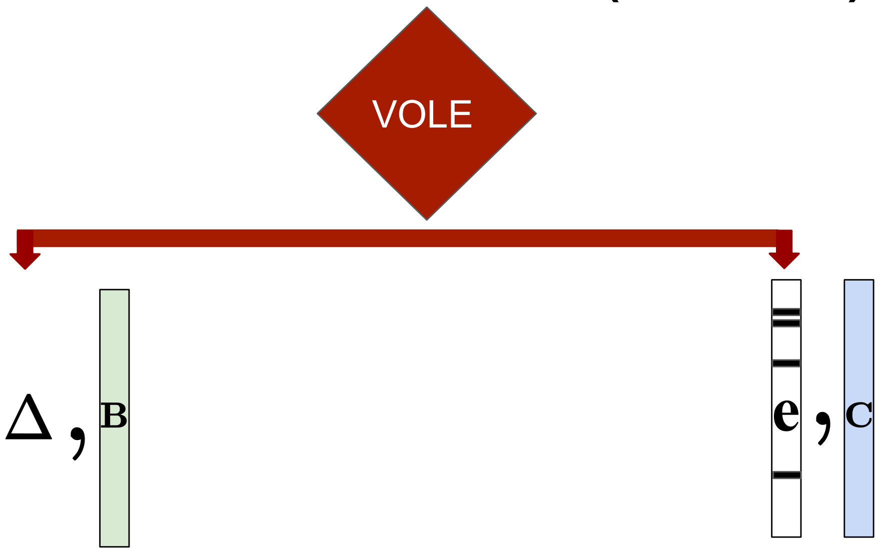
Vector Oblivious Linear Evaluation (VOLE)

$[\Delta\mathbf{e}]$



Vector Oblivious Linear Evaluation (VOLE)

$$[\Delta \mathbf{e}] = \mathbf{B} + \mathbf{C}$$



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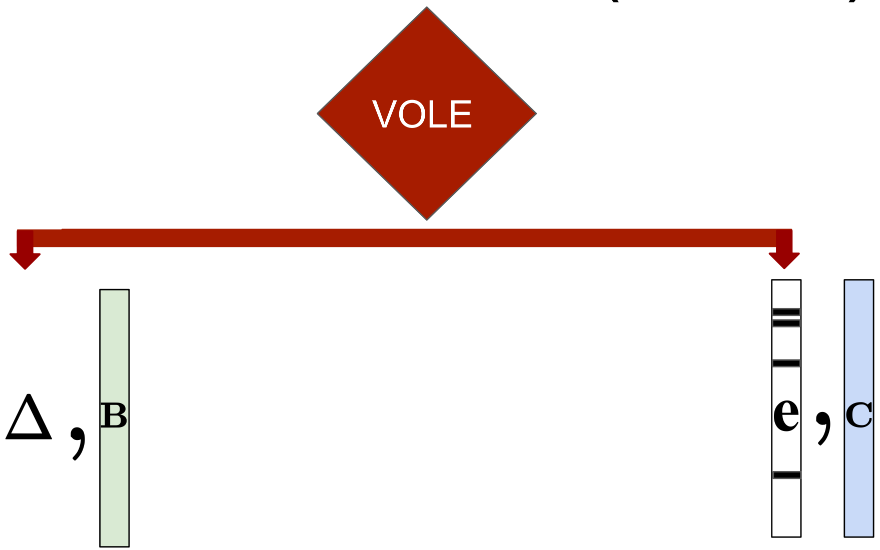
Interactive (sublinear comm.)



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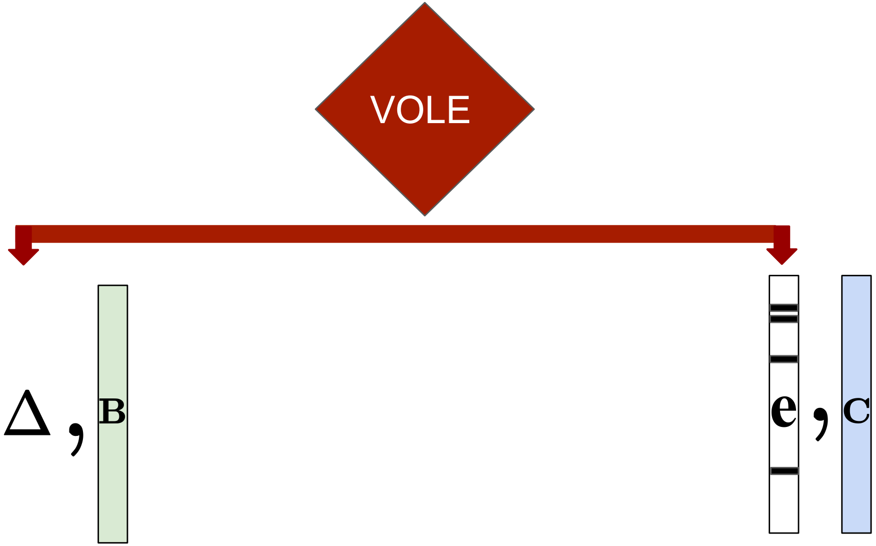
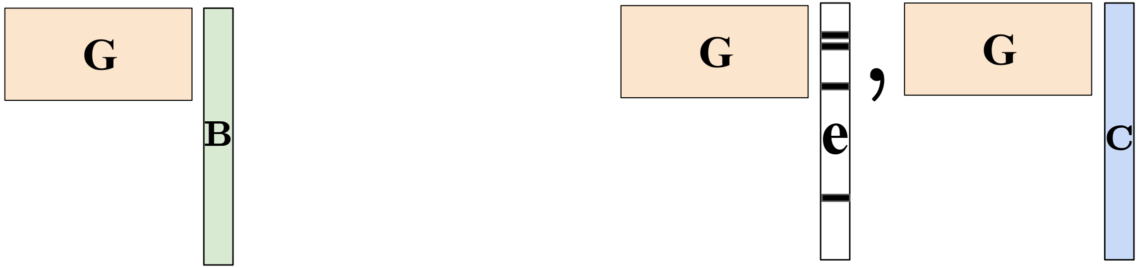


$$[\Delta \mathbf{e}] = \mathbf{B} + \mathbf{C}$$

Interactive (sublinear comm.)

$$[\mathbf{G}\Delta \mathbf{e}]$$



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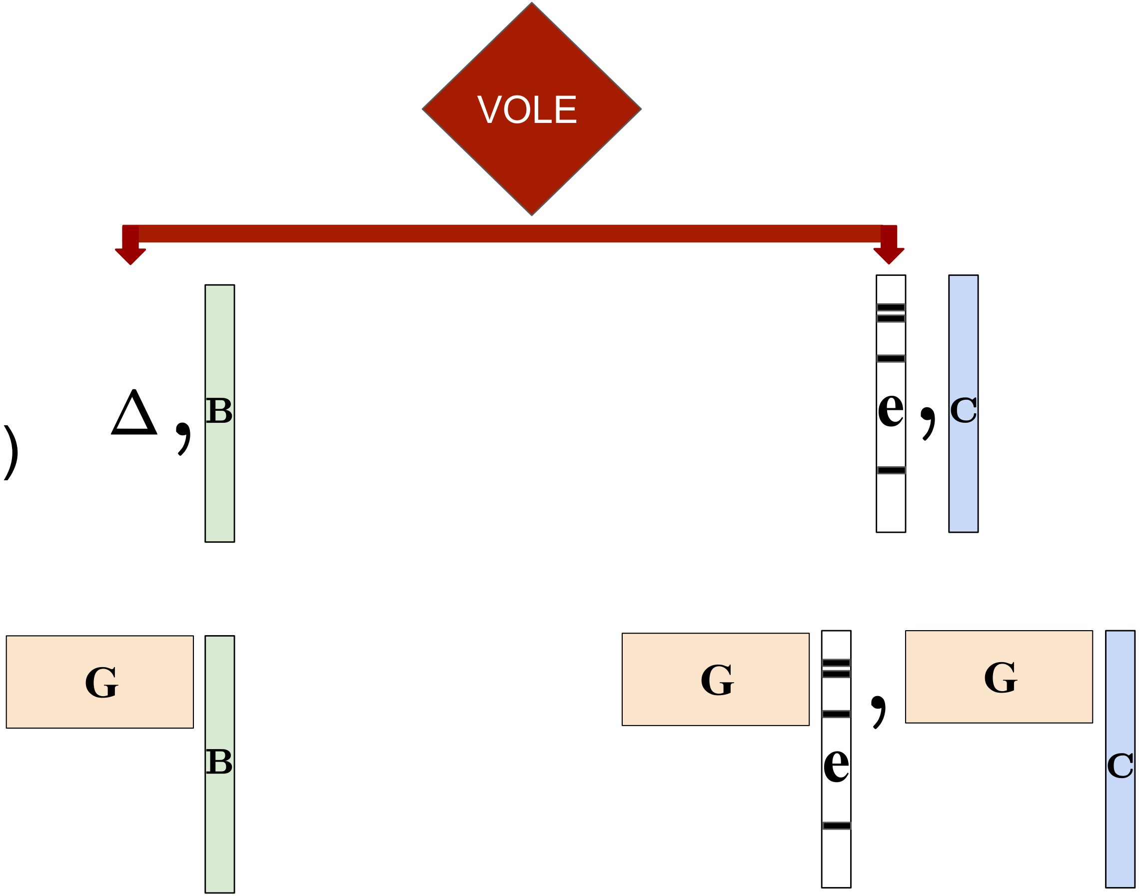

$$[\mathbf{G}\boldsymbol{\Delta}\mathbf{e}]$$

Local

Vector Oblivious Linear Evaluation (VOLE)

Similar for OT

$$[\Delta \mathbf{e}] = \mathbf{B} + \mathbf{C}$$

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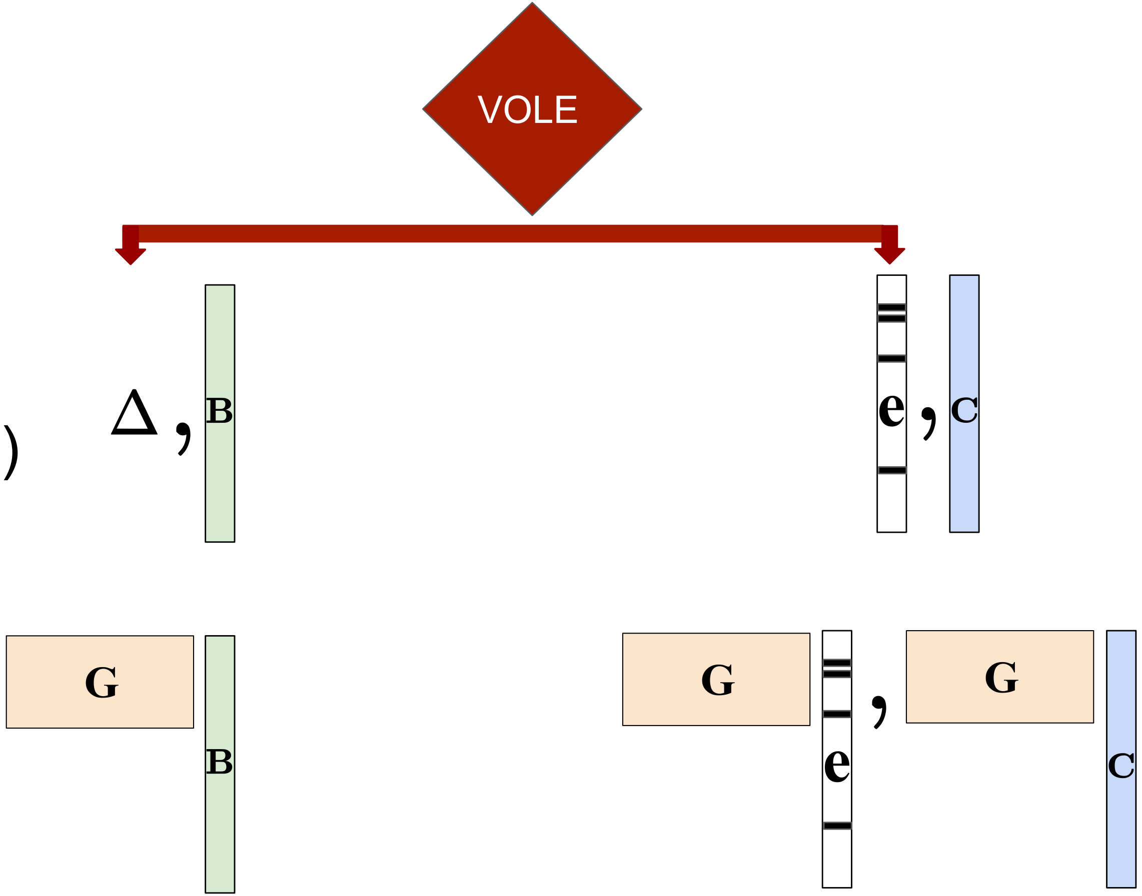
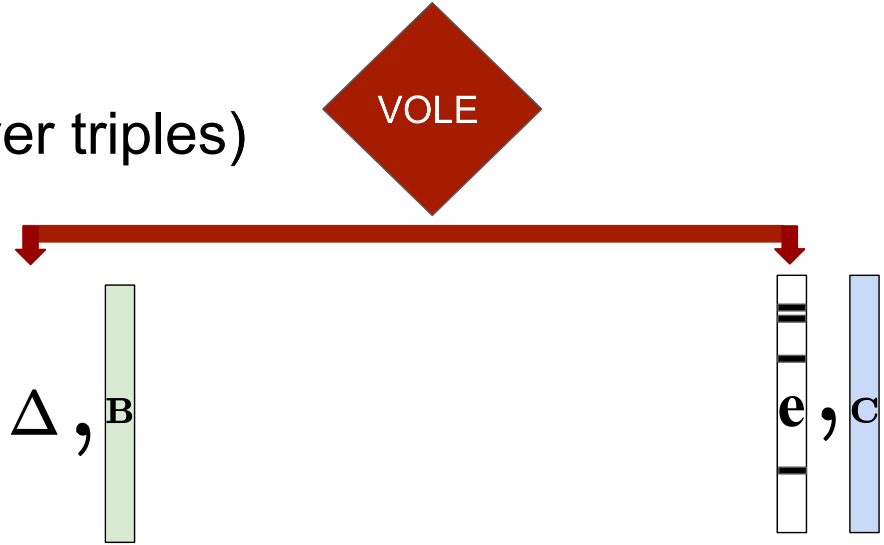


Vector Oblivious Linear Evaluation (VOLE)

Similar for OT
Other correlations (Beaver triples)



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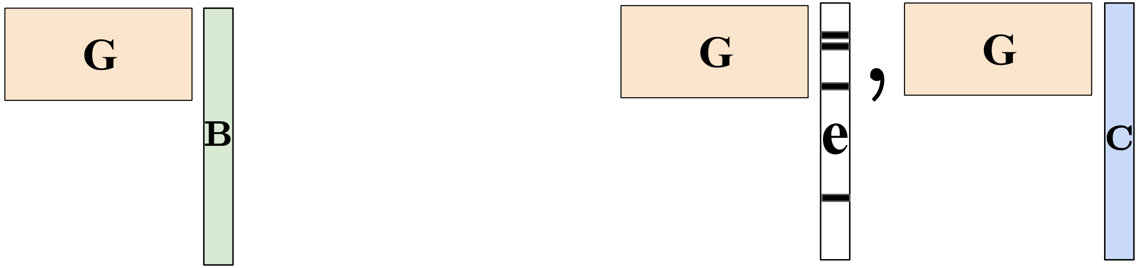

Pseudorandom Correlation Generators (PCGs)

Sublinear communication, compelling computation

State of the art for generating correlated randomness

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Sublinear communication, compelling computation

State of the art for generating correlated randomness

Correlated randomness is essential for MPC

Vector Oblivious Linear Evaluation (VOLE)

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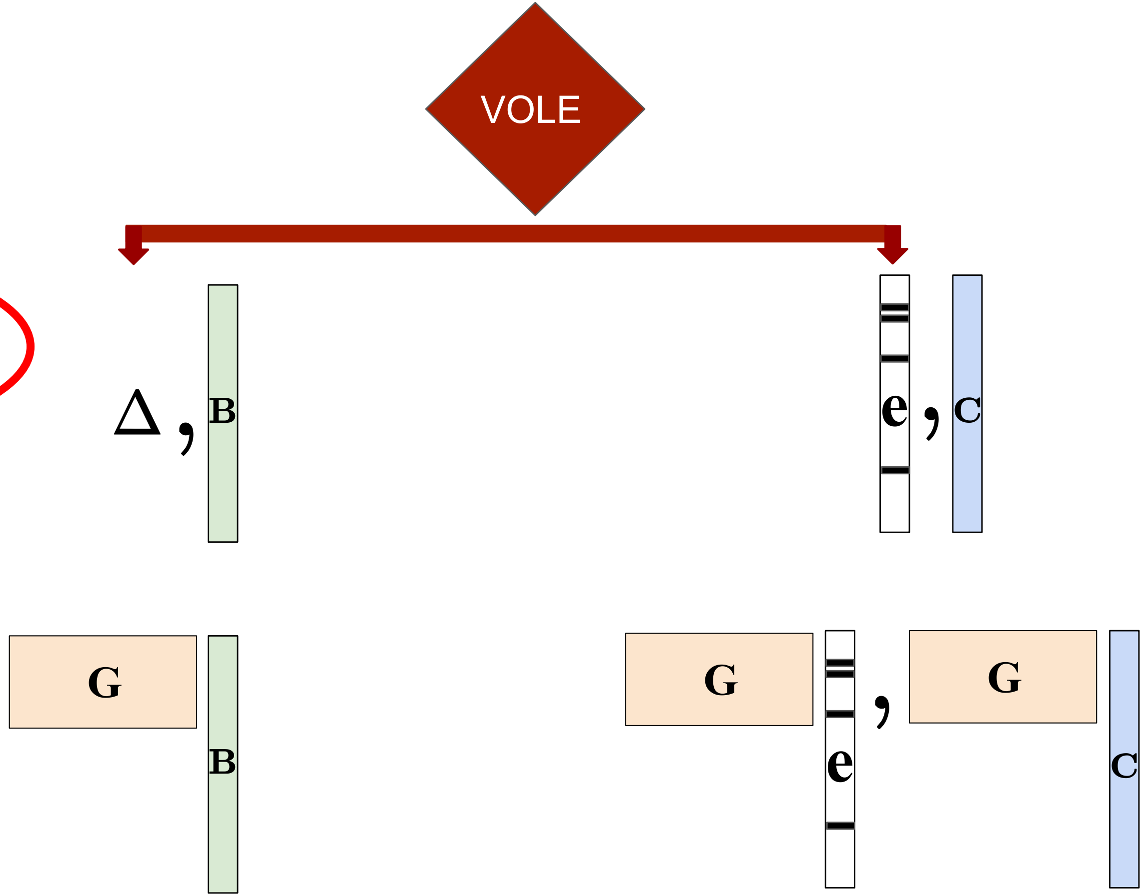
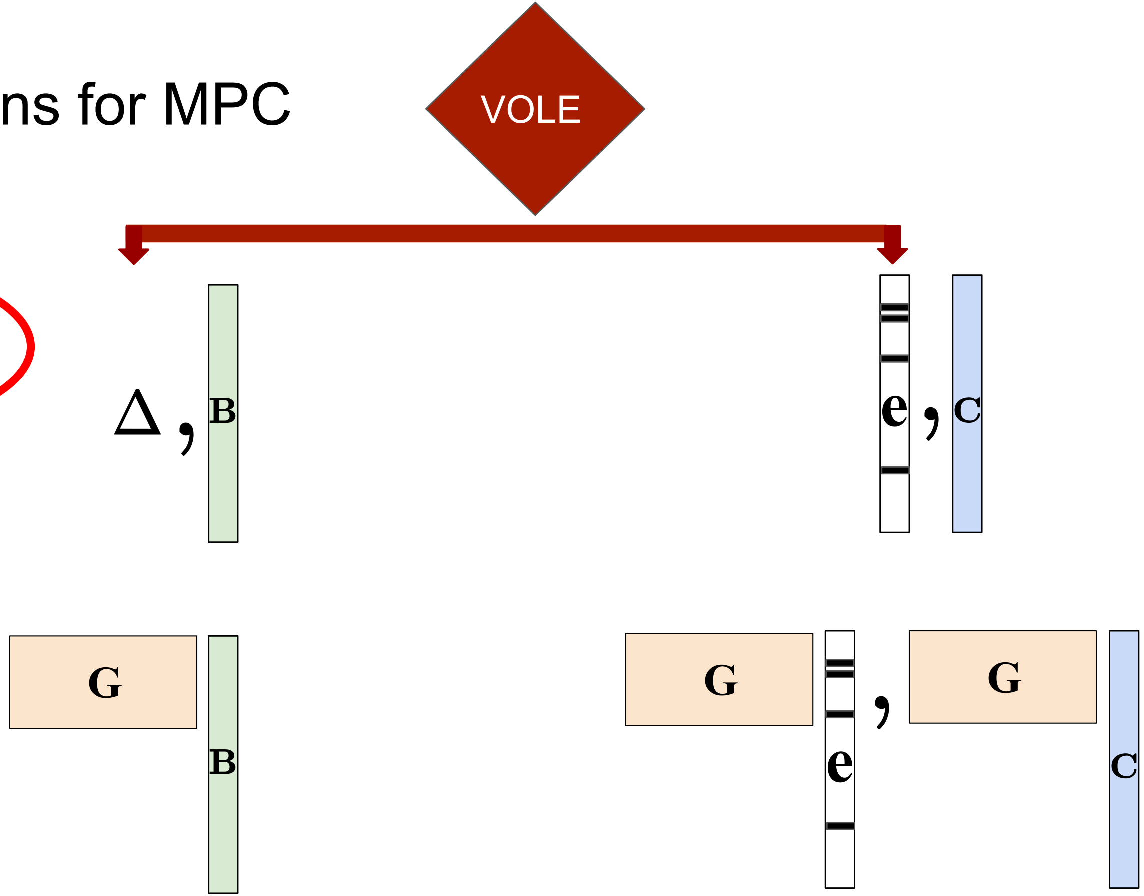


Vector Oblivious Linear Evaluation (VOLE)

- Need many correlations for MPC

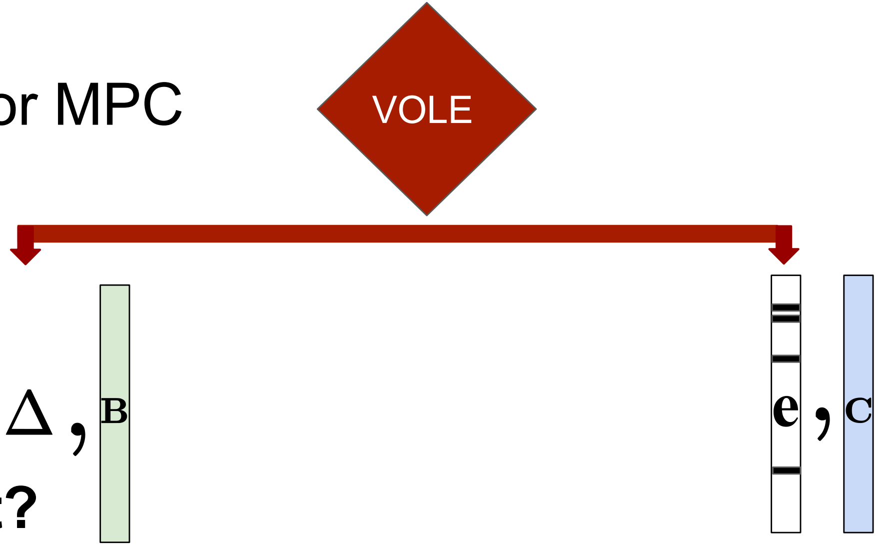
$$[\Delta \mathbf{e}] = \mathbf{B} + \mathbf{C}$$

$$[\mathbf{G}\Delta \mathbf{e}]$$



Vector Oblivious Linear Evaluation (VOLE)

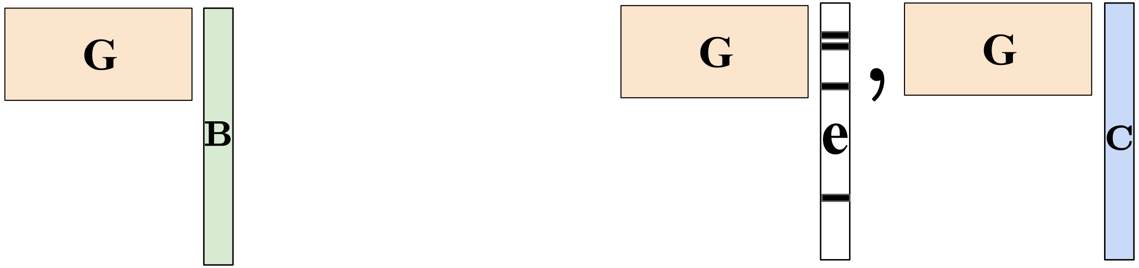
➡ Need many correlations for MPC



$$[\Delta \mathbf{e}] = \mathbf{B} + \mathbf{C}$$

Can we amortize this cost?

$$[\mathbf{G}\Delta \mathbf{e}]$$



LPN

Syndrome Decoding (SD)

LPN

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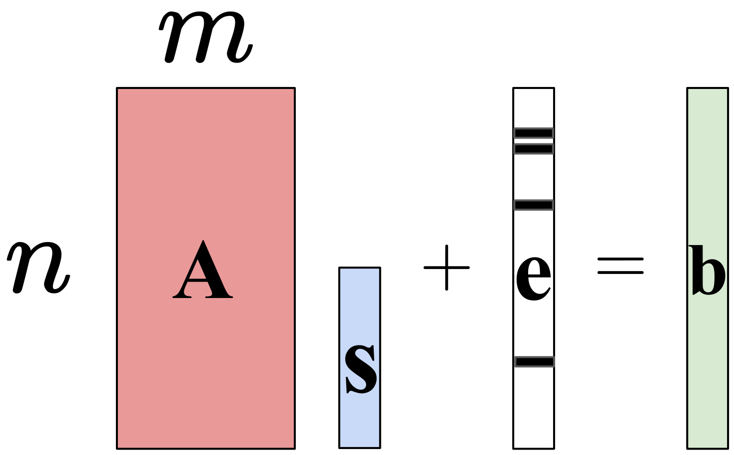
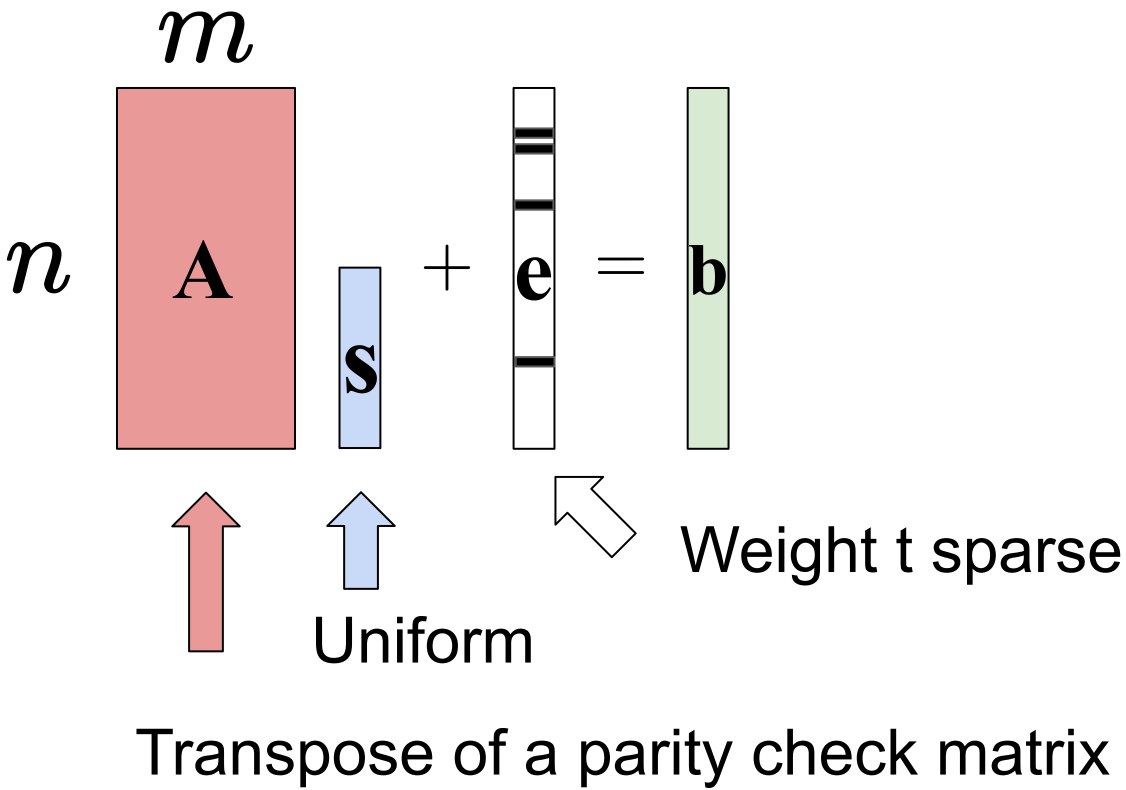


LPN | Syndrome Decoding (SD)



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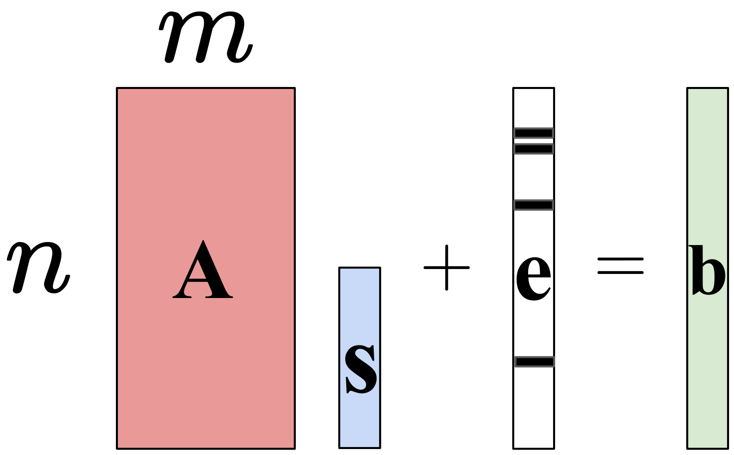
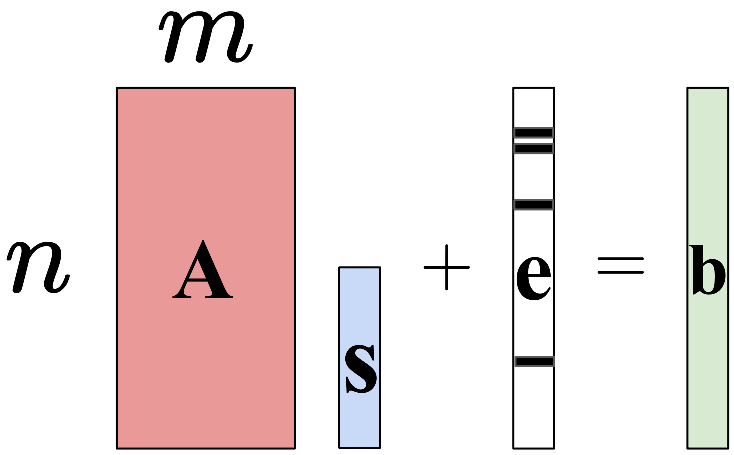


$$(\mathbf{A}, \mathbf{b}) \approx (\mathbf{A}, \$)$$

LPN



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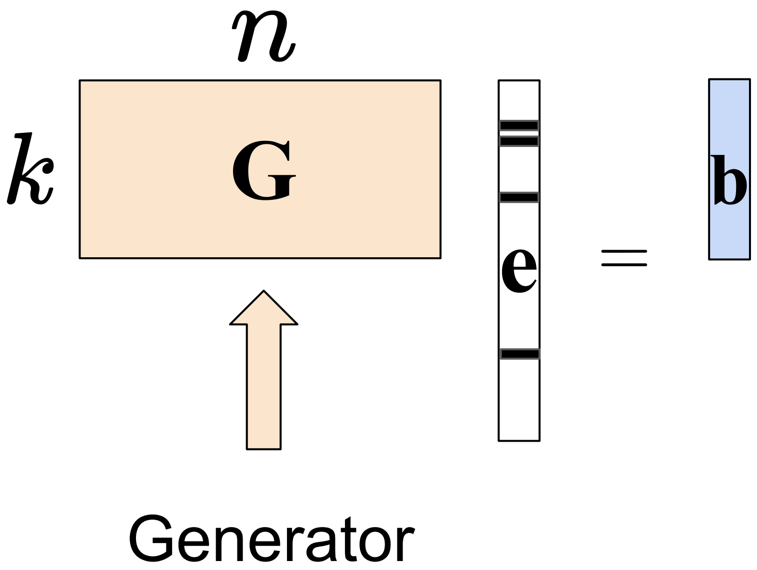

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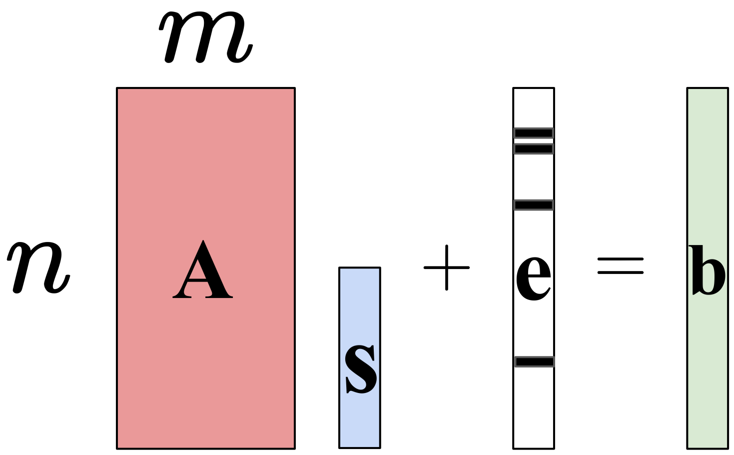

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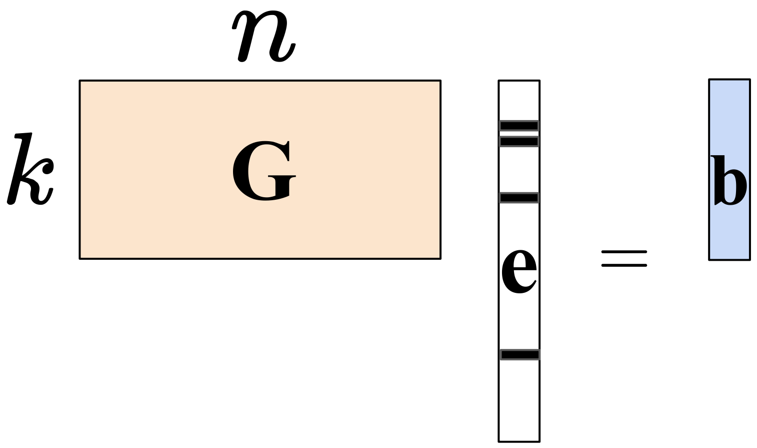
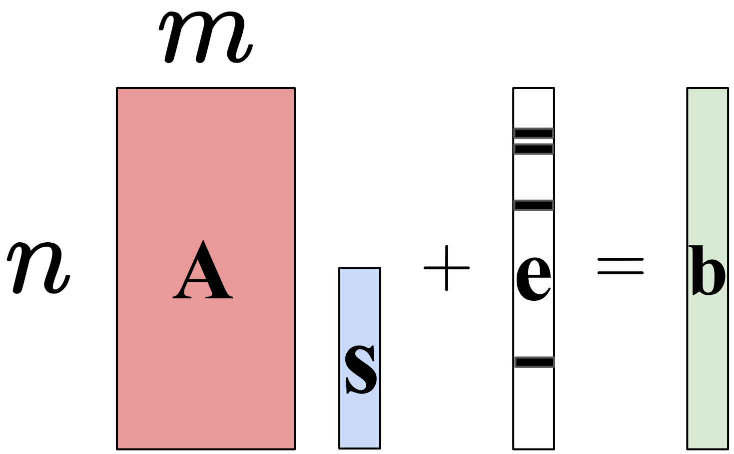


$$(\mathbf{G}, \mathbf{b}) \approx (\mathbf{G}, \$)$$

LPN



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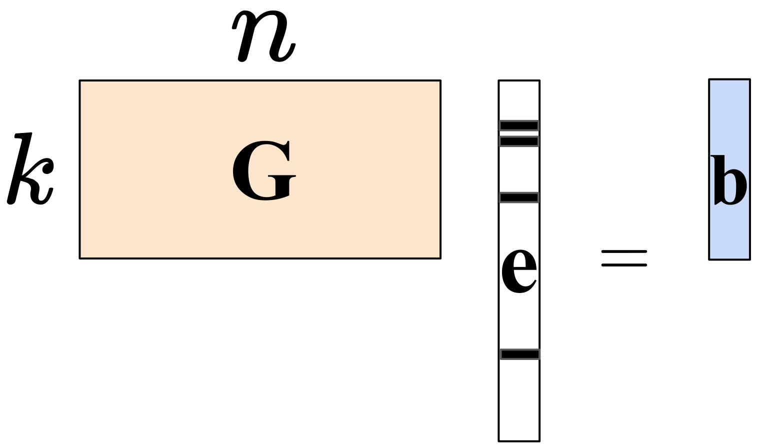

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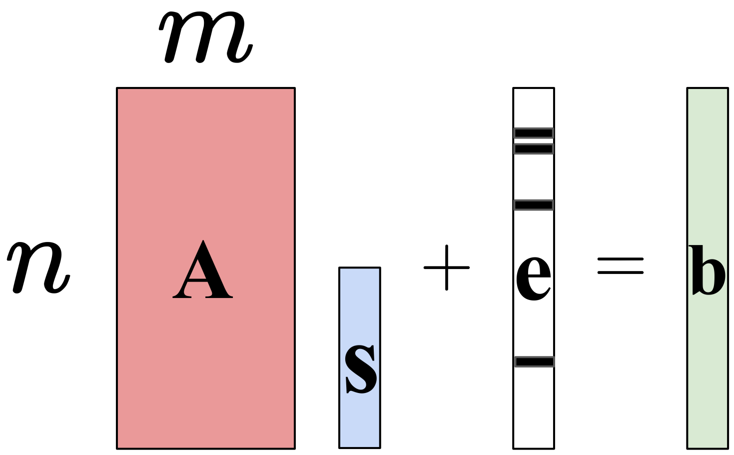
LPN and SD are equivalent

LPN



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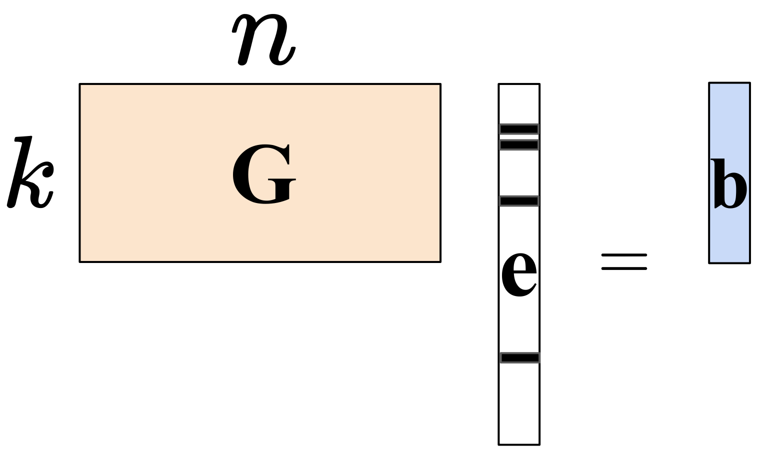


$$(\mathbf{G}, \mathbf{b}) \approx (\mathbf{G}, \$)$$

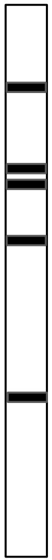
Used for PCGs

Syndrome Decoding (SD)

Known to be false for some choices of **G** and **e**

Noise Distributions

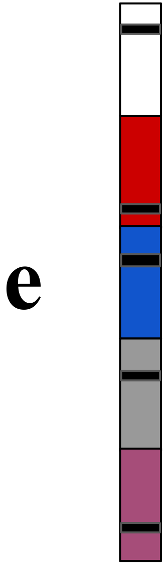
e



Bernoulli - classic, sample $\mathbf{e}_i$ with $\text{Ber}_{t/n}$

Exact - fixes Hamming weight to t

Noise Distributions



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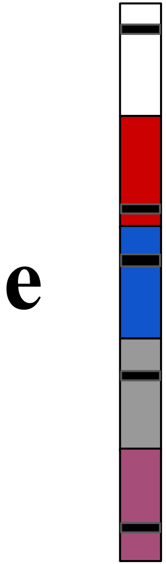

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All of these improve for 1 instance

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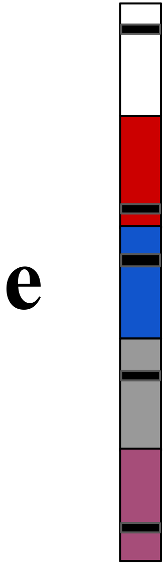


Bernoulli - classic, sample $\mathbf{e}_i$ with $\mathrm{Ber}_{t/n}$

Exact - fixes Hamming weight to t

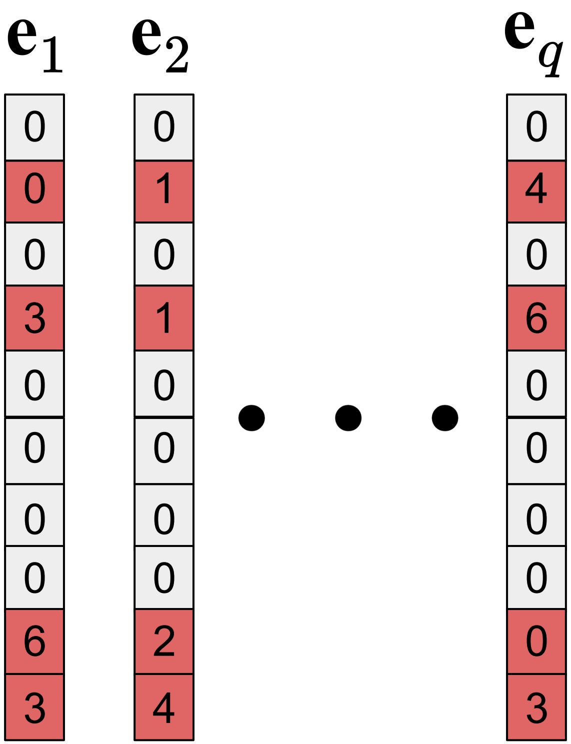
Regular - t same-size blocks, each a random unit vector

All of these improve for 1 instance

We amortize the cost of $[\Delta \mathbf{e}]$ across q SD instances

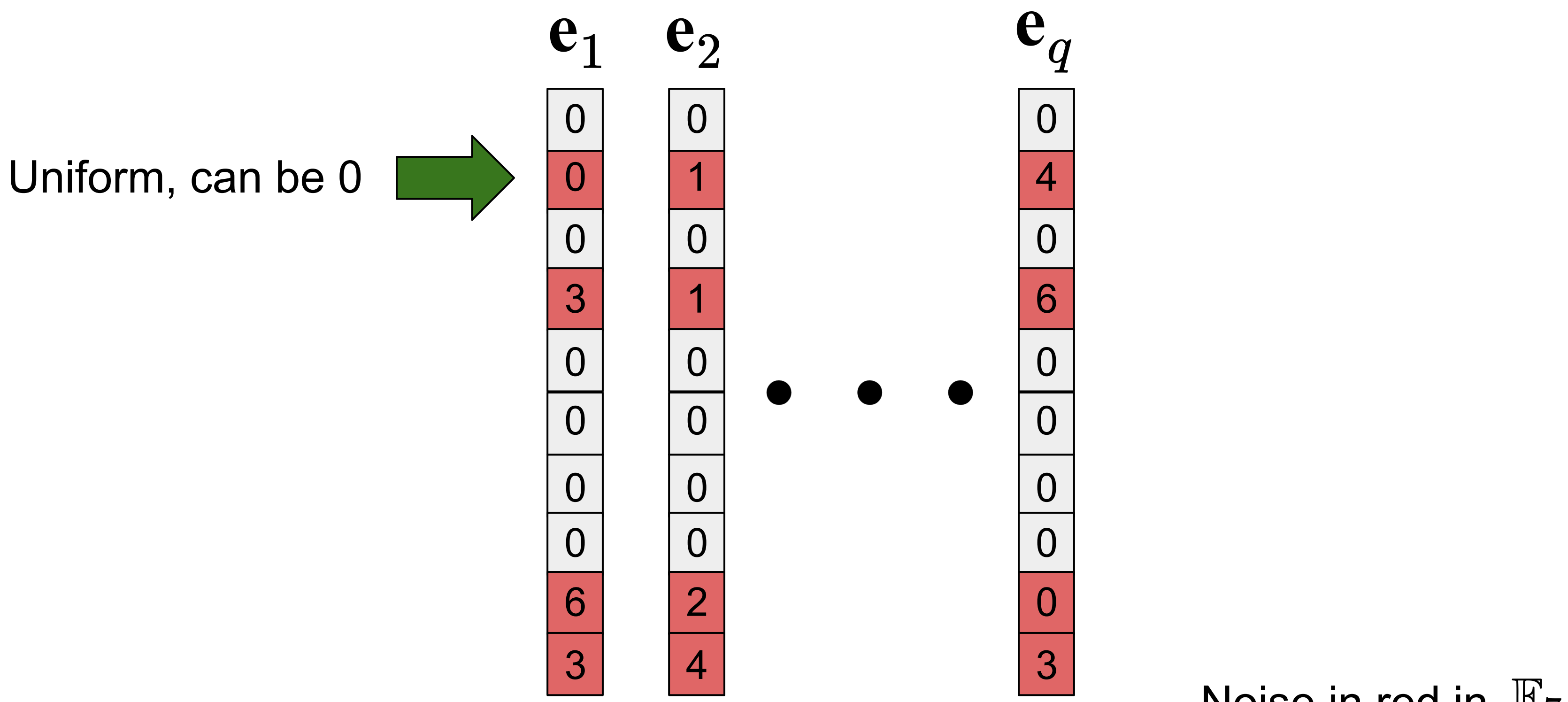
Stationary Syndrome Decoding (SSD)

Noisy coordinates reused

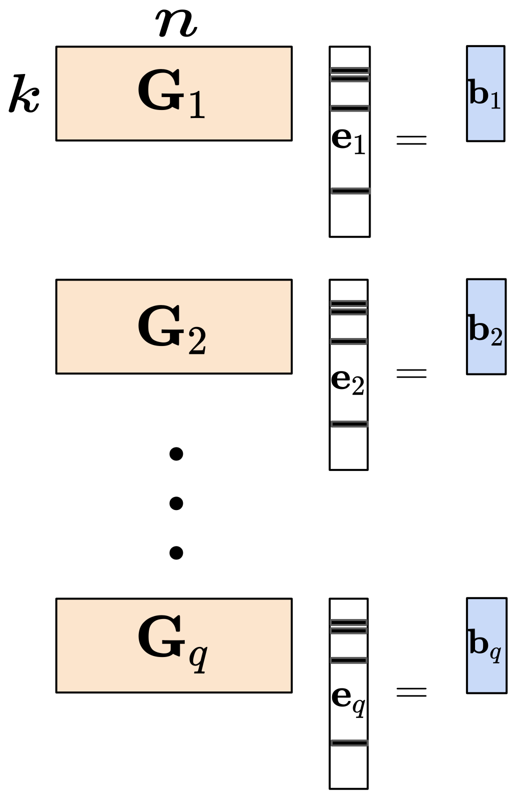


Noise in red in $\mathbb{F}_7$

Stationary Syndrome Decoding (SSD)

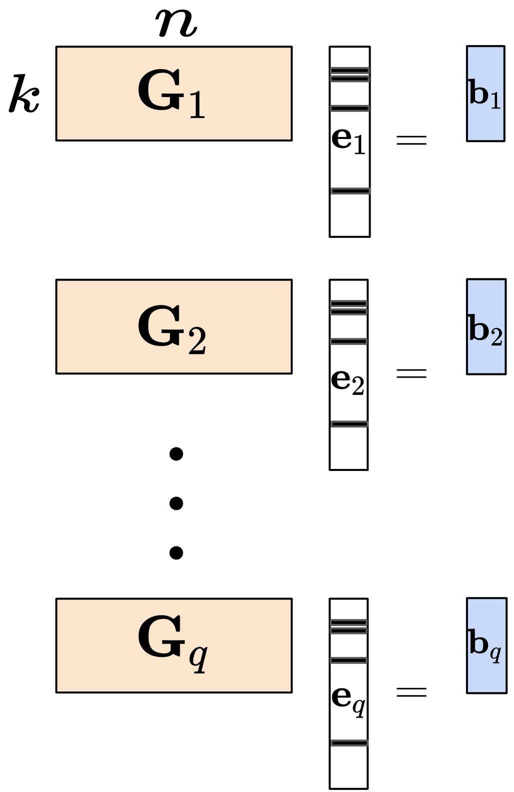


Stationary Syndrome Decoding



$$(\mathbf{G}_i\,, \mathbf{b}_i) \approx (\mathbf{G}_i\,, \$)$$

Stationary Syndrome Decoding



$$(\mathbf{G}_i, \mathbf{b}_i) \approx (\mathbf{G}_i, \$)$$

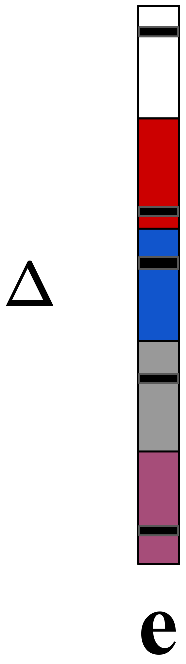
We define SLPN similarly

Stationary Syndrome Decoding (SSD)

We cryptanalyze for $\mathbf{G}_i$

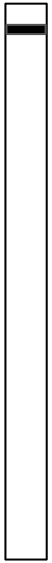
with high minimum distance and regular $\mathbf{e}_i$

How Does SSD Help with PCGs?



How Does SSD Help with PCGs?

Δ



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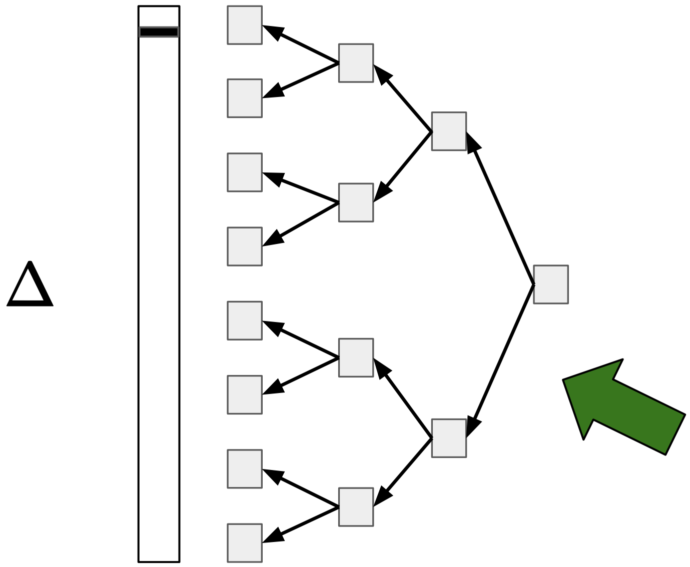
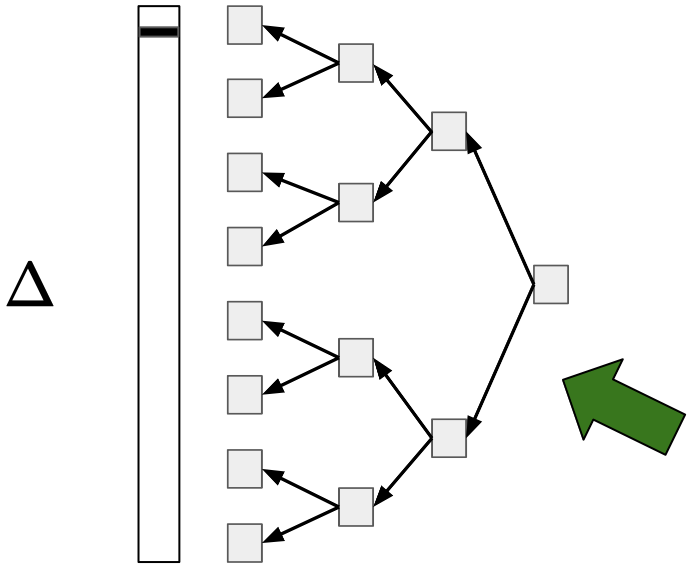


OT at each level

How Does SSD Help with PCGs?



OT at each level

SSD allows for reusing OTs across all q noise vectors

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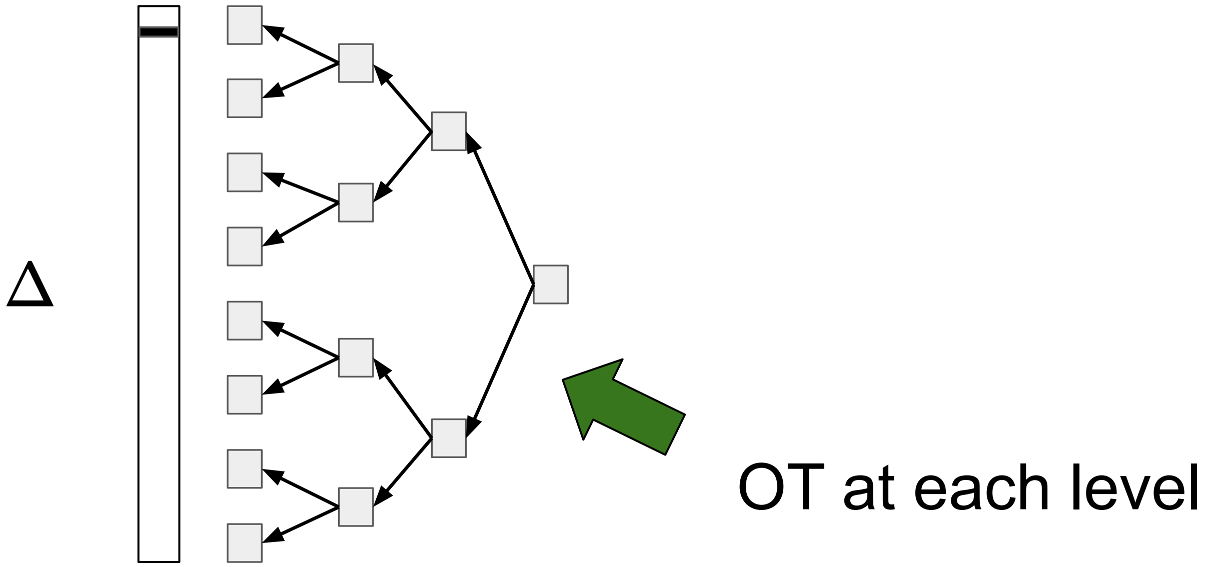

SSD allows for reusing OTs across all q noise vectors

Better cache and memory utilization

Presentation Outline

SSD’s Resilience to Linear Attacks

Other Linear Attacks

SSD’s Resilience to Algebraic Attacks

Experimental Evaluation

Linear Attacks

Gaussian Eliminations [BKW00, Lyu05, LF06, EKM17]

Information Set Decoding [Pra62, Ste88, FS09, BLP11, MMT11, BJMM12, MO15, EKM17, BM18]

Cover Sets [ZW16, BV16, BTV16, GJL20]

Statistical Decoding Attacks [AJ01, FKI06, Ove06, DAT17]

Generalized Birthday Attacks [Wag02, Kir11]

Linearization Attacks [BM97, Saa07]

Low Weight Code [Zic17]

…

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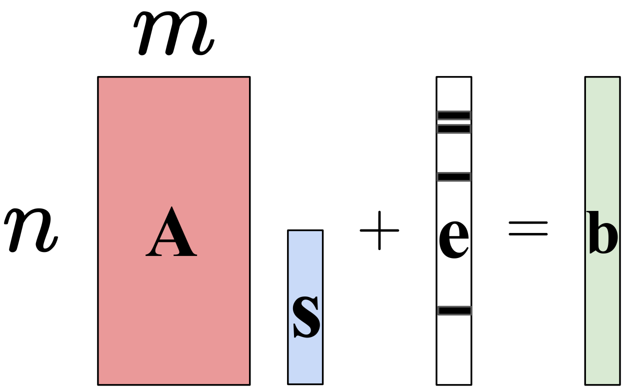
…

Tedious to go through each attack

Linear Test Framework

LPN



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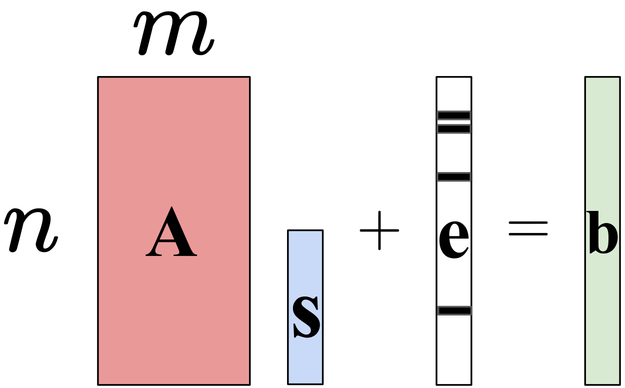
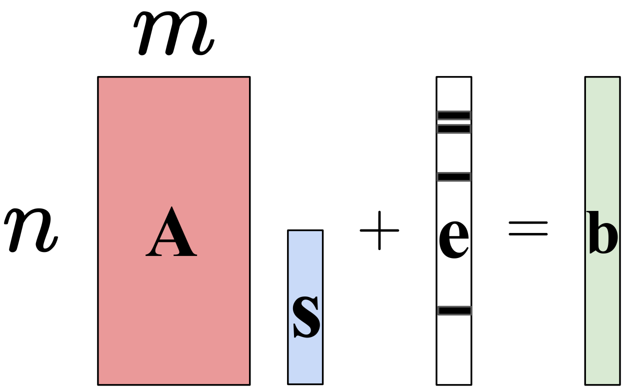
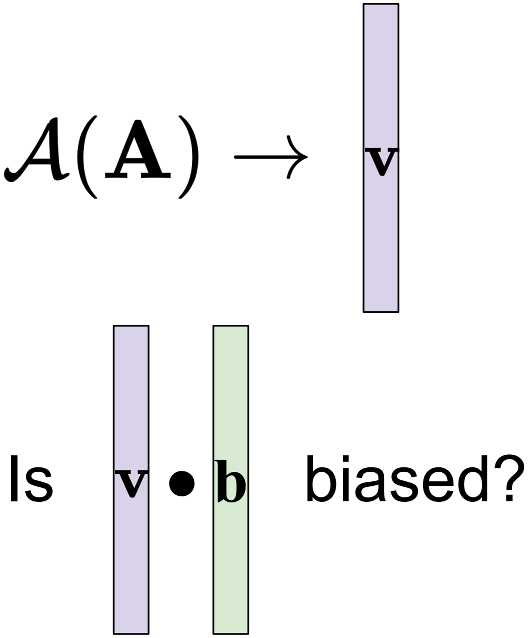


$$\mathcal{A}(\mathbf{A}) \to \mathbf{v}$$

Linear Test Framework

LPN





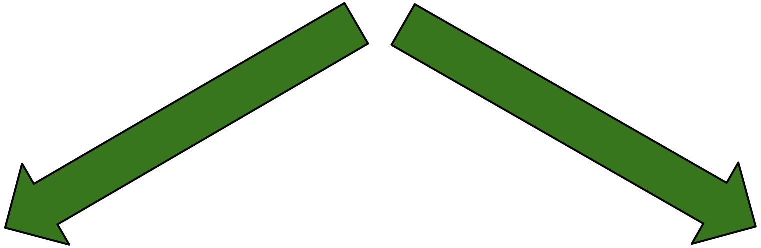
$$\mathbf{v} \cdot (\mathbf{A}\mathbf{s} + \mathbf{e})$$

High-Level Proof Template

For SLPN with **regular** noise

Given equivalence of SLPN and SSD, security for SSD is straightforward

High-Level Proof Template



$q = 1$

$q > 1$

High-Level Proof Template

$q = 1$

$q > 1$

Differs from plain LPN
with regular noise

High-Level Proof Template

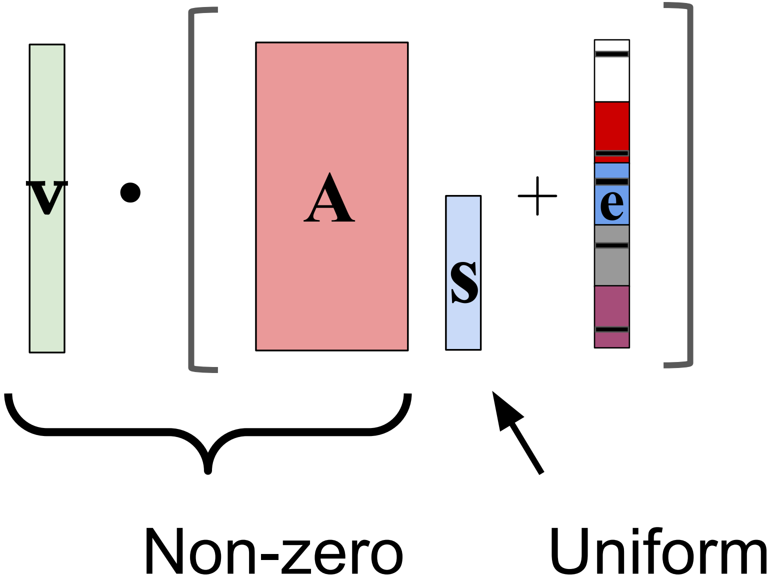
$$q = 1$$

non-codeword **v**

High-Level Proof Template

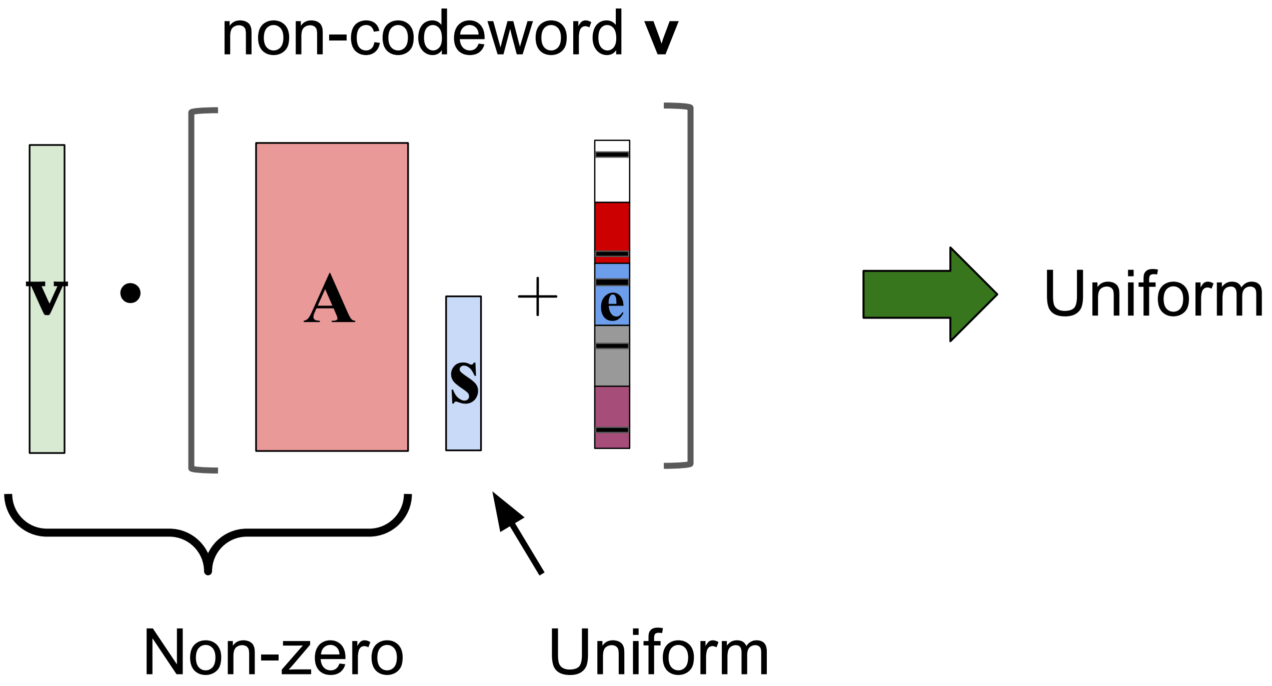
$$q = 1$$

non-codeword **v**



High-Level Proof Template

$$q = 1$$



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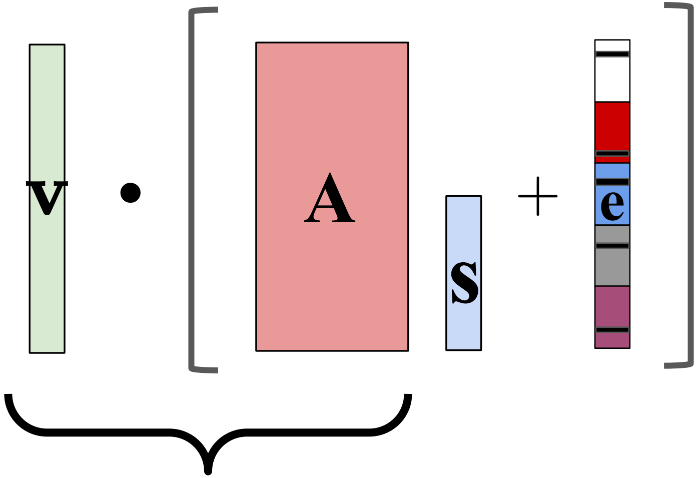
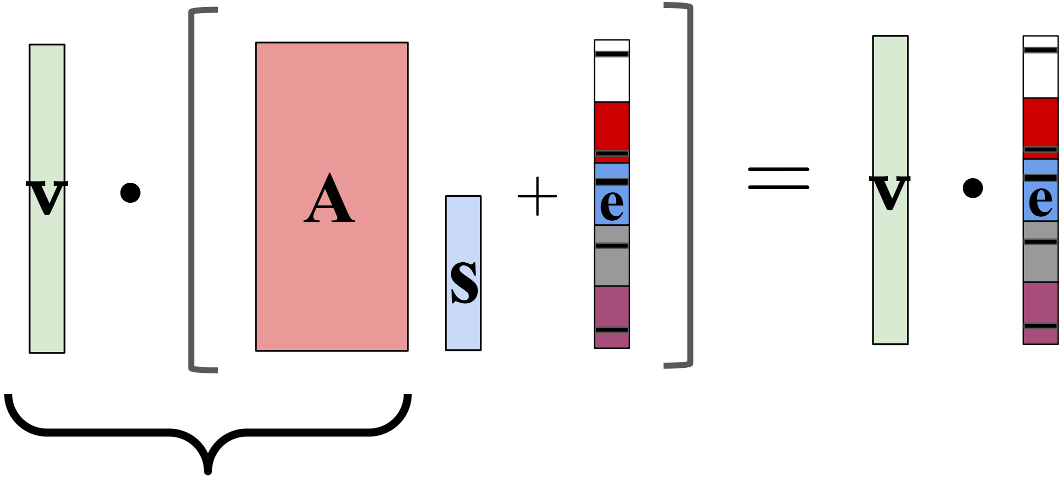


Zero, randomness by **s** vanishes

High-Level Proof Template

$$q = 1$$

codeword **v**



Zero, randomness by **s** vanishes

High-Level Proof Template

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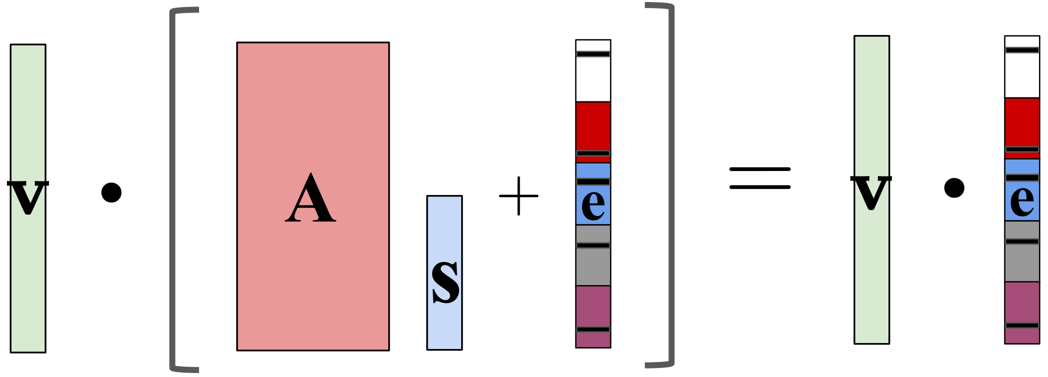
codeword $\mathbf{v}$

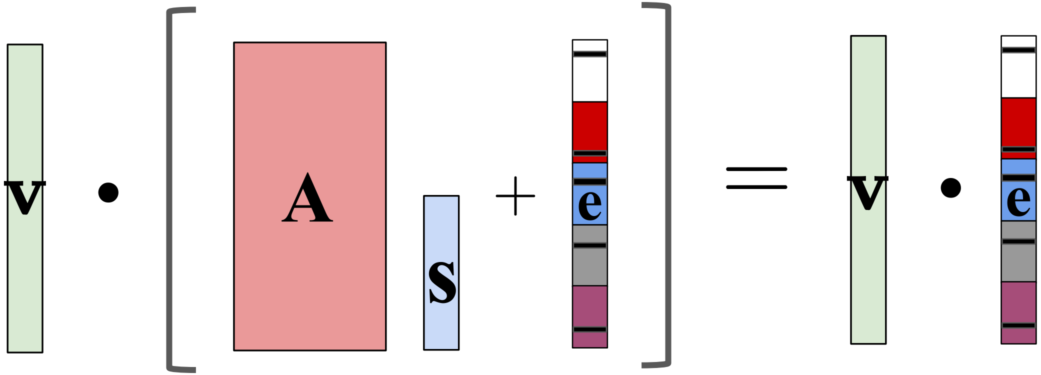


→ Need to show $\mathbf{v} \cdot \mathbf{e}$ has negligible bias

High-Level Proof Template

$$q = 1$$

codeword **v**

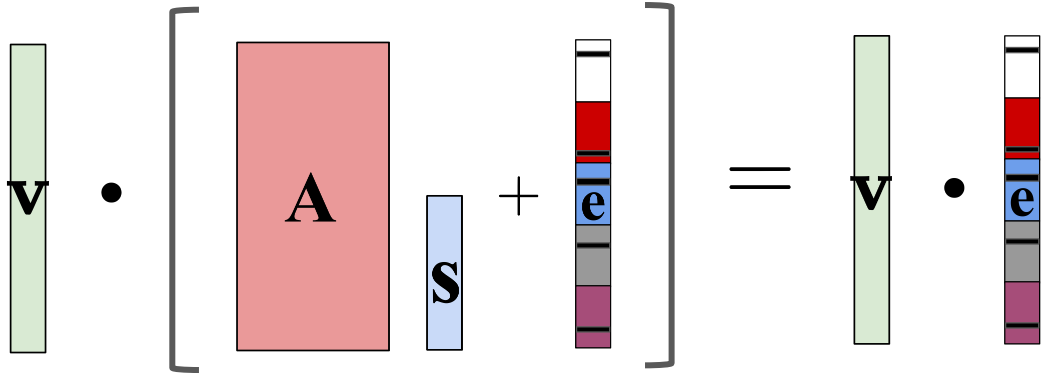


$$\leq \left(1 - \frac{d}{n}\right)^t$$

High-Level Proof Template

$$q = 1$$

codeword **v**

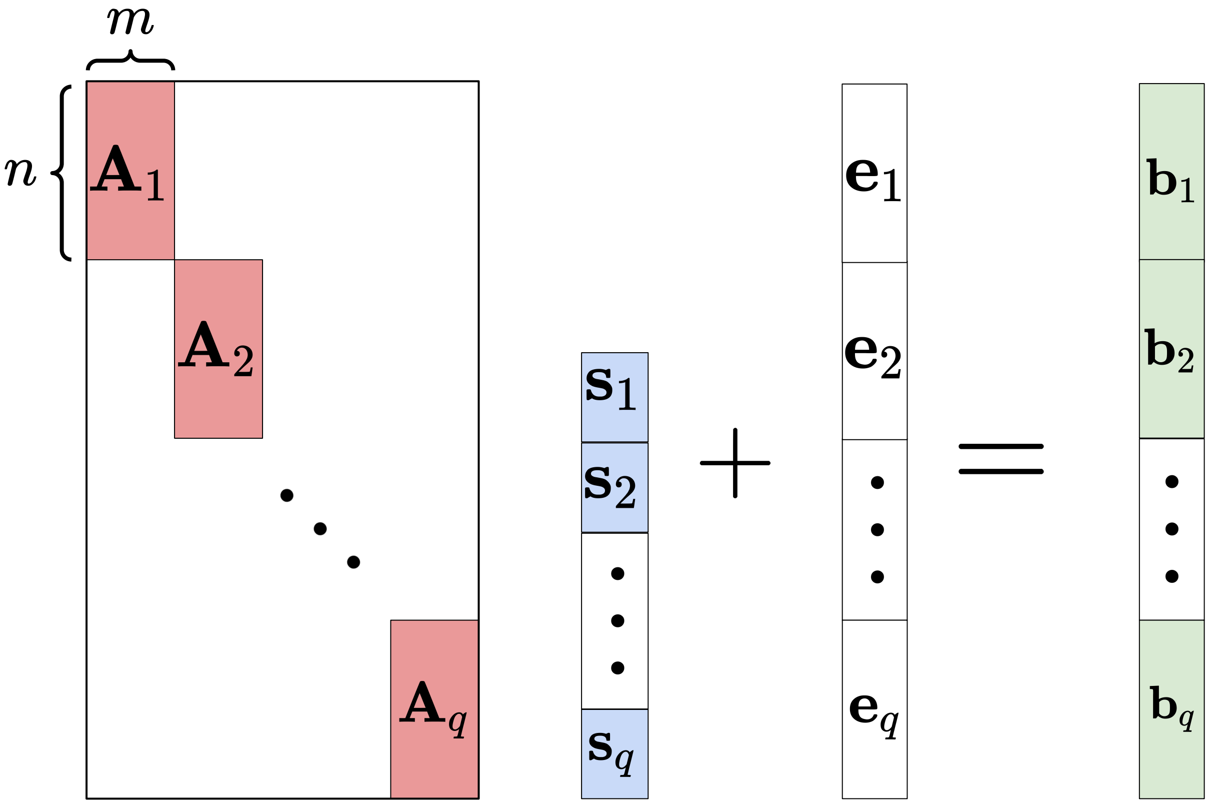


Regular LPN with $\mathbb{F}_2$ noise

$$\le \left(1 - \frac{2d}{n}\right)^t$$

High-Level Proof Template

Consider canonical representation for q > 1:



High-Level Proof Template

$$q > 1$$

v is not a concatenation of q codewords

High-Level Proof Template

$$q > 1$$

$\mathbf{v}$ is not a concatenation of q codewords

$\mathbf{v} \cdot (\mathbf{A}\mathbf{s} + \mathbf{e})$ is uniform because $\mathbf{s}$ is not mapped to 0

High-Level Proof Template

$$q > 1$$

v is a concatenation of q codewords

High-Level Proof Template

$$q > 1$$

v is a concatenation of q codewords

$$\mathbf{v} \cdot (\mathbf{A}\mathbf{s} + \mathbf{e}) = \mathbf{v} \cdot \mathbf{e}$$

High-Level Proof Template

$$q > 1$$

v is a concatenation of q codewords

$$\mathbf{v} \cdot (\mathbf{As} + \mathbf{e}) = \mathbf{v} \cdot \mathbf{e}$$

$$\leq \left(1 - \frac{d}{n}\right)^t$$

Other Linear Attacks

Explored new attacks that could be considered linear but do not fit into the linear test framework

Algebraic Attacks

Solve for $\mathbf{e}_1$, …, $\mathbf{e}_q$ in a polynomial system

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Adapted [BØ23]'s attack to use SSD's additional structure

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Bounds on the running time of XL algorithm

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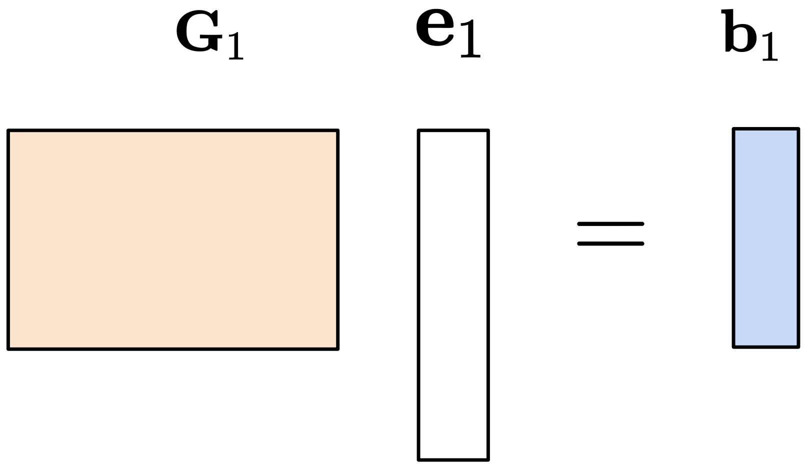
Adapted [BØ23]'s attack to use SSD's additional structure

Bounds on the running time of XL algorithm

We do not find $q > 1$ reduces security (for PCG parameters)

Not competitive with linear attacks

Polynomial System



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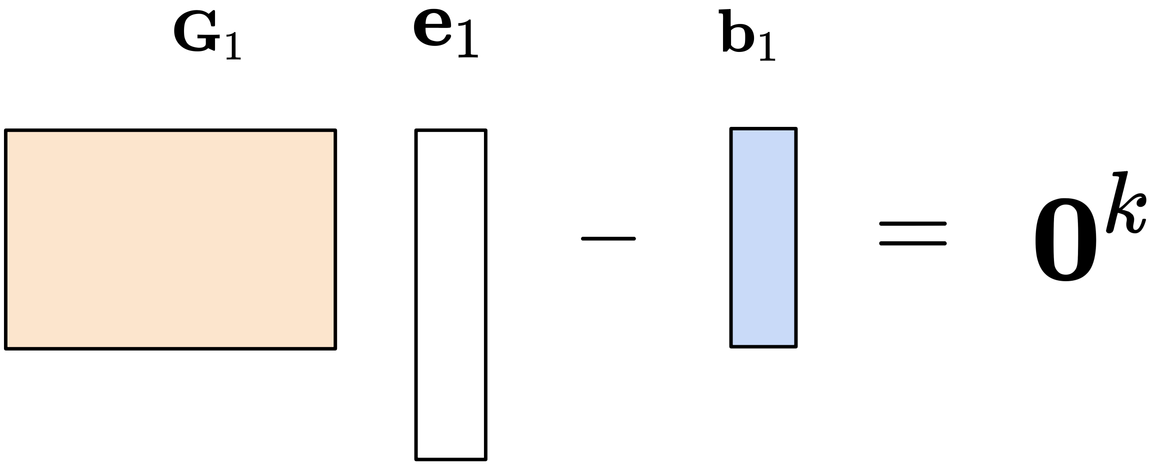

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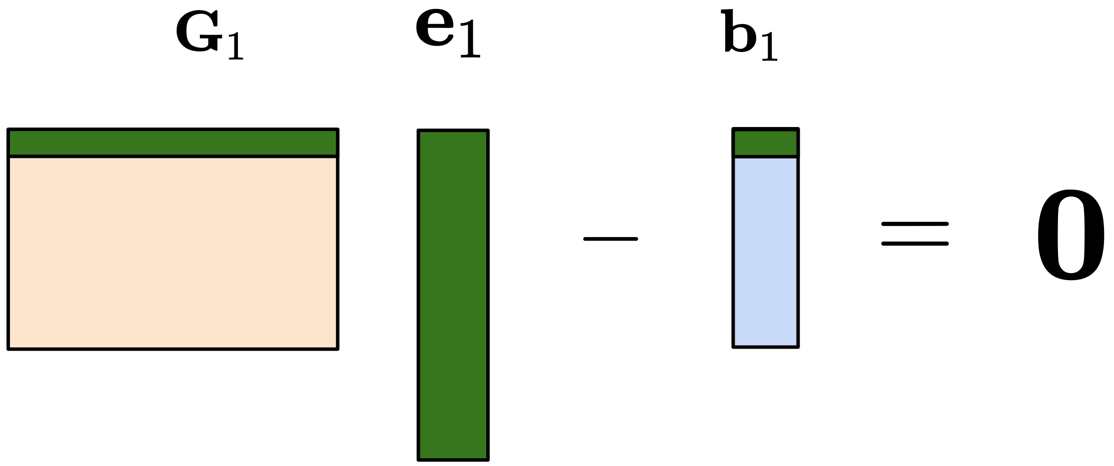

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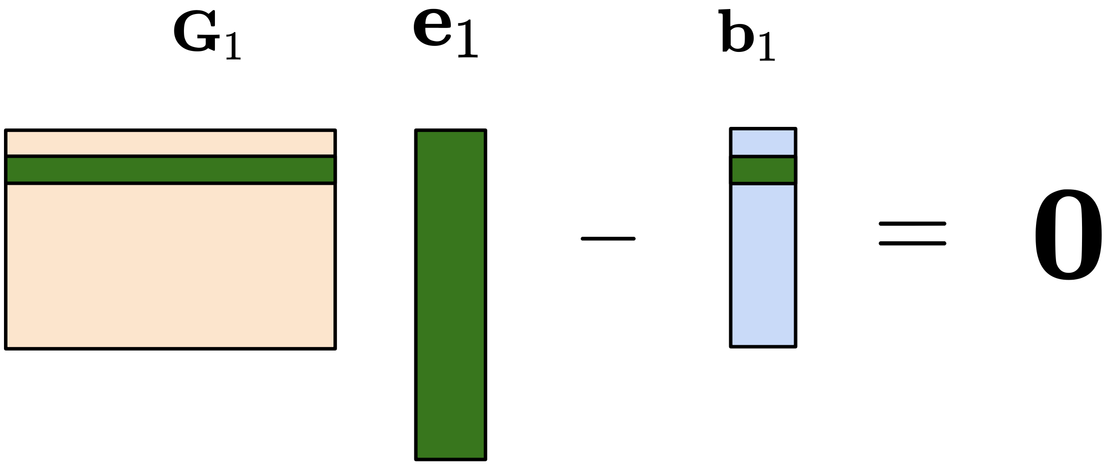

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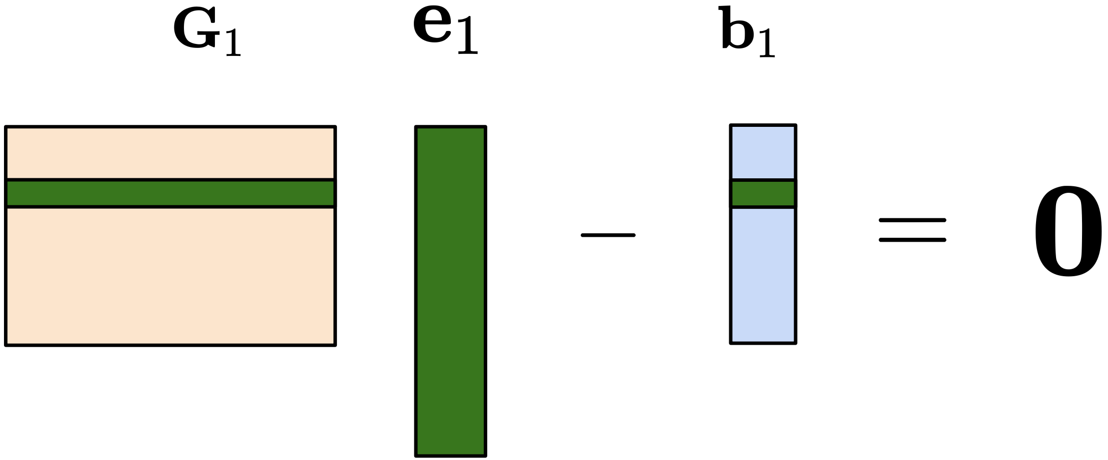
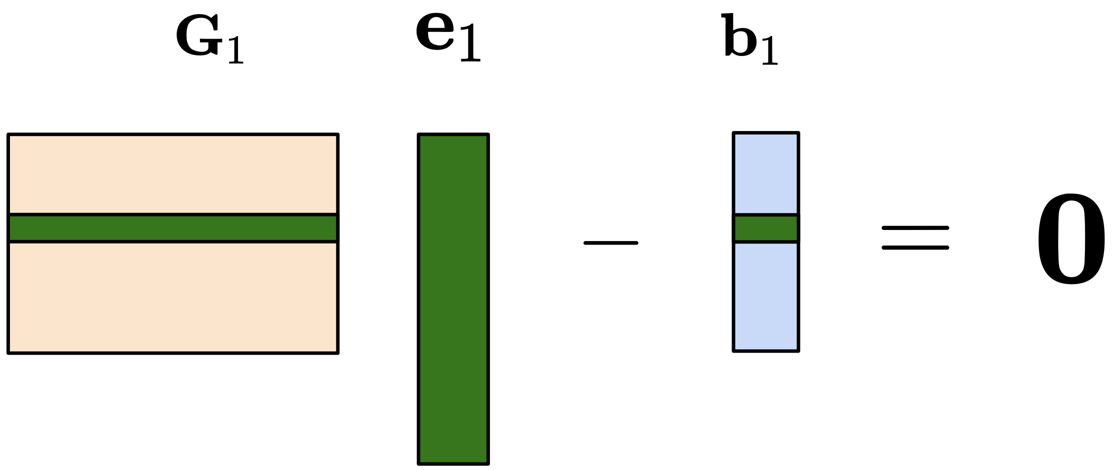


$$\mathbf{G}_1 \mathbf{e}_1 - \mathbf{b}_1 = \mathbf{0}$$

Polynomial System



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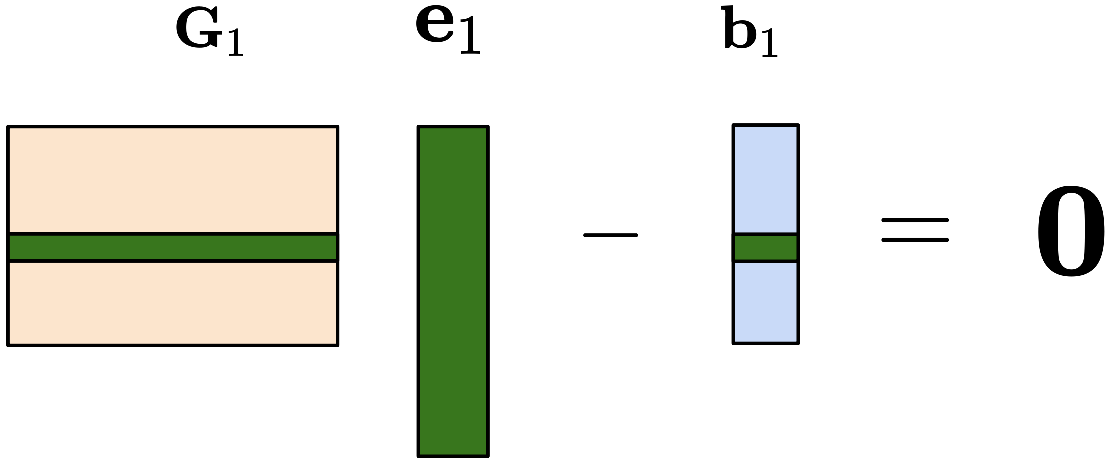

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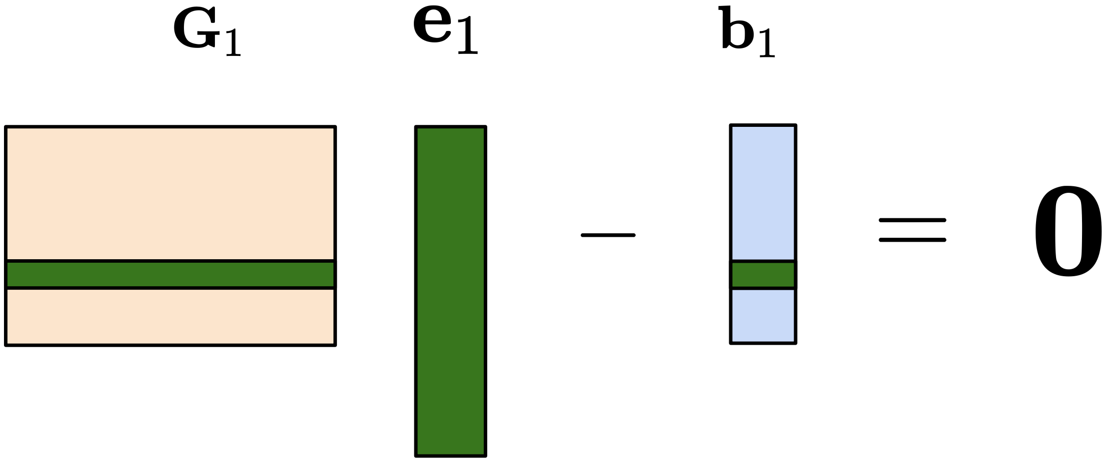

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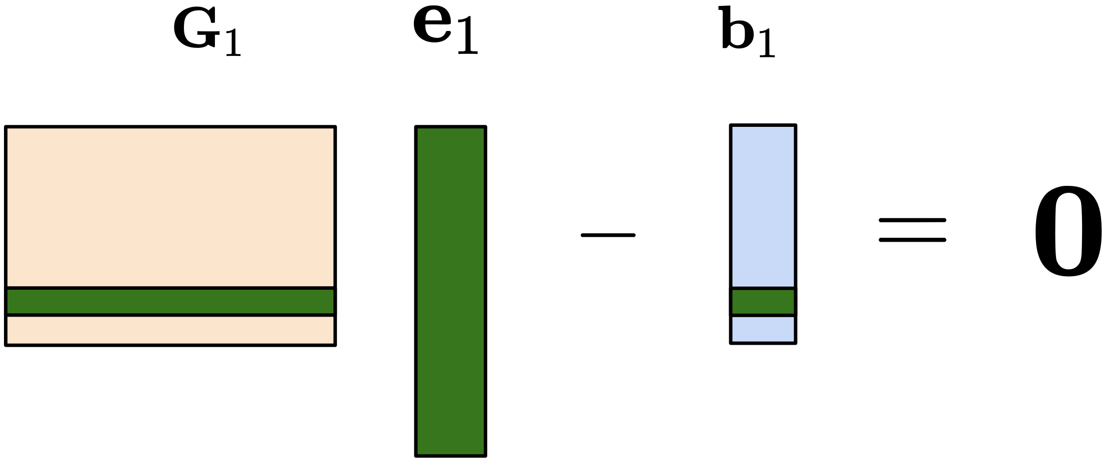

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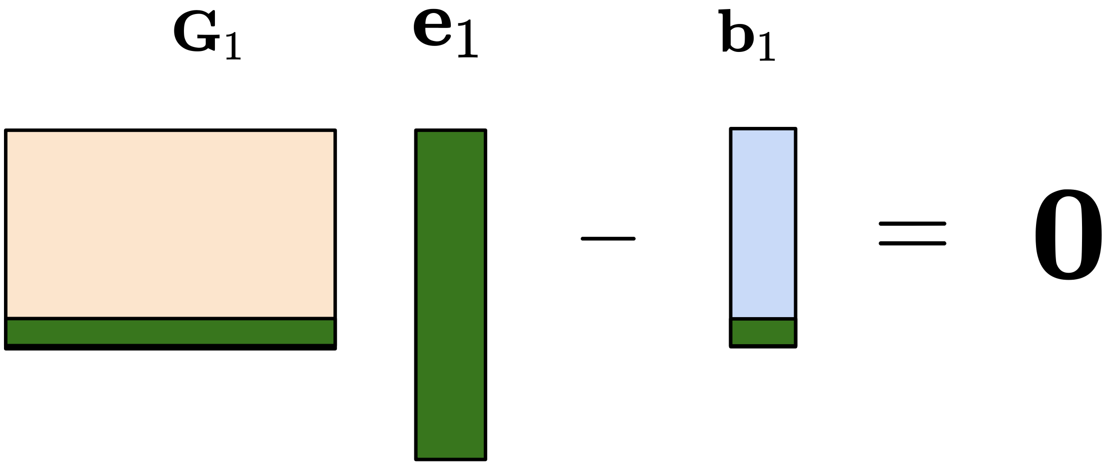

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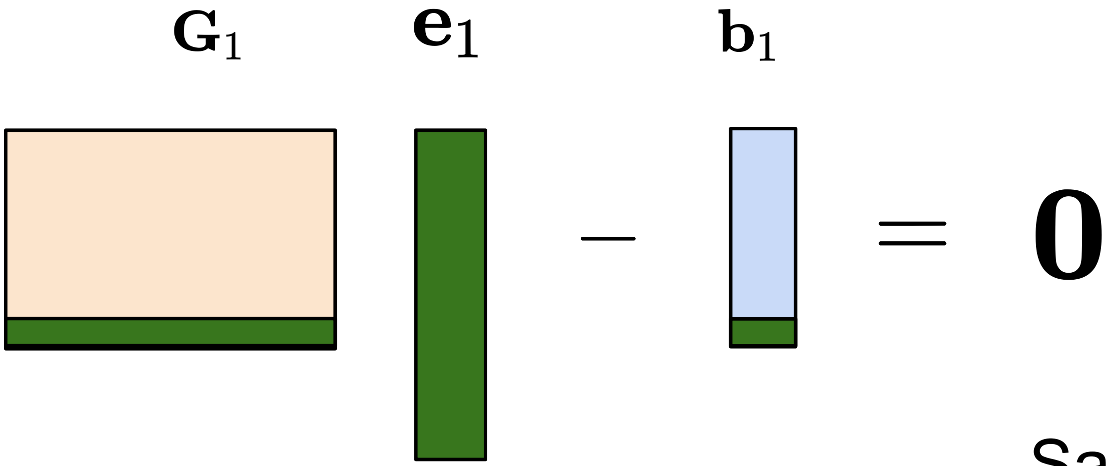
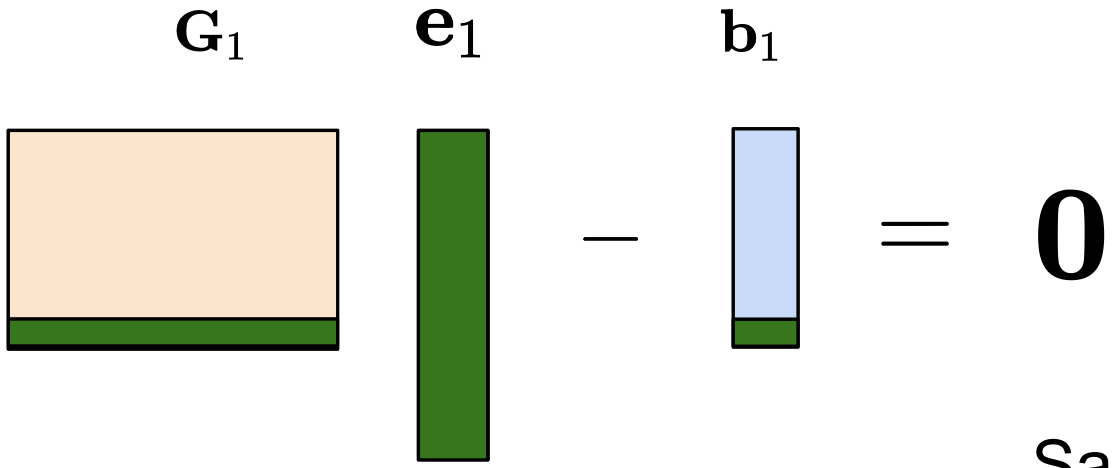


Same for $i = 2, \ldots, q$

Polynomial System



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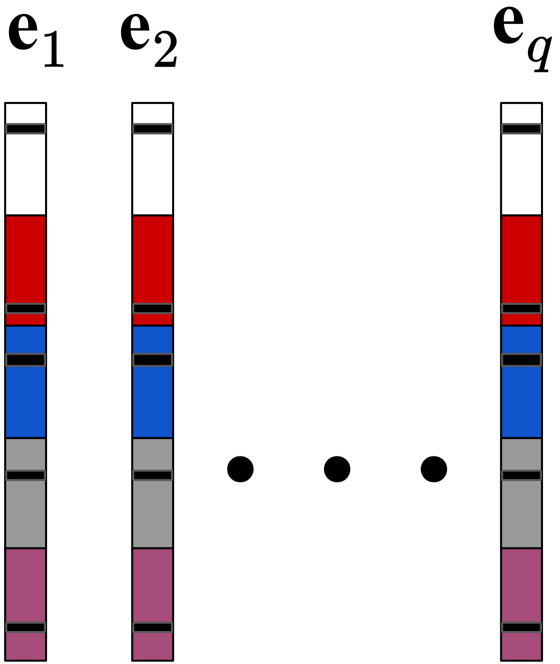
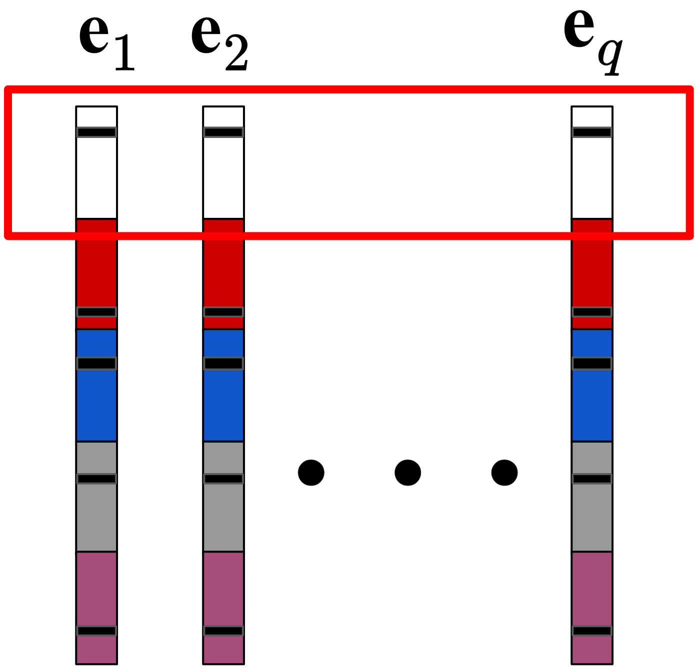
Linear

Polynomial System



Encode stationary structure

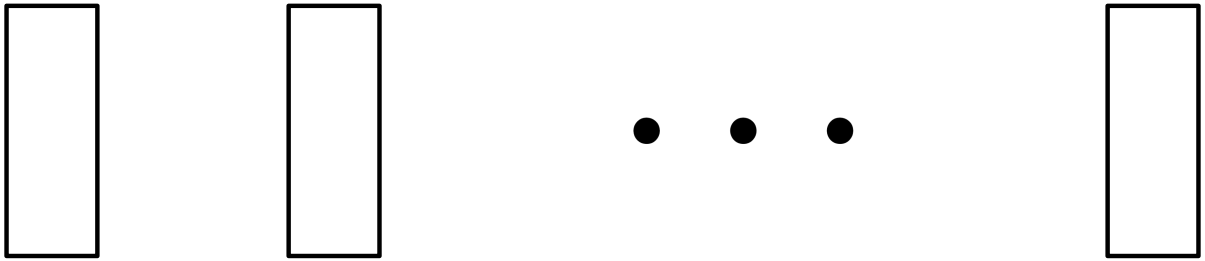
Polynomial System



Encode stationary structure

Polynomial System

Can we multiply 2 elements in the blocks such that their output is 0?



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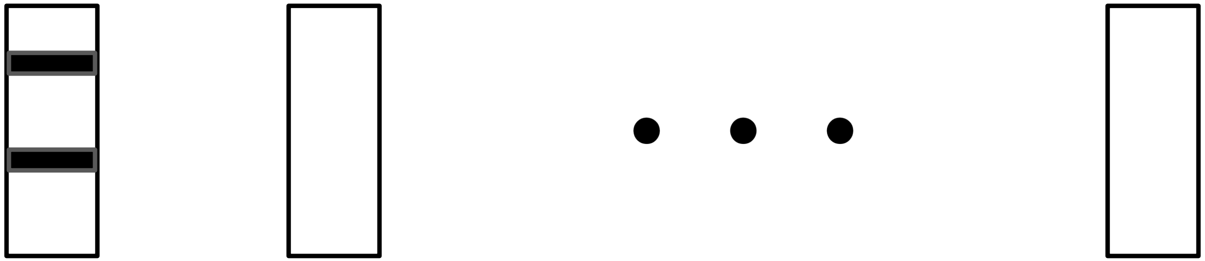

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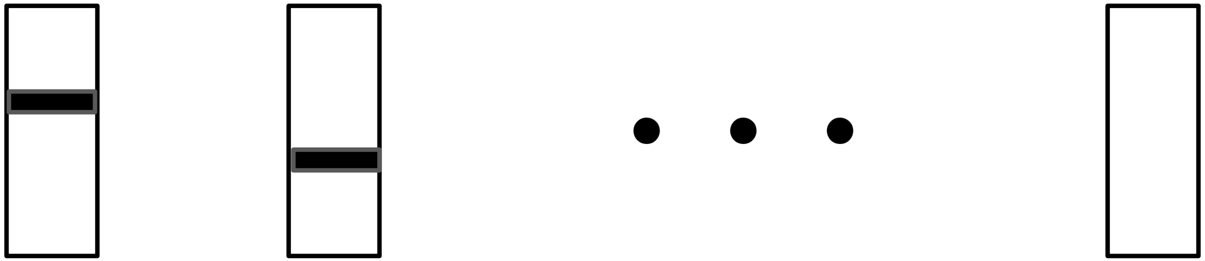

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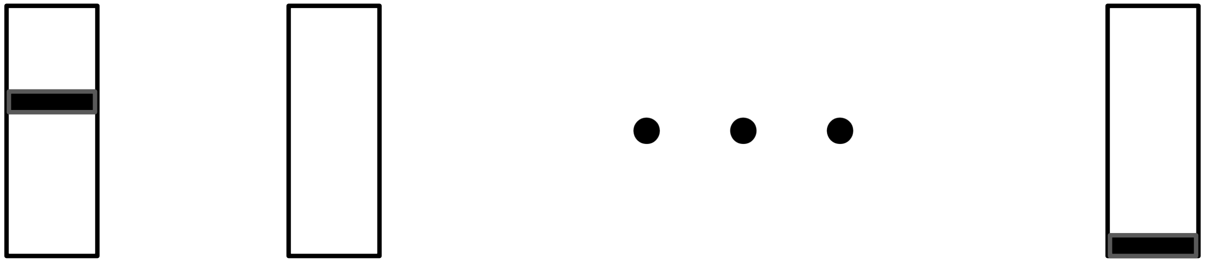

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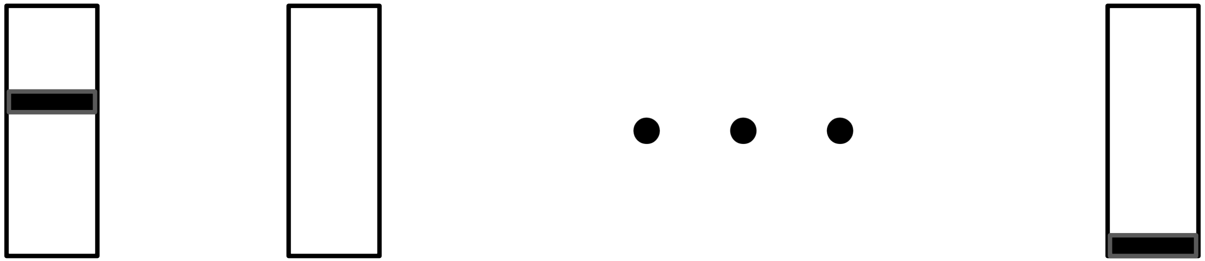

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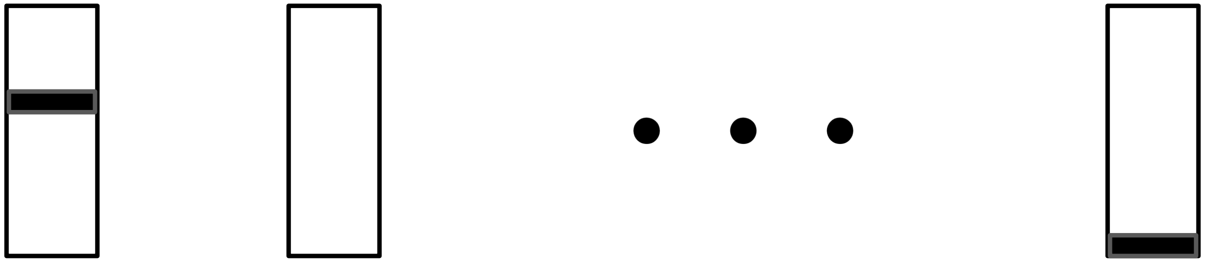

- They just cannot be in the same row
- Quadratic

Polynomial System

In $\mathbb{F}_2$, we also add field equations

High-Level Approach

Construct the system of polynomials $F=\{f_1, \ldots, f_p\}$

Apply the XL Algorithm [CKPS00]

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→ Computing d is the key challenge (from Hilbert series)

Experimental Evaluation

Implemented OT and VOLE from SD/SSD

Reduce communication 6-18x

Reduce runtime 1.5x

Work in Submission

Our new work significantly accelerates multiplication by **G**

Thus, the cost of generating $[\Delta \mathbf{e}]$ becomes even more significant

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Another work closely relies on SSD to generate Beaver triples

Stationary Syndrome Decoding (SSD)

Allows reusing noisy coordinates of **e** across q SD instances

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Excited to see novel applications of SSD

We invite the community to analyze SSD and its variants