A reduction from Hawk to the principal ideal problem in a quaternion algebra

May 5th, Eurocrypt 2025, Madrid

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A bit of context...

2022: module-LIP and **Hawk** (Ducas, Postlethwaite, Pulles, van Woerden, [2]).

- NIST candidate (additional call for signatures)
- based on module-LIP over **cyclotomic fields**
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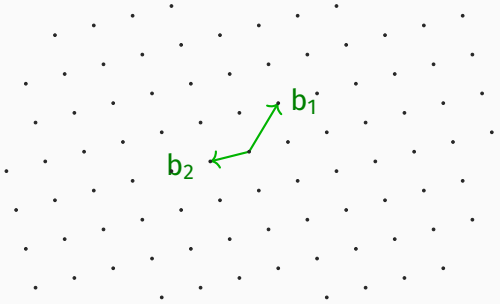

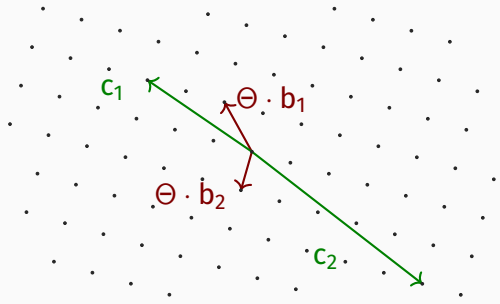

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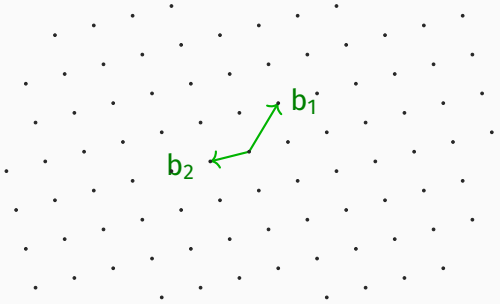

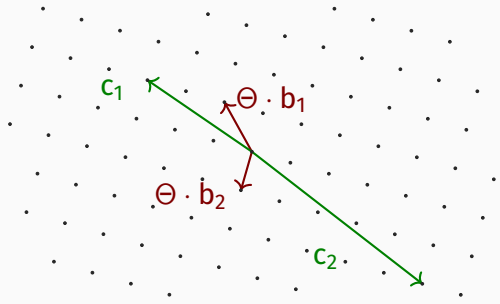

Input: Bases **B** and **C**

Goal: Compute Θ orthogonal or $\mathbf{U} \in \mathbf{GL}_n(\mathbb{Z})$ such that $\mathbf{C} = \Theta\mathbf{B}\mathbf{U}$.

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For efficiency and compactness, we use **module-lattices.** That is, we consider

$$\mathcal{L} = \mathbf{b_1}\mathbb{Z}_K + \cdots + \mathbf{b_n}\mathbb{Z}_K,$$

where K is a **number field**, $\mathbb{Z}_K$ its **ring of integers** and $\mathbf{b_1}, \ldots, \mathbf{b_n} \in K^n$ are K-linearly independent. $\mathbf{B} = (\mathbf{b_1}|\cdots|\mathbf{b_n})$ is a **basis** of $\mathcal{L}$.

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Example: $K = \mathbb{Q}[X]/(X^{2^m} + 1)$ and $\mathbb{Z}_K = \mathbb{Z}[X]/(X^{2^m} + 1)$ a **power-of-two cyclotomic** number field. It is equipped with an automorphism $a \mapsto \overline{a}$ "complex conjugation".

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Remark: Consider only power-of-two cyclotomic fields in the talk. Everything works for a larger family of fields (**CM** number fields).

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Remark: The security of Hawk is related to the hardness of module-LIP on $\mathcal{H}$.
$\longrightarrow$ In the following, we focus on the module-lattice $\mathcal{H}$.

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Put $x = \mathbf{x}_1 + \mathbf{i}\mathbf{x}_2$, $y = \mathbf{y}_1 + \mathbf{i}\mathbf{y}_2$, first coefficient of $\mathbf{H}$ is a **sum of four squares** in $\mathbb{Z}_F$:

$$x\bar{x} + y\bar{y} = \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{y}_1^2 + \mathbf{y}_2^2.$$

A reduction from Hawk to nrdPIP

Introduce the **quaternion algebra**

$$\mathcal{A} = F + F \cdot i + F \cdot j + F \cdot ij,$$

with basis $\{1, i, j, ij\}$ and rules $i^2 = j^2 = -1$, $ji = -ij$.

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- Non-commutative F-algebra containing $K = F + F \cdot i$.
- Has complex conjugation $\alpha \mapsto \overline{\alpha}$ extending the one on K.
- The **reduced norm** of $\alpha = x + yi + zj + tij$ is $\mathrm{nrd}(\alpha) := \alpha\overline{\alpha} = x^2 + y^2 + z^2 + t^2$.

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See the column $\binom{x}{y} \in \mathbb{Z}_K^2$ of the secret matrix **U** as a quaternion $\alpha = x + y \cdot j \in \mathcal{A}$.
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$\longrightarrow$ Compute **all** solutions $\alpha \in \mathbb{Z}_K + \mathbb{Z}_K \cdot j$ to $\mathrm{nrd}(\alpha) = q$ to recover **U**.

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Remark: In a previous work [4], for rank-2 module-LIP over a totally real field (when $\mathcal{L}, \mathcal{L}' \subset F^2$), we solved $\mathrm{nrd}(\alpha) = q$ with $\alpha \in \mathbb{Z}_K = \mathbb{Z}_F + \mathbb{Z}_F \cdot i$.

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$\longrightarrow$ Use extra information on $\mathbf{H}$ to **reduce the solution space**.

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Denote $\mathbf{H} = \begin{pmatrix} q_1 & q_2 \\ \overline{q_2} & q_3 \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$. Set $\alpha = x + y \cdot j$ and $\beta = z + t \cdot j \in \mathcal{A}$.

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$$\text{We prove:} \quad \mathsf{U}^*\mathsf{U} = \mathsf{H} \iff \begin{cases} \mathrm{nrd}(\alpha) = q_1 \\ \alpha\beta^{-1} = q_3^{-1}(\overline{q_2} - j) \end{cases}$$

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- Proof is direct computation.
- From $\mathbf{H}$ (public), can compute $\alpha\beta^{-1}$ (α and β fully determine a solution).
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Question: Can we retrieve α from $\alpha\beta^{-1}$?

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Example: Let $a, b \in \mathbb{Z}$, $b \neq 0$. Suppose $\gcd(a, b) = 1$. Can we get a from $\frac{a}{b}$?

[1] Precisely, take a **maximal order** containing it.

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Example: Let $a, b \in \mathbb{Z}$, $b \neq 0$. Suppose $\gcd(a, b) = 1$. Can we get a from $\frac{a}{b}$? Build the **ideal** of $\mathbb{Z}$:

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and then take a generator $\rightarrow$ get a up to sign.

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The "gcd" of α and β is 1, morally because $\alpha \leftrightarrow \binom{x}{y}$ and $\beta \leftrightarrow \binom{z}{t}$ generate $\mathbb{Z}_K^2$.

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- Use $\alpha\beta^{-1}$ to build the **principal ideal** $\mathcal{I} = \alpha\mathcal{O}$.

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- Use $\alpha\beta^{-1}$ to build the **principal ideal** $\mathcal{I} = \alpha\mathcal{O}$.

We have reduced to the reduced norm Principal Ideal Problem (**nrdPIP**):

Input: A (right) $\mathcal{O}$-ideal $\mathcal{I}$ of $\mathcal{A}$ and $q \in F$.

Goal: A (left) generator α of $\mathcal{I}$ with $\mathrm{nrd}(\alpha) = q$, if it exists.

A reduction from Hawk to nrdPIP

Sum-up:

- Columns of the secret matrix **U** $\leftrightarrow$ secret quaternions $\alpha, \beta \in \mathcal{A}$.
- From the public Gram matrix **H**, compute $\alpha\beta^{-1}$.
- Use $\alpha\beta^{-1}$ to build the **principal ideal** $\mathcal{I} = \alpha\mathcal{O}$.

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$\longrightarrow$ We obtain a polynomial-time reduction from module-LIP on $\mathcal{H}$ to nrdPIP in $\mathcal{O}$.

$\longrightarrow$ Can be adapted to any rank-2 module $\mathcal{M} \subset K^2$.

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About nrdPIP:

- State-of-the-art: compute a shortest vector in $\mathcal{I}$ (rank-$2\dim_{\mathbb{Q}}(K)$ lattice) [3].

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Conclusion:

- New angle of attack on module-LIP and Hawk.
- Connection between a lattice problem and the world of quaternions.
- A lot to do on nrdPIP. Any improvements would impact Hawk.

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Thanks a lot! Any questions?

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