Constructing Quantum Implementations with the Minimal T-depth or Minimal Width and Their Applications

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Outline

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Motivation

- Quantum Cryptanlysis for Symmetric Ciphers
  - Grover's algorithm: the attacker needs to construct a Grover oracle to search the key.
  - Simon's algorithm (Kuwakado and Mori, ISIT 2010; Kaplan et al. Crypto 2016): The attacker needs to access an online quantum encryption oracle.
  - Offline Simon's algorithm (Bonnetain et al. Asiacrypt 2019): the attacker needs to construct different quantum encryption oracles for different keys.
  - **The quantum circuit that implements the cipher** is a primary component of the Grover oracle or the quantum encryption oracle.
- NIST's call for proposals for PQC
  - **The complexity of quantum key search circuit for AES** is used as a baseline to categorize the post-quantum public-key schemes.

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- NIST's call for proposals for PQC
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Circuit Complexity

Cost Metrics:

- **Width** (the number of qubits); Gate count, especially the T-count;
- Depth: The number of layers of the circuit (gates acting on disjoint sets of qubits can be applied in parallel)
- T-**depth**: the number of layers for T gates, which dominates the running time of a circuit in fault-tolerant quantum computation (Surface code).

Previous Works (PQCrypto 2016, Asiacrypt 2020, Eurocrypt 2020, Aisacrypt 2022, Asiacrypt 2023, Asiacrypt 2024, etc):

- AES Width: $2953 \rightarrow 512 \rightarrow 374$, T-depth: $120 \rightarrow 80 \rightarrow 60$
- Heuristic improvements that are only applicable for AES.

Both T-depth and Width have theoretical minimums.

Generic methods for constructing circuits that achieve these minimums?

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Convert Classical Circuits to Quantum Circuits

Boolean Function Implementation Problem

Given a vectorial Boolean function $F(x)$, construct a quantum circuit that maps

- $|x, 0\rangle \rightarrow |F(x), 0\rangle$: An **in-place** implementation of an invertible F.
- $|x, y\rangle \rightarrow |x, y \oplus F(x)\rangle$ for any y (sometimes only for $y = 0$): An **out-of-place** implementation of F.

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From classical circuits to quantum circuits:

- Classicial gates: NOT, XOR, AND $\Rightarrow$ NCT gate set: NOT(Pauli-X), CNOT, Toffoli

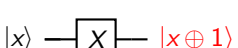


Figure: NOT gate

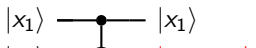


Figure: CNOT gate

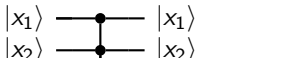
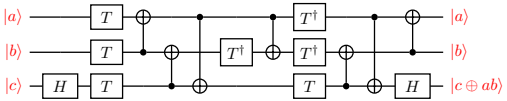


Figure: Toffoli gate

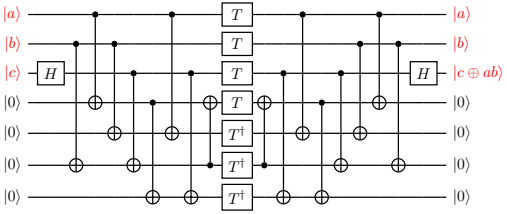
$\Rightarrow$ Clifford+T gates: {Pauli gates, CNOT, S, H} + T

Clifford+T Implementations for the Toffoli gate

- No ancilla qubit:



- T-depth-1:



Design NCT circuits that achieve the minimal Toffoli-depth or minimal width

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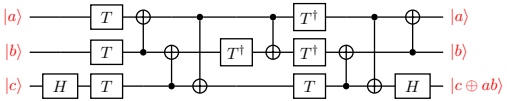

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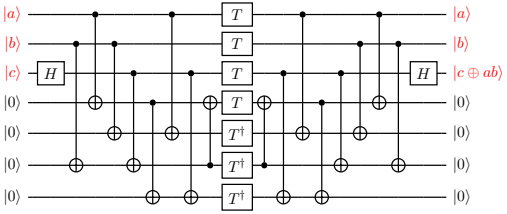
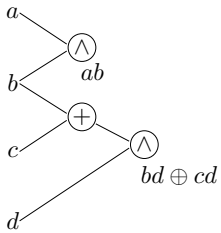


Design NCT circuits that achieve the minimal Toffoli-depth or minimal width

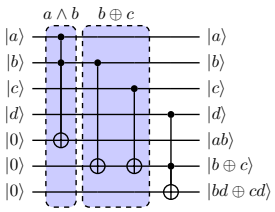
Outline

Convert Classical Circuits to NCT Circuits

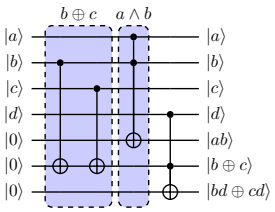
- Problems of the trivial conversion (XOR→CNOT, AND→Toffoli):
  - Need lots of ancilla qubits.
  - The depth may change, especially **Toffoli-depth≠ AND-depth** in sometimes.



(1) AND-depth-1



(2) Toffoli-depth-2



(3) Toffoli-depth-1

Figure: Quantum implementations of a classical circuit with AND-depth 1

Quantum Circuit with the Minimal Toffoli-depth

Theorem (Asiacrypt 2022, Huang & Sun)

Given a classical circuit with **AND-depth s**, the Toffoli-depth of the quantum circuit implementing all the nodes of the classical circuit **is not smaller than s**. Moreover, **with sufficiently many ancillae**, we can construct a quantum circuit implementing all the nodes of the classical circuit with **Toffoli-depth s**.

1. Construct a classical circuit with the minimal AND-depth ($= \lceil \log_2(D) \rceil$);
   - D is the algebraic degree of the vectorial Boolean function F (With an AND-layer, one can mostly double the degree).
   - AES S-box (degree 7): the minimal AND-depth is 3.
2. Convert it to a Toffoli-depth-$\lceil \log_2(D) \rceil$ NCT circuit.

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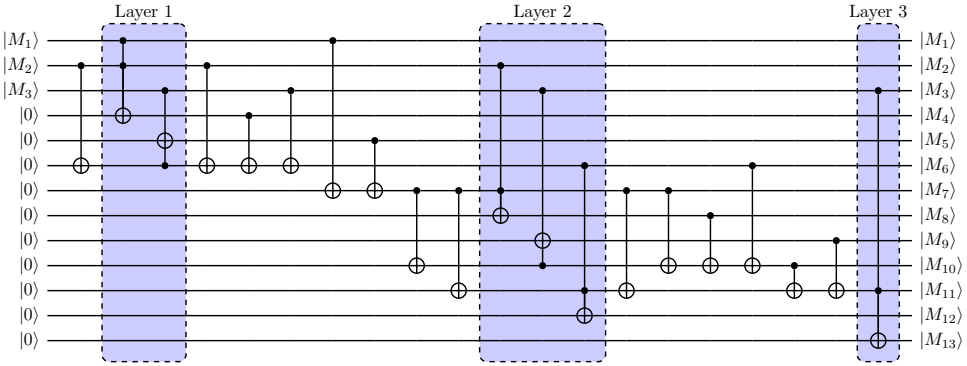
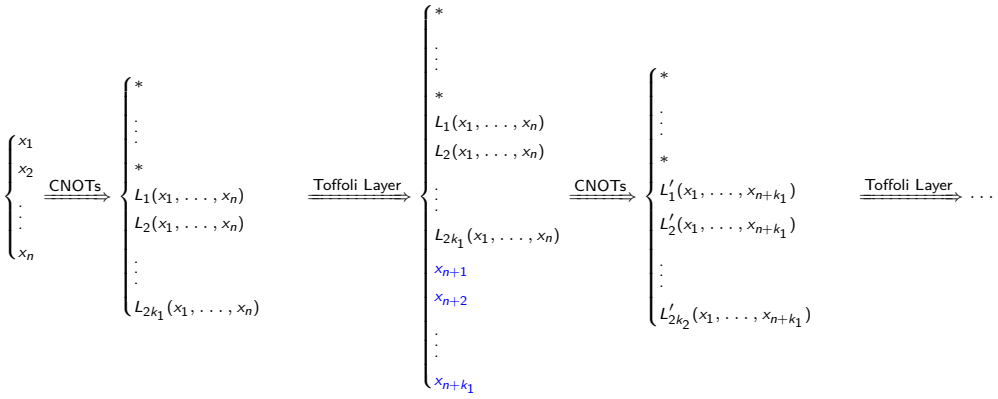
Automatic Conversion with Low Quantum Resource Cost



- Our idea: **Reconstruct the CNOT sub-circuits** that connect the Toffoli layers while maintaining the **lowest width and a low gate-count**.

Inputs and Outputs of the CNOT Sub-circuits

- CNOT gates generate linear expressions.
- A Toffoli layer generates some new Boolean variables.



Lemma 2

Let $\{L_1, L_2 \ldots, L_t\}$ and $\{T_1, T_2 \ldots, T_m\}$ be two sequences of linear functions w.r.t Boolean variables $x_1, x_2, \ldots, x_n$. Suppose **the rank of $L_1, L_2, \ldots, L_t$ is n, and the rank of $T_1, T_2, \ldots, T_m$ is k.** If $|L_1, L_2, \ldots, L_t\rangle$ is the input of a t-qubit register, then to **output the state $|T_1, T_2, \ldots, T_m\rangle$ using a CNOT circuit**, **$m - k - (t - n)$ additional qubits are necessary and sufficient**. Additionally, if $m - k - (t - n) < 0$, it means no additional qubits are required. Instead, $t - n - m + k$ qubits can be returned to $|0\rangle$.

Table: Different NCT circuits originated from an AND-depth-4 classical circuit for the AES S-box. Toffoli-depth=2× AND-depth, since uncomputation is included.

Type	#NOT	#CNOT	#Toffoli	Toffoli-depth	Full Depth	Width	Source
$\mathfrak{C}^*$	4	312	68	8	78	90	Aisacrypt 2023
$\mathfrak{C}^*$	4	368	68	8	105	76	Aisacrypt 2023
$\mathfrak{C}^0/\mathfrak{C}^*$	4	**227/240**	**60**	8	**60**	**66**	This work

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Construct Classical Implementations with the Minimal AND-depth

- Trivial approach:
  - 1st AND layer: generate monomials with **degree** 2 simultaneously: $x_1x_2, \ldots, x_{n-1}x_n$;
  - 2rd AND layer: generate monomials with **degree** $3, 4$ simultaneously;
    $\ldots$
  - k-th AND layer: generate monomials with **degree from** $2^{k-1}+1$ **to** 2^k;
  - Construct F from these monomials by XOR gates.

- χ function of SHA3 : $(f_1, f_2, f_3, f_4, f_5) =$
  $(x_1 + (x_2 + 1)x_3, x_2 + (x_3 + 1)x_4, x_3 + (x_4 + 1)x_5, x_4 + (x_5 + 1)x_1, x_5 + (x_1 + 1)x_2)$.
  Construct $x_2 \cdot x_3, x_3 \cdot x_4, x_4 \cdot x_5, x_5 \cdot x_1, x_1 \cdot x_2$ in **one AND layer**.

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- **Top-down approach**:
  - $\mathbb{M}_s$: **Polynomials in layer** s. Boolean polynomials with degree in the range $[2^{s-1}+1, 2^s]$, whose minimal AND-depth is s.
  - $f = f_d + f_{d-1} + \cdots + f_1$, where $f_i \in \mathbb{M}_i$. Generate $f_d \to f_{d-1} \to \cdots$
  - Compute **the max-depth cover** $\{C_1, C_2, \ldots, C_k\} \subset \mathbb{M}_d$ of f:

    $$f = C_1 + C_2 + \cdots + C_k + R,$$

    where $C_i = D_{i,1} \cdot D_{i,2}$, and $D_{i,j}, R \in \mathbb{M}_{d-1}$.

    Then recursively compute the covers of these $D_{i,j}$ and R.
  - Trivial cover (all monomials contained in f_d) is equivalent to the trivial approach.

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Construct Nontrivial Covers

Example 1

Let

$$f = x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_6 + x_1x_3x_4x_5x_6 + x_1x_2x_4x_5x_6 + x_1x_2x_3x_5x_6 + x_2x_3x_4x_5x_6.$$

$$\mathcal{C} = \{(x_1x_2x_3 + x_2x_3 + x_1x_2 + x_1x_3) \cdot (x_4x_5x_6 + x_4x_5 + x_4x_6 + x_5x_6)\}$$

is a max-depth cover of f with size 1.

- Greedy approach: gradually enlarge the monomial sets S_1, S_2, such that the product $(\sum_{p_i \in S_1} p_i)(\sum_{q_j \in S_2} q_j)$ covers more monomials in F.
- SAT-based Method: encode the relation $f = \sum_{i=1}^{k} \mathrm{D}_i^1 \cdot \mathrm{D}_i^2 + \mathrm{R}$ to Boolean equations, then solve them by an off-the-shelf SAT-solver.

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Applications

- AES S-box (degree 7): AND-depth 3,**AND-count 76**.
  - Asiacrypt 2022: AND-depth 3, **AND-count 78**, modified from an AND-depth-4 implementation with some heuristics.
- SKINNY S-box (degree 7): AND-depth 3, AND-count 10.

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Minimal AND-depth for Implementing the Inversion

The problem

- $\alpha \in \mathbb{F}_{2^n}$: $x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n$, where $x_1, \ldots, x_n \in \mathbb{F}_2$.
- α^{-1} can be written as
  $f_1(x_1, x_2, \ldots, x_n)\beta_1 + f_2(x_1, x_2, \ldots, x_n)\beta_2 + \cdots + f_n(x_1, x_2, \ldots, x_n)\beta_n$.
- Goal: implementing $F = (f_1, f_2, \ldots, f_n)$.

- It is easy to prove that $deg(F) = n - 1$.

Theorem

The minimal AND-depth for implementing the multiplicative inversion in $\mathbb{F}_{2^n}$ is $\lceil \log_2(n-1) \rceil$.

Minimal AND-depth for Implementing the Inversion

The problem

- $\alpha \in \mathbb{F}_{2^n}$: $x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n$, where $x_1, \ldots, x_n \in \mathbb{F}_2$.
- α^{-1} can be written as
  $f_1(x_1, x_2, \ldots, x_n)\beta_1 + f_2(x_1, x_2, \ldots, x_n)\beta_2 + \cdots + f_n(x_1, x_2, \ldots, x_n)\beta_n$.
- Goal: implementing $F = (f_1, f_2, \ldots, f_n)$.

- It is easy to prove that $deg(F) = n - 1$.

Theorem

The minimal AND-depth for implementing the multiplicative inversion in $\mathbb{F}_{2^n}$ *is* $\lceil \log_2(n-1) \rceil$.

Addition Chain and Exponentiation

- $\alpha^{2^n-2} \cdot \alpha = \alpha^{2^n-1} = 1$, and if $\alpha = 0$, $\alpha^{2^n-2} = 0 \Rightarrow \alpha^{-1} = \alpha^{2^n-2}$.
- Compute $\alpha^{-1} = \alpha^{62} \in \mathbb{F}_{2^6}$:

  $1_2(1) \xrightarrow{\wedge 2} 10_2(2) \xrightarrow{\times} 11_2(3) \xrightarrow{\wedge 2} 110_2(6) \xrightarrow{\times} 111_2(7) \xrightarrow{\wedge 4} 11100_2(28) \xrightarrow{\times} 11111_2(31) \xrightarrow{\wedge 2} 111110_2(62)$

  - Squaring can be implemented without AND gate, squaring will not change the Hamming weight.
  - The change of the Hamming weight: $1 \to 2 \to 3 \to 5$
  - 3 additions correspond to 3 multiplications in $\mathbb{F}_{2^6}$.

Lemma

If $k = 2^r s$, for some positive number r and odd number s, then the multiplication of two elements in $\mathbb{F}_{2^k}$ can be implemented by **one AND layer and $\omega(k) = 3^r s^2$ AND gates**.

  - AND-depth 3, AND-count $3 \cdot 27$.

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Parallel Addition Chain

- Shortest addition chain $\neq$ Minimal-AND-depth implementation:
  - $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 7 \Rightarrow$ 4 AND layers
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- Parallel Addition Chain:

$$A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

$$\mathcal{C}_1 : 1 \xrightarrow{+1} 2 \xrightarrow{+2} 4 \xrightarrow[\mathcal{C}_2]{+3} 7$$
$$\mathcal{C}_2 : 1 \longrightarrow 1 \xrightarrow[\mathcal{C}_1]{+2} 3$$

$$\alpha \xrightarrow{\wedge 2} \alpha^{10_2} \xrightarrow{\times \alpha^{1}} \alpha^{11_2} \xrightarrow{\wedge 4} \alpha^{1100_2} \xrightarrow{\times \alpha^{11_2}} \alpha^{1111_2} \xrightarrow{\wedge 16} \alpha^{11110000_2} \xrightarrow{\times \alpha^{1110_2}} \alpha^{11111110_2}$$
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Lemma

For any n, there is a parallel addition chain for n with the minimal depth $\lceil \log_2(n) \rceil$ and involving $\mathrm{HW}(n) + \lfloor \log_2(n) \rfloor - 1$ additions.

$$\begin{bmatrix} 1 & 2 & \cdots & 2^{k_1} & 2^{k_1+1} & \cdots & 2^{k_2} & 2^{k_2+1} & \cdots & 2^{k_3} & \cdots & 2^{k_m} & 0 \\ 0 & 0 & \cdots & 2^{k_1} & 0 & \cdots & 0 & \sum_{i=1}^{2} 2^{k_i} & \cdots & 0 & \cdots & 0 & \sum_{i=1}^{m} 2^{k_i} \end{bmatrix}$$

Theorem

There is a classical circuit implementing the inversion in $\mathbb{F}_{2^n}$ with AND-depth $\lceil \log_2(n-1) \rceil$ and AND-count $\omega(n)(\mathrm{HW}(n-1) + \lfloor \log_2(n-1) \rfloor - 1)$.

- RAIN-128 (an MPC-friendly block cipher, CCS 2022) S-box, the inversion in $\mathbb{F}_{2^{128}}$:

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 & 32 & 64 & 127 \\ 1 & 0 & 3 & 7 & 15 & 31 & 63 & 0 \end{bmatrix},$$

  AND-depth 7 and AND-count 24057.

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Implementing the Inversion in $\mathbb{F}_{2^{2m}}$

- Tower field structure: $\alpha = a_0\beta^{2^m} + a_1\beta$,

$$\alpha^{2^m} = a_1\beta^{2^m} + a_0\beta, \quad b = \alpha \cdot \alpha^{2^m} = a_0a_1\beta^{2^{m+1}} + (a_0^2 + a_1^2)\beta^{2^m+1} + a_0a_1\beta^2$$

  and $\alpha^{-1} = b^{-1}a_1\beta^{2^m} + b^{-1}a_0\beta$.

- Three Steps: (1) compute $b \in \mathbb{F}_{2^m}$; (2) b^{-1}; (3) $b^{-1}a_1, b^{-1}a_0$;

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- **Merge Step (2)** (compute a parallel addition chain of $m-1$) **and (3)**:

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow A_2 = \begin{bmatrix} 1 & 2 & 3 \boxplus a_1 \\ 1 & 1 \boxplus a_0 & 3 \boxplus a_0 \\ 1 & 1 \boxplus a_1 & 0 \end{bmatrix}$$

$$\begin{array}{l} b \xrightarrow{\wedge 4} b^{100_2} \xrightarrow{\times b^1} b^{101_2} \xrightarrow{\wedge 2} b^{1010_2} \xrightarrow{\times b^{100_2}a_1} b^{1110_2}a_1 = b^{-1}a_1 \\ \quad\quad b^{100_2} \xrightarrow{\times a_0} b^{100_2}a_0 \rightarrow b^{100_2}a_0 \xrightarrow{\times b^{1010_2}} b^{1110_2}a_0 = b^{-1}a_0 \\ \quad\quad b^{100_2} \xrightarrow{\times a_1} b^{100_2}a_1 \rightarrow b^{100_2}a_1 \end{array}$$

Theorem

Let $m-1=\sum_{i=1}^{s}2^{k_i}$ *with* $0\le k_1<k_2<\cdots<k_s$ *and* $s\ge 2$. *If* $k_s>\lceil\log_2(\sum_{i=1}^{s-1}2^{k_i})\rceil$, *then* $\alpha^{-1}\in\mathbb{F}_{2^{2m}}$ *can be implemented by a classical circuit with the minimal AND-depth* $\lceil\log_2(2m-1)\rceil$ *and AND-count* $\omega(m)(HW(m-1)+\lfloor\log_2(m-1)\rfloor+3)$.

$$A=\begin{bmatrix} 1 & 2 & \cdots & 2^{k_1} & 2^{k_1+1} & \cdots & 2^{k_2} & 2^{k_2+1} & \cdots & 2^{k_{s-1}} & 2^{k_{s-1}+1} & \cdots & \mathbf{2^{k_s}} & 0 \\ 0 & 0 & \cdots & 2^{k_1} & 0 & \cdots & 0 & \sum_{i=1}^{2}2^{k_i} & \cdots & 0 & \sum_{i=1}^{s-1}\mathbf{2^{k_i}} & \cdots & 0 & m-1 \end{bmatrix}.$$

- AES S-box: AND-depth 3 , **AND-count 42** (Asiacrypt 2022, AND-count 78).

- Convert to Clifford+T circuit with our new conversion algorithm.

Table: Clifford+T circuits (including uncomputation) for the AES S-box with T-depth 3.

Type	#CNOT	#1qClifford	#T	#M	T-depth	Full Depth	Width	Source
$\mathfrak{C}^0/\mathfrak{C}^*$	1396/1398	494	312	78	3	119	218/226	Asiacrypt 2022
$\mathfrak{C}^*$	1110	448	264	66	3	92	129	IEEE TC 2024
$\mathfrak{C}^0/\mathfrak{C}^*$	**827/856**	**266/298**	**168**	**34/42**	3	**85/87**	**89/97**	This work

New Results for Implementing Quantum Oracles for AES

Table: The Costs of Grover Oracles based on the Pipeline Structure.

#CNOT	#1qClifford	#T	#M	T-depth	Full Depth	Width	Source
456040	179200	105600	26400	60	1802	3796	IEEE TC 2024
353160	**119200**	**67200**	**16800**	60	**1782**	**3156**	This work

Table: The Costs of Encryption Oracles based on the Interlacing-Uncompute Structure.

#CNOT	#1qClifford	#T	#M	T-depth	Full Depth	Width	Source
364360	144584	84480	21120	33	1078	4128	IEEE TC 2024
281896	**96584**	**53760**	**13440**	33	**1066**	**3104**	This work

Outline

Minimal-width NCT implementations

- An invertible vectorial Boolean function F is **a permutation $\mathcal{P}$ on $\mathbb{F}_2^n$.**
- It always has an in-place NCT implementation:
  - $\mathcal{P}$ is even: 0 ancilla qubit, minimal-width$= n$
  - $\mathcal{P}$ is odd: 1 ancilla qubit, minimal-width$= n + 1$

Vivek V. Shende, Aditya K. Prasad, Igor L. Markov, John P. Hayes: Synthesis of reversible logic circuits. *IEEE Trans. Comput. Aided Des. Integr. Circuits Syst.* 22(6): 710-722 (2003)

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SAT-based Method

- ≤ 5 qubit: at most one Toffoli gate in each layer.
- $F = \mathcal{S}_k \circ \mathcal{T}_k \circ \mathcal{S}_{k-1} \circ \cdots \circ \mathcal{S}_2 \circ \mathcal{T}_2 \circ \mathcal{S}_1 \circ \mathcal{T}_1 \circ \mathcal{S}_0$: $\mathcal{T}$ corresponds a Toffoli gate, $\mathcal{S}$ corresponds to all possible affine transformation.
  - **Affine Layers**: $\forall i \in \{1, \ldots, w\}, \forall j \in \{0, \ldots, k\}$, $B_{i,j} = c_{1,i}^{(j)} A_{1,j} + \cdots + c_{w,i}^{(j)} A_{w,j} + d_i^{(j)}$, for some Boolean variables $c_{1,i}^{(j)}, \ldots, c_{w,i}^{(j)}$ and $d_i^{(j)}$.
  - **Toffoli layers**: $\forall j \in \{1, \ldots, k\}$, $A_{3,j} = B_{3,(j-1)} + B_{1,(j-1)} \cdot B_{2,(j-1)}$, and $A_{i,j} = B_{i,(j-1)}$ if $i \neq 3$.
  - **Inputs and outputs**: $\forall i \in \{1, \ldots, n\}$, $A_{i,0} = x_i$, $B_{i,k} = f_i(x_1, x_2, \ldots, x_n)$.
  - **Ancilla qubit (for an odd F)**: If $w = n + 1$, $A_{w,0} = 0$ and $B_{w,k} = 0$.

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Improve the Encoding Scheme

- Exclude equivalent solutions (Fix the Toffoli gate):
  - $\mathcal{T}_{i,j,k} = \text{Rewire} \circ \mathcal{T}_{1,2,3} \circ \text{Rewire}^{-1}$

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$$\mathcal{S}_1 \circ \mathcal{T}_{i,j,k} \circ \mathcal{S}_2 = (\mathcal{S}_1 \circ \text{Rewire}) \circ \mathcal{T}_{1,2,3} \circ (\text{Rewire}^{-1} \circ \mathcal{S}_2) = \mathcal{S}_1' \circ \mathcal{T}_{1,2,3} \circ \mathcal{S}_2'$$

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Applications

Table: Costs of the 5-qubit quantum circuits for the χ function of SHA3.

Type	#NOT	#CNOT	#Toffoli	Width	Toffoli-depth	Full Depth	Source
NCT	12	0	**7**	**5**	7	10	This work
Type	#CNOT	#1qClifford	$\#T$	Width	T-depth	Full Depth	Source
Clifford+T	79	24	70	12	30	103	eprint 2023
	49	**24**	**49**	**5**	**21**	**66**	This work

- Implementing χ requires at least 5 Toffoli gates without width limit. Toffoli-count 7 is almost optimal.

Table: Costs of the 5-qubit quantum circuit for the S-box of ASCON.

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Method Based on MCT Implementations

- C^mX gate: an MCT (Multiple Controlled Toffoli) gate with m control qubits, maps $|x_1, x_2, \cdots, x_k\rangle |x_{k+1}\rangle$ to $|x_1, x_2, \cdots, x_k\rangle |x_{k+1} \oplus x_1 x_2 \cdots x_k\rangle$
- $C^0X =$ NOT, $C^1X =$ CNOT, $C^2X =$ Toffoli.
- **Our idea**:
  1. Construct an MCT implementation without ancilla qubit;
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Decomposition of C^mX

Lemma (Physical Review A 1995, Barenco et al.)

A C^mX gate can be implemented by two C^pX gates and two (one) C^qX gate and one dirty (clean) ancilla qubit, where $p + q = m + 1$

Figure: Implementing a C^7X gate with one ancilla qubit.

Decomposition of C^mX

Lemma (Physical Review A 1995, Barenco et al.)

A C^mX gate with $m > 3$ can be implemented by $4(m-2)$ Toffoli gates and $m-2$ ancilla qubits(dirty qubits).

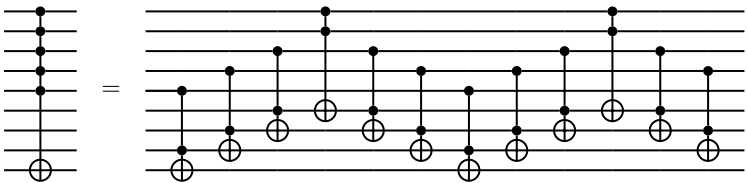

Figure: Implementing a C^5X gate with three **dirty ancilla qubits**.

- With on ancilla qubit (dirty or clean), one can decompose C^mX into Toffoli gates.

Decomposition of C^mX

Lemma (Physical Review A 1995, Barenco et al.)

A C^mX gate with $m > 3$ can be implemented by $4(m-2)$ Toffoli gates and $m-2$ ancilla qubits(dirty qubits).

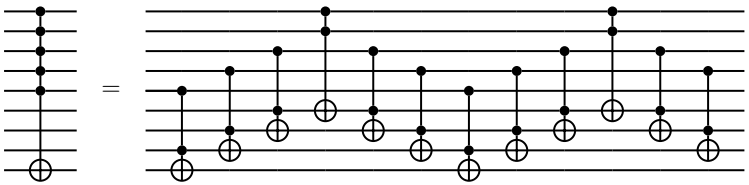

Figure: Implementing a C^5X gate with three **dirty ancilla qubits**.

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MCT Implementations from Tensor Decomposition

Lee et al.: An Algorithm for Reversible Logic Circuit Synthesis Based on Tensor Decomposition. arXiv:2107.04298, 2021

- $M_k M_{k-1} \cdots M_1 \mathcal{P}_n = \mathcal{P}_{n-1} \otimes I_2$
- $\mathcal{P}_{n-1}$ only involve the first $n-1$ wires. Recursively decompose $\mathcal{P}_{n-1}$.
- We can obtain a MCT decomposition consisting of $\{X, C^1X, C^2X, \ldots, C^{n-1}X\}$
  - For an even permutation, the decomposition does not contain $C^{n-1}X$ in most cases.
  - Combing the decomposition of C^mX, we can obtain the minimal-width NCT circuit for $\mathcal{P}_n$.

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Minimal-width Implementations for the AES S-box

- Tensor Decomposition based method + SAT-based method (for 5-qubit sub-circuits)

Table: Costs of the 9-qubit NCT circuit for the AES S-box.

#NOT	#CNOT	#Toffoli	Width	Toffoli-depth	Full Depth
233	885	833	9	793	1594

Table: Costs of different 9-qubit Clifford+T circuits for the AES S-box.

#Clifford (CNOT, 1qClifford)	#T	Width	T-depth	Full Depth	Source
$\leq$ 12631(- , -)	$\leq$ 9295	9	-	-	PQCrypt 16
7465 (6028, 1437)	**3783**	**9**	**1501**	**7180**	This work
13008 (10633, 2375)	**3447**	**9**	**1274**	**9954**	This work (T-par)

Minimal-width Implementations for a Pair of S-boxes

- Implementing a pair of S-box, (S_1, S_2): use a qubit allocated for implementing S_2 as the dirty ancilla qubit when implementing S_1, and vice versa.

Table: Costs of the 16-qubit quantum circuits for a pair of AES S-boxes.

Type	#NOT	#CNOT	#Toffoli	Width	Toffoli-depth	Full Depth
NCT	502	1770	2140	16	1714	2990
Type	#1qClifford	#CNOT	#T	Width	T-depth	Full Depth
Clifford+T	3628	14786	9008	16	2949	15253
Clifford+T (T-par)	5066	23976	8360	16	2774	18883

- We can construct a 256-qubit quantum circuit of AES-128, achieving the theoretical minimum.

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Thank you for your attention!

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