Fully Homomorphic Encryption for Cyclotomic Prime Moduli

The Generalized BFV scheme

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Eurocrypt 2025

BFV versus CLPX

- BFV scheme
  - Computations over $\mathbb{F}_{p^d}$ for small p and d
- CLPX scheme
  - Computations over $\mathbb{Z}_p$ for huge p

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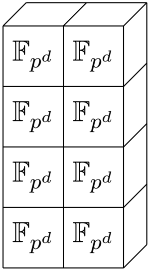

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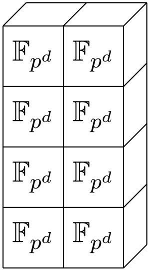

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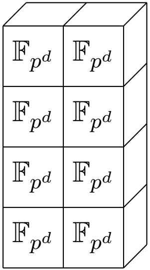
  - Computations over $\mathbb{F}_{p^d}$ for small p and d
  - Pack full hypercube in a ciphertext: ℓ slots
  - Arbitrary computations via bootstrapping
  - Example: $p^d = 2^{20}$ and $\ell = 1200$



- CLPX scheme
  - Computations over $\mathbb{Z}_p$ for huge p
  - No packing
  - No bootstrapping
  - Example: $p = 2^{2^{14}} + 1$

Research questions

- Can we define something in between BFV and CLPX?
  - Computations over $\mathbb{F}_{p^d}$ with large p and small d
  - Possibility of packing
  - Bootstrapping

Notations

- We use the m-th cyclotomic polynomial $\Phi_m(x)$
  - The corresponding cyclotomic field is $\mathcal{K} = \mathbb{Q}[x]/(\Phi_m(x))$
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  - Define **plaintext ring** $\mathcal{R}_t = \mathcal{R}/t\mathcal{R}$
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  - So $\mathcal{R}_t = \mathcal{R}/t\mathcal{R} = \mathbb{Z}[x]/(\Phi_m(x), t(x))$
  - BFV uses integer t and CLPX uses $t(x) = x - b$

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$$(\boldsymbol{c}_0, \boldsymbol{c}_1) = \left(\left\lfloor \frac{q}{t} \cdot \boldsymbol{m} \right\rceil + \boldsymbol{a} \cdot \boldsymbol{s} + \boldsymbol{e}, -\boldsymbol{a} \right) \pmod{q}$$

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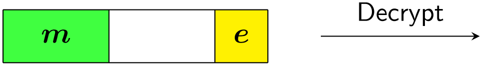
- Decryption via scale-and-round:



$$\boldsymbol{m} = \left\lfloor \frac{t}{q} \cdot (\boldsymbol{c}_0 + \boldsymbol{c}_1 \cdot \boldsymbol{s}) \right\rceil$$

The GBFV scheme: flattening

- New function to choose small representatives modulo t:

$$\mathsf{Flatten}_t\colon \mathcal{R}_t \to \mathcal{R}\colon \boldsymbol{m} \mapsto t \cdot \left[\frac{\boldsymbol{m}}{t}\right]_1$$

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$$||\hat{\boldsymbol{m}}||_\infty^{\mathsf{can}} \leq (\varphi(m)/2) \cdot ||t||_\infty^{\mathsf{can}}$$

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- $\text{Flatten}_{(x-4)^2}(256 \cdot x + 512) = -2 \cdot x^6 + 11 \cdot x^5 - 6 \cdot x^4$

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- We want to multiply ciphertext $(\boldsymbol{c}_0, \boldsymbol{c}_1)$ with plaintext $\boldsymbol{m}$
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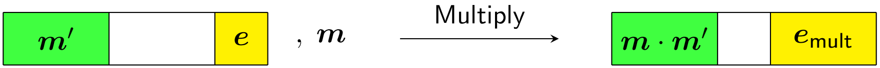

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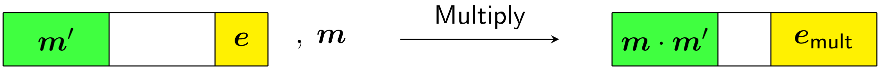

- New noise $\boldsymbol{e}_{\mathsf{mult}} = \hat{\boldsymbol{m}} \cdot \boldsymbol{e}$ satisfies

$$||\boldsymbol{e}_{\mathsf{mult}}||_\infty^{\mathsf{can}} \leq (\varphi(m)/2) \cdot ||t||_\infty^{\mathsf{can}} \cdot ||\boldsymbol{e}||_\infty^{\mathsf{can}}$$

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- Native arithmetic modulo an integer $p = \Phi_r(b^{m/(rk)})$
  - If p is a prime number then we call it a **cyclotomic prime**

Packing-noise trade-off: Fermat family

- Consider $m = 2^{15}$ and **Fermat prime** $p = \Phi_2(2^{16}) = 2^{16} + 1$
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- Trade-off between #slots and noise:
  - Number of slots: k
  - Noise: increases with b

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Number of slots	1024	2048	4096	8192	16384
Mult noise (bits)	10.5	11.2	13.0	17.3	25.1

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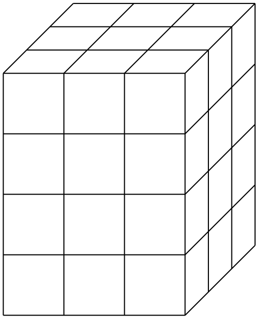
Number of slots	256	512	1024	2048	4096	8192	16384
Mult noise (bits)	10.3	11.3	13.1	17.2	25.2	41.3	73.0

The BFV hypercube

- Slots of $\mathbb{F}_{p^d}$-elements are arranged in hypercube



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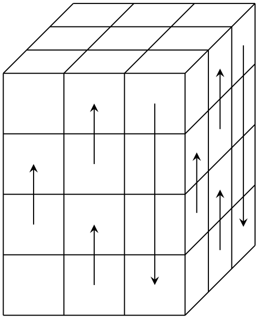

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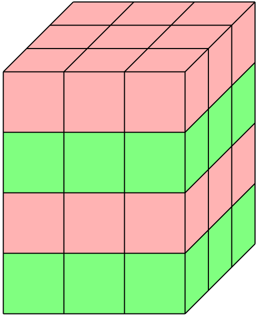

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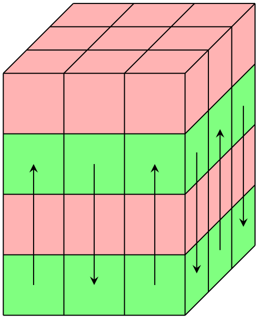

Ring switching

- Change cyclotomic ring *during computation* and select subset of the slots
  - Example for Goldilocks prime:

$$m = 3 \cdot 2^4,\ t(x) = x^2 - 256 \quad \longrightarrow \quad m' = 3 \cdot 2^{3},\ t'(x) = x^{1} - 256$$

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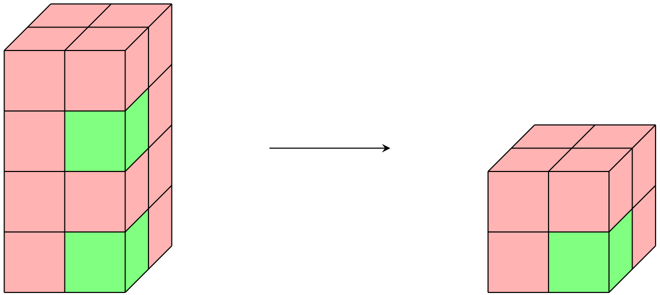

Conversion to BFV

- GBFV ciphertext $(\boldsymbol{c}_0, \boldsymbol{c}_1)$ satisfies $\boldsymbol{c}_0 + \boldsymbol{c}_1 \cdot \boldsymbol{s} = (q/t) \cdot (\boldsymbol{m} + t \cdot \boldsymbol{a} + \boldsymbol{v})$
  - Divide by $p/t \in \mathcal{R}$ and round:

$$\left\lfloor \frac{t}{p} \cdot \boldsymbol{c}_0 \right\rceil + \left\lfloor \frac{t}{p} \cdot \boldsymbol{c}_1 \right\rceil \cdot \boldsymbol{s} \approx \frac{q}{p} \cdot (\boldsymbol{m} + t \cdot \boldsymbol{a} + \boldsymbol{v})$$

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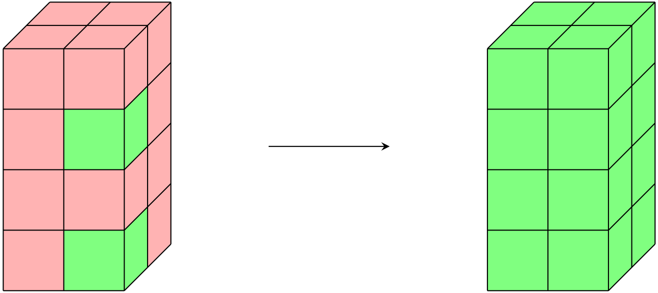

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- Conversion uses available space inefficiently
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  1. Convert GBFV to BFV and put zero in garbage slots
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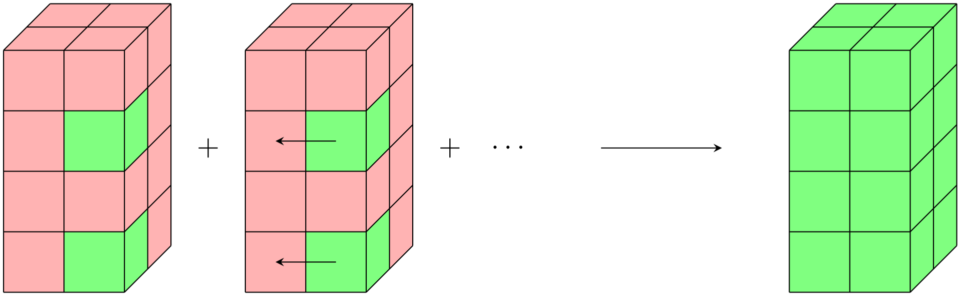

GBFV bootstrapping

$$\begin{array}{c}
\mathsf{Enc}_t(m_1, \ldots, m_{\ell'}) \\
\downarrow \text{GBFV to BFV} \\
\mathsf{Enc}_p(m_1, \ldots, m_{\ell'}, \ldots, m_\ell) \\
\downarrow \text{Noisy expansion} \\
\mathsf{Enc}_{p^2}(p \cdot m_1 + e_1, \ldots, p \cdot m_{\ell'} + e_{\ell'}, \ldots, p \cdot m_\ell + e_\ell) \\
\downarrow \text{BFV to GBFV} \\
\mathsf{Enc}_{t^2}(p \cdot m_1 + e_1, \ldots, p \cdot m_{\ell'} + e_{\ell'}) \\
\downarrow \text{Digit removal} \\
\mathsf{Enc}_t(m_1, \ldots, m_{\ell'})
\end{array}$$

Bootstrapping results

Table: results for $m = 2^{15}$ and $p = 2^{16} + 1$

Number of slots ℓ'		1024	2048	4096	8192
Bits per multiplicative level		11	12	14	18
Noise (bits)	Noisy expansion	111	111	114	118
	Digit removal	82	91	113	161
	Remaining	**124**	**115**	**90**	**38**
Execution time (sec)	Noisy expansion	1.41	1.44	1.44	1.46
	Digit removal	0.53	0.54	0.54	0.55
	Total	**1.94**	**1.98**	**1.98**	**2.01**

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- Bootstrapping converts to regular BFV

Blog post:

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Thank you for listening!