Cryptanalysis of rank-2 module-LIP: a single real embedding is all it takes

Bill Allombert, Alice Pellet-Mary (Université de Bordeaux),
Wessel van Woerden (Université de Bordeaux & PQShield) .

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This talk: What about other number fields?

Number fields and module-LIP

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- Cause of most technical problems in this work

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[MPMPW24]

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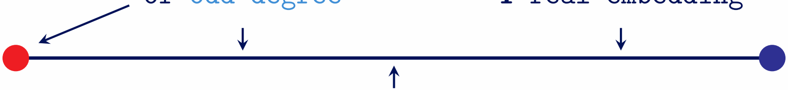

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- This is a lattice problem!

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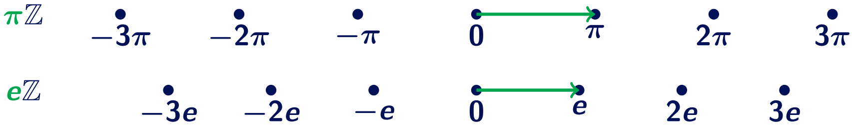

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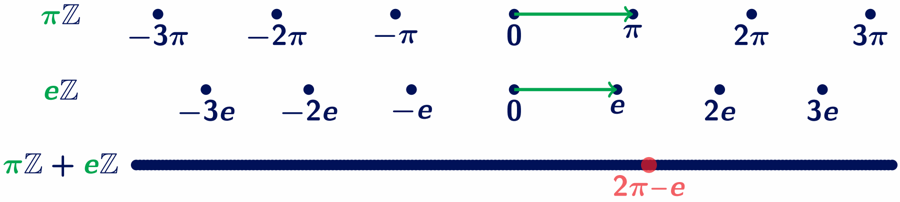

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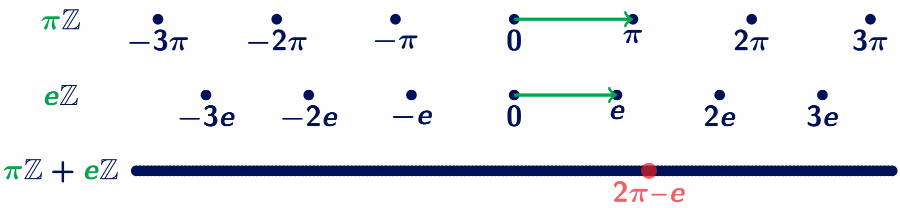

Small combinations only ($S = \{-2, -1, 0, 1, 2\}$)

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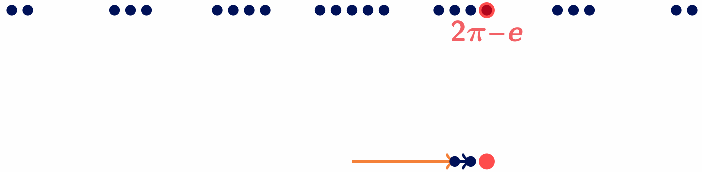

integer combination π and e close to v

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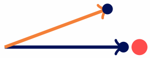

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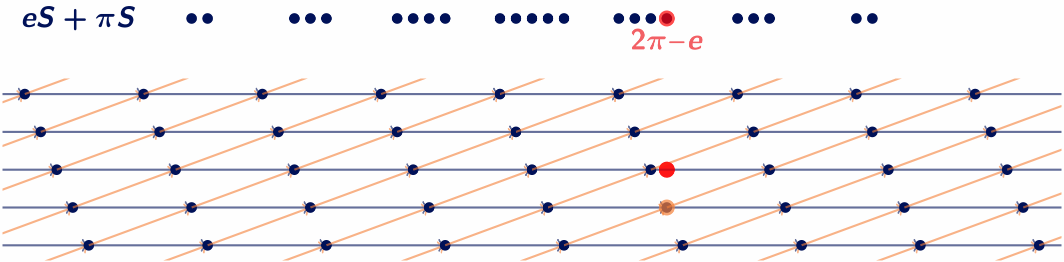

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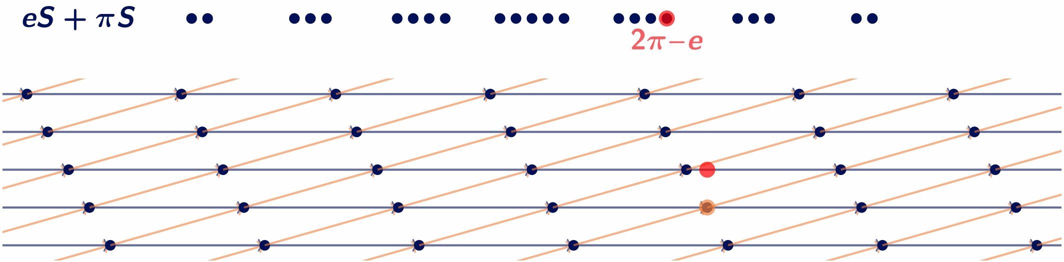

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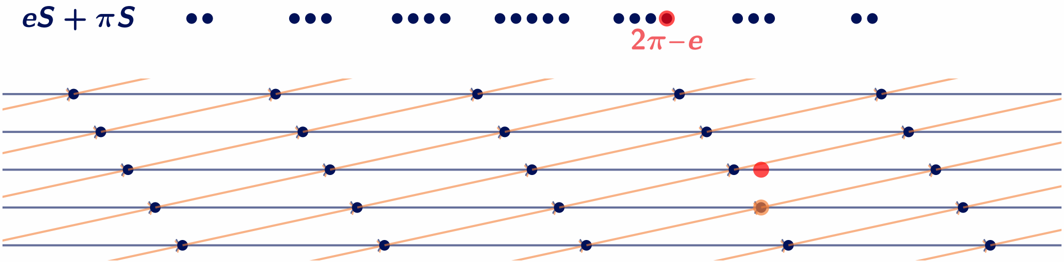

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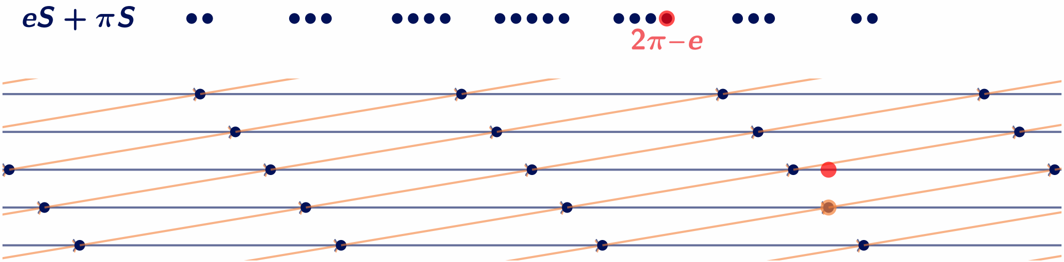

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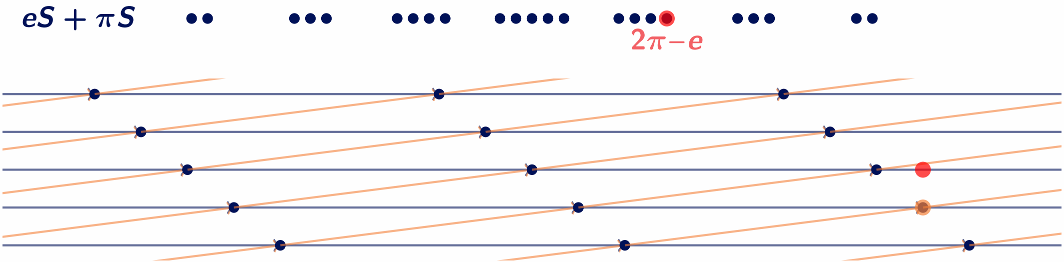
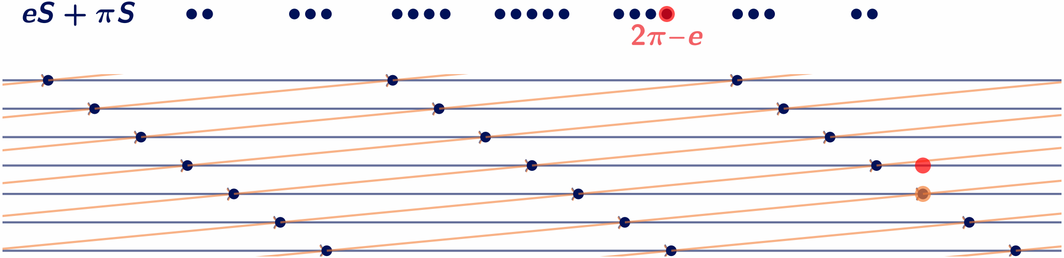


$$\text{integer combination } \begin{pmatrix} 2^\lambda \pi \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2^\lambda e \\ 1 \end{pmatrix} \text{ close to } \begin{pmatrix} 2^\lambda v \\ 0 \end{pmatrix}$$

- Increasing λ moves wrong combinations further away
- Distance to correct combination $\approx$ same

Recovery from a single real embedding

$eS + \pi S$



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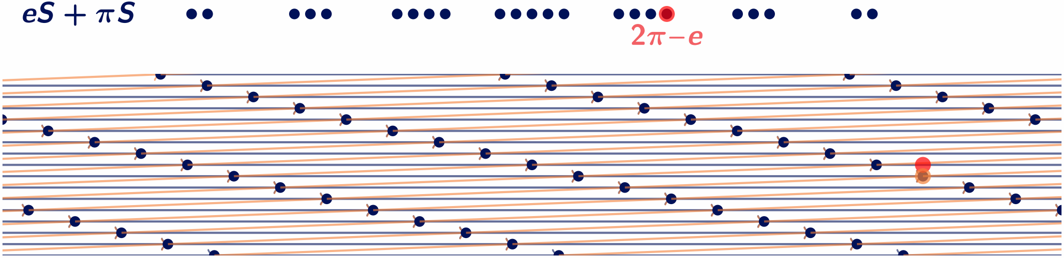

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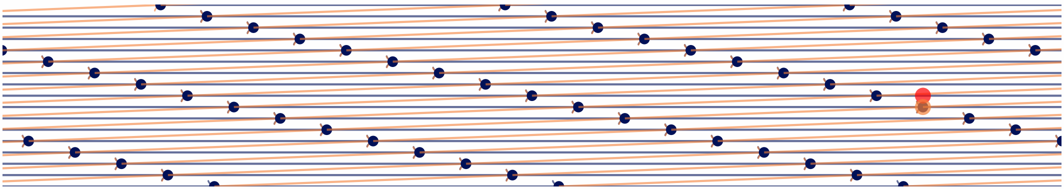

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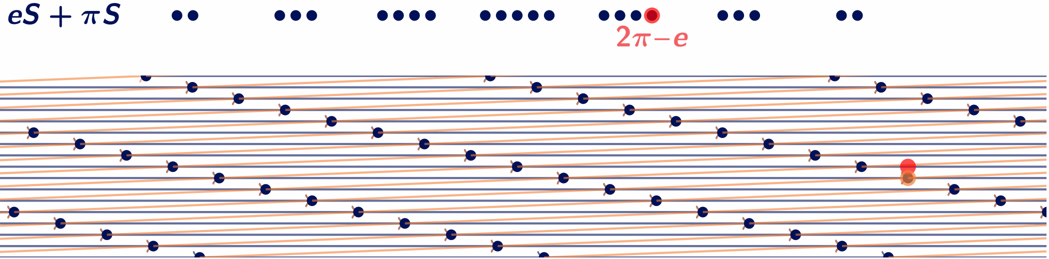

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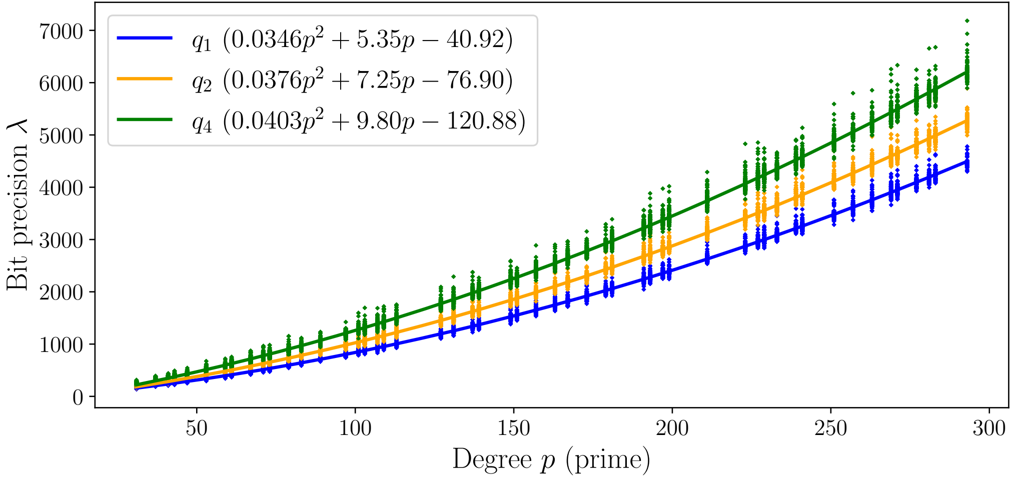
Required precision for NTRU Prime field

Lemma [PMS21]: $\boldsymbol{\lambda \geq \mathrm{poly}(\log(|\Delta_K|), \log\|P\|, \log(\|x\|))}$ is sufficient



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- Use trick from previous talk to recover ideal $(a+bi)\mathcal{O}_L$

Notation: $z = a+bi$, $z' = c+di$

Lemma: Ideal recovery (previous talk!)

$$z\mathcal{O}_L = \mathcal{O}_L \cap zz'^{-1}\mathcal{O}_L = \mathcal{O}_L \cap q_1(\det(B)i + q_2)^{-1}\mathcal{O}_L.$$

Generalized Gentry-Szydlo algorithm

When L is a CM field Gentry-Szydlo recovers z from $z\mathcal{O}_L$ and $\bar{z}z$.

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(experimentally verified for $K =$ NTRU Prime and $L =$ cyclotomics or random field)

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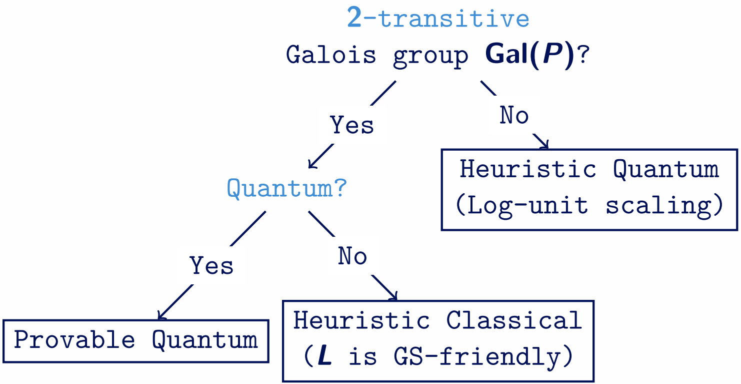
We have 3 approaches to solve this.

Make your choice



Make your choice

2-transitive
Galois group $\mathbf{Gal}(P)$?

- Yes — Can recover all $|\tau_j(z)|$
  - Quantum?
    - Yes → Provable Quantum
    - No → Heuristic Classical (L is GS-friendly)
- No → Heuristic Quantum (Log-unit scaling)

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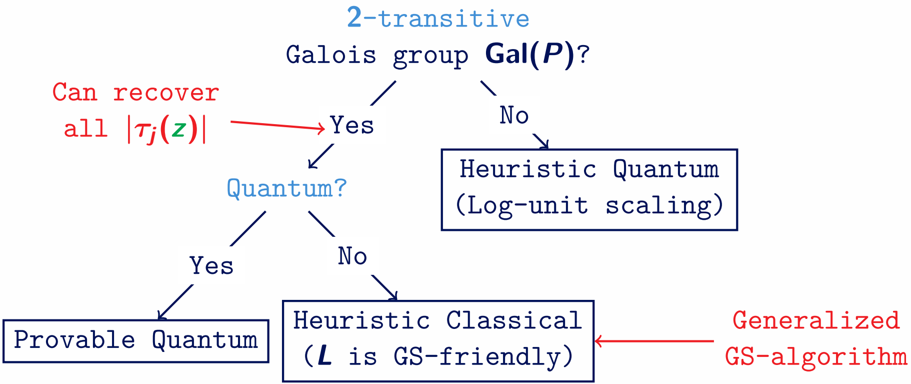

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Generalized
GS-algorithm

PIP + log-unit lattice

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2-transitive
Galois group $\mathsf{Gal}(P)$?

Use only one $|\tau_j(z)|$ & LLL
(similar to first step)

Can recover all $|\tau_j(z)|$

Yes

No

Heuristic Quantum
(Log-unit scaling)

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Yes

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Conclusion

- Security of rank-2 module-LIP depends on the number field
- Real embeddings cause problems

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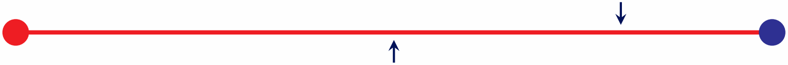
New state of cryptanalysis:

NTRU Prime field

1 real embedding



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Broken!
[MPMPW24]

What about here?

also broken!

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HAWK (CM field)
previous talk:
quaternions!

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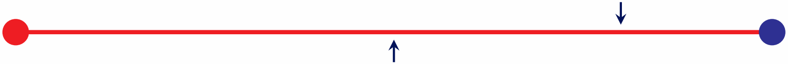

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