On Algebraic Homomorphic Encryption and its Applications to Doubly-Efficient PIR

Hiroki Okada<sup>1</sup>, Rachel Player<sup>2</sup>, Simon Pohmann<sup>2</sup>, Christian Weinert<sup>2</sup>

<sup>1</sup> KDDI Research and The University of Tokyo, Japan

<sup>2</sup> Royal Holloway, University of London, UK

Some slides adapted from slides of Simon Pohmann



# Background and prior work



# Homomorphic encryption





- Client can outsource computation of F(x) on their data x
- Server does not learn x or F(x)



















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#### Goals:

- Server does not learn *i*
- Communication cost is less than downloading whole database (*N*)

PIR can be achieved from HE with data x encoding i and polynomial  $f_{DB}(x)$  encoding function "retrieve data at location x"



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- What about computational cost?



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Preprocessing?



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- What about computational cost?

No

Preprocessing?

Computational cost must be  $\Omega(N)$  [ВІМОО]



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[KU11, Theorem 2.1] Let  $f \in R[X_1, ..., X_m]$  be a degree-dpolynomial. Then we can build a datastructure that allows us to compute  $f(x_1, ..., x_m)$  in time poly $(d, m, \log \# R)$ 



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Problem: While  $f_{DB}$  is a polynomial, Eval( $f_{DB}$ ,  $\cdot$ ) may not be (e.g. due to modulus switching, relinearisation)



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Algebraic Somewhat Homomorphic Encryption (ASHE)

A somewhat homomorphic encryption scheme such that for polynomial f:

$$\mathsf{Eval}(f, ct_1, \dots, ct_m) = f(ct_1, \dots, ct_m)$$

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[KU11] K. S. Kedlaya and C. Umans. Fast Polynomial Factorization and Modular Composition. SIAM J. Comput. 40.6 (2011) [LMW23] W. Lin, E. Mook, D. Wichs. Doubly Efficient Private Information Retrieval and Fully Homomorphic RAM Computation from Ring LWE. STOC 2023.



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The ciphertext space must be a ring!

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# Our contributions





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<u>Theorem</u> (informal) A post-quantum ASHE scheme that can evaluate polynomials of degree d has ciphertext ring size  $\Omega(2^d)$ 



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Lower bound on performance of post-quantum ASHE



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Lower bound on performance of post-quantum ASHE

Scheme	ASHE?	Ciphertext ring/space size	
BGV, BFV	×	O(poly(d))	
BV	$\checkmark$	$O(2^{d^2})$	

[BV11] Z. Brakerski, V. Vaikuntanathan. Fully Homomorphic Encryption from Ring-LWE and Security for Key Dependent Messages. CRYPTO 2011.
 [BGV12] Z. Brakerski, C. Gentry, V. Vaikuntanathan. (Leveled) Fully Homomorphic Encryption without Bootstrapping. ITCS 2012.
 [Bra12] Z. Brakerski. Fully Homomorphic Encryption without Modulus Switching from Classical GapSVP. CRYPTO 2012.
 [FV12] J. Fan and F. Vercauteren. Somewhat Practical Fully Homomorphic Encryption. Eprint 2012/144.



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#### **Idea:** Use double CRT to work over $\mathbb{F}_p[T]$ instead of $R_q[T]$





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Idea: Decompose the polynomial f being evaluated as

$$f(b_1 + a_1T, \dots, b_m + a_mT) = \sum_i T^i f_i(a_1, \dots, a_m, b_1, \dots, b_m)$$

to evaluate 2m-variate polynomials over  $\mathbb{F}_p$ 

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#### **Definition (** $\{0,1\}$ **-CRT set)**

Define S as the set of elements in  $R_q$  such that  $S := \iota^{-1}(\{0, 1\}^{nr})$  where  $\iota$  is the double-CRT isomorphism and r is the number of divisors of q.

# $\{0,1\}$ -CRT-RLWE





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Let  $s \in R_q$  be secret. Distinguish samples  $(a_i, b_i)$  from uniform, where  $b_i \in S$ and  $a_i = s^{-1}(b_i - e_i)$  for  $e_i$  chosen from some small error distribution.

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Let q satisfy some technical constraints. If standard RLWE with preprocessing on  $R_q$  is hard, then {0,1}-CRT-RLWE with preprocessing on  $R_q$  is also hard.

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The adversary is allowed to run a preprocessing phase with only R and q as input before querying the RLWE oracle





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Reduction in read queries:

- 8× for parameters we can run
  - 23× for larger parameters



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#### Performance compared to [OPPW24] Reduction in read queries: Speedup of $8 \times$ for parameters we can run more than $4 \times$ 23× for larger parameters $2^{53}$ [OPPW24], m = 4 $2^{49}$ [OPPW24], m = 5new, m = 4Total read queries $2^{45}$ [OPPW24], m = 6new, m = 5 $2^{41}$ $2^{37}$ One variable "for free" $2^{33}$ $2^{29}$ $2^{13}$ $2^{19}$ $2^{21}$ $2^{23}$ $2^{25}$ $2^{27}$ $2^{29}$ $2^{11}$ $2^{15}$ $2^{17}$

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Database size N

43





45











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Lower bound on performance of post-quantum ASHE



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Modified BV scheme based on {0,1}-CRT-RLWE



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Additional improvement to polynomial evaluation datastructure

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Improved DEPIR implementation



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Reduction from RLWE to certain reduced entropy variants of RLWE, including {0,1}-CRT-RLWE



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Preliminary cryptanalysis of {0,1}-CRT-RLWE



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Are there other approaches for more efficient DEPIR?



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Is {0,1}-CRT-RLWE concretely as hard as RLWE?



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Is {0,1}-CRT-RLWE concretely as hard as RLWE?

Are our reduced entropy RLWE variants useful for other applications?

# Thank you! Any questions?

# Full version on eprint: 2024/1307 Implementation: github.com/FeanorTheElf/ashe-depir

### rachel.player@rhul.ac.uk https://rachelplayer.github.io



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