Low-Bandwidth Mixed Arithmetic in VOLE-Based ZK from Low-Degree PRGs

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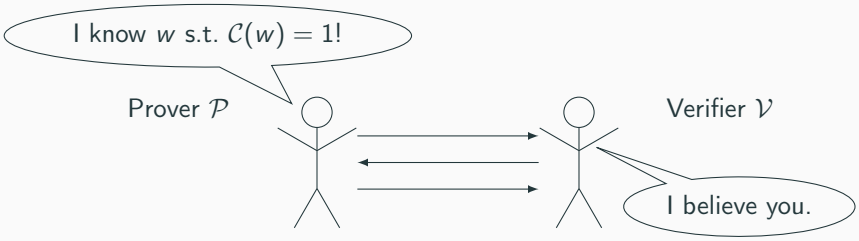
INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

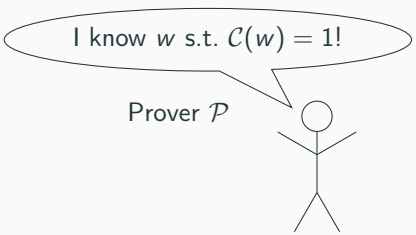
AARHUS UNIVERSITY

Zero-Knowledge Proofs (of Knowledge)



- Security Properties:
  - Soundness
  - Zero-Knowledge

Zero-Knowledge Proofs (of Knowledge) for Circuits



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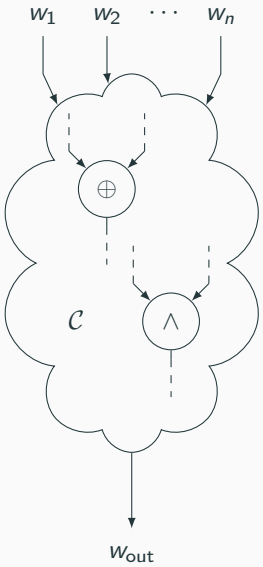
- $\mathcal{C}$ is a circuit
  - Boolean over $\mathbb{F}_2 = \{0, 1\}$



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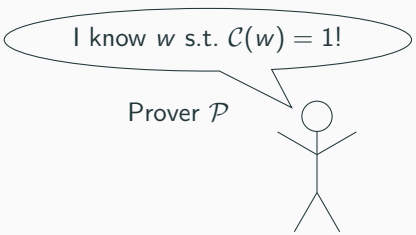

- Security Properties:
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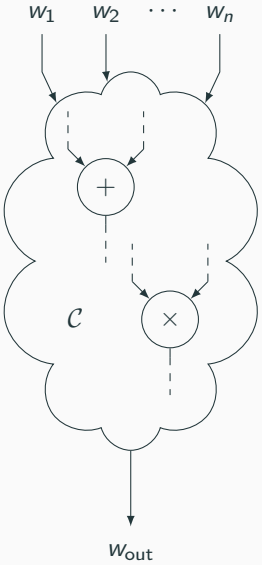
  - Boolean over $\mathbb{F}_2 = \{0, 1\}$
  - arithmetic over a larger ring or field



Hybrid Circuits and Mixed Arithmetic

Boolean and arithmetic circuits have different strengths:

- Boolean over $\mathbb{F}_2$: e.g., simple comparisons
- arithmetic over $\mathbb{F}_p$: cheap addition/multiplication

$\Longrightarrow$ combine both into a hybrid circuit

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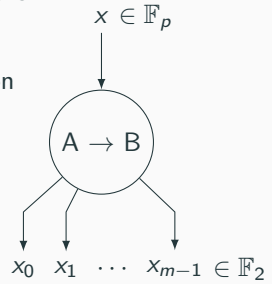

such that $x = \sum_{i=0}^{m-1} 2^i \cdot x_i$

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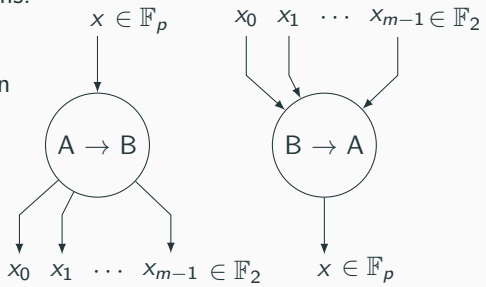

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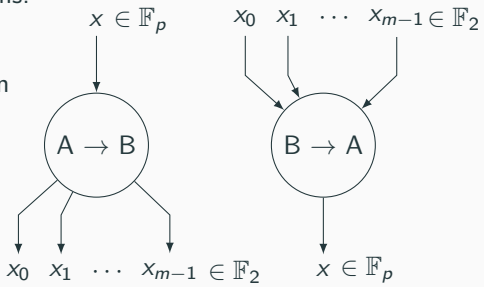
More Gadgets:

- (fixed-point) truncation: $x \mapsto \lfloor x/2^k \rfloor$
- MSB extraction: $x \mapsto x_{m-1}$
- ReLU: $x \mapsto (1 - x_{m-1}) \cdot x$

Conversion Gates



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1. Introduction to VOLE-based Zero-Knowledge

This Talk

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2. Proving Conversions

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3. Committed Bits from Low-Degree PRGs

VOLE-based Zero-Knowledge

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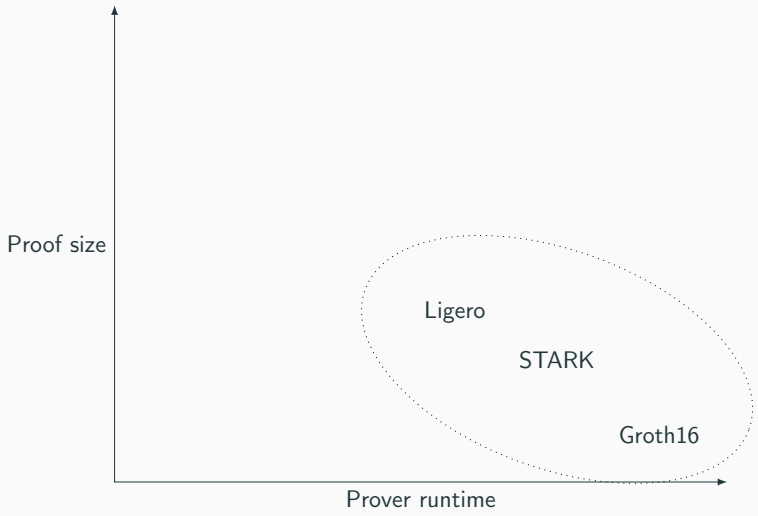

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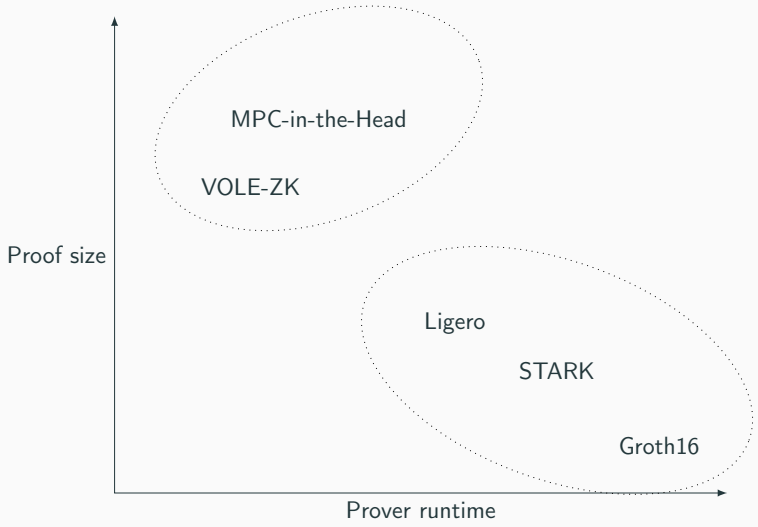

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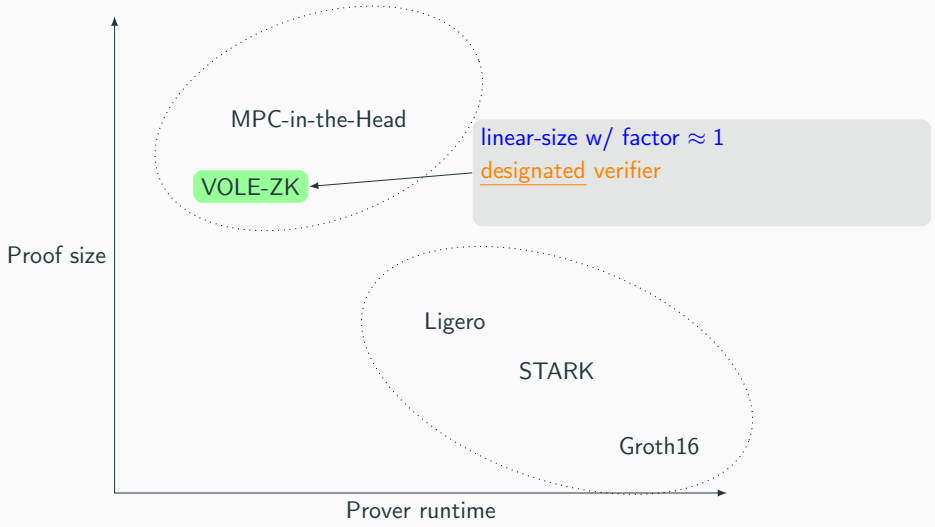

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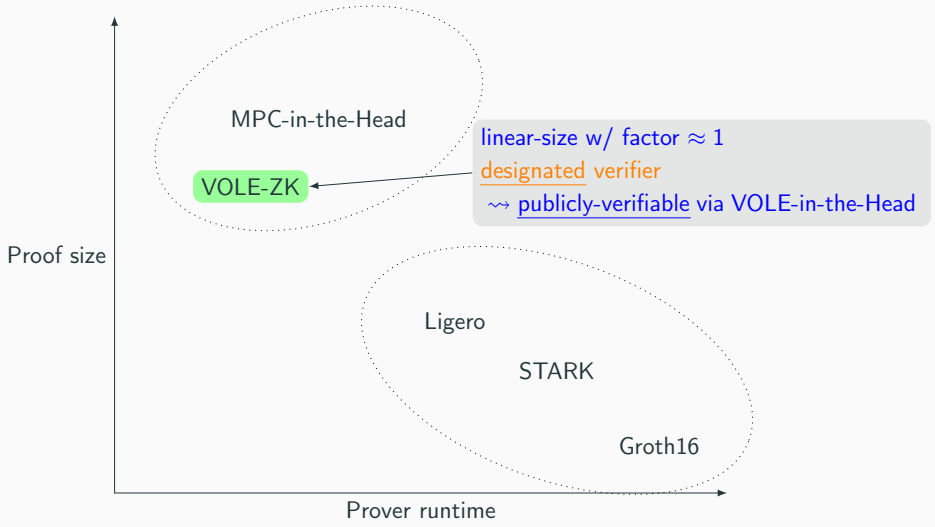
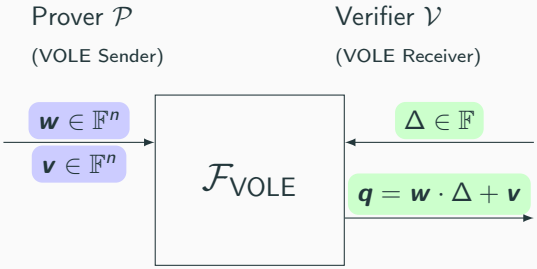
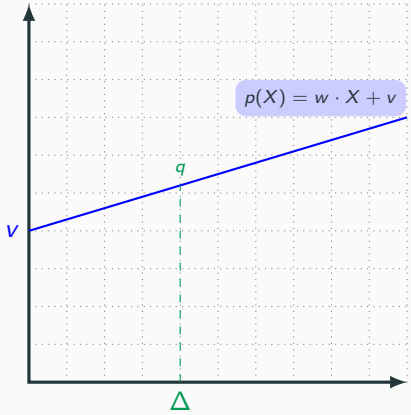


Vector Oblivious Linear Evaluation (VOLE)





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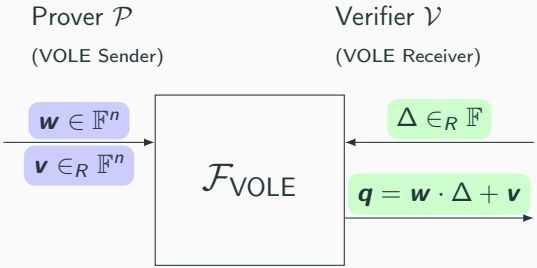
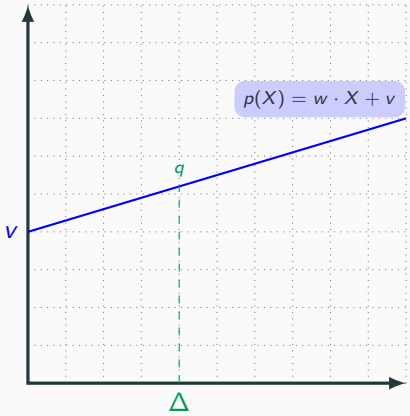
Vector Oblivious Linear Evaluation (VOLE) as Homomorphic Commitments





Linearly Homomorphic Commitments (cf. [CF13]; [BDOZ11])
use $q_i = w_i \cdot \Delta + v_i$ as information-theoretic MAC on w_i

- hiding since v_i is random
- breaking binding $\implies$ guessing Δ $\implies$ prob. $1/|\mathbb{F}|$

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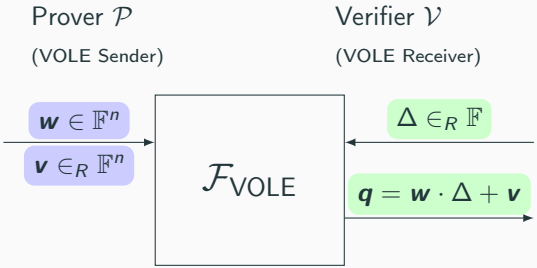

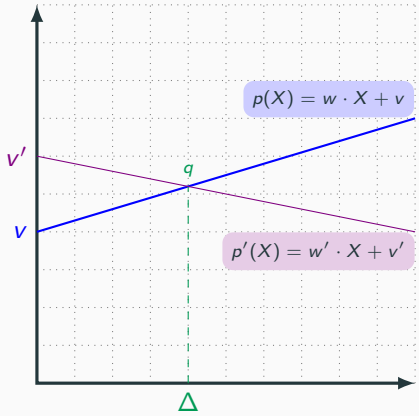

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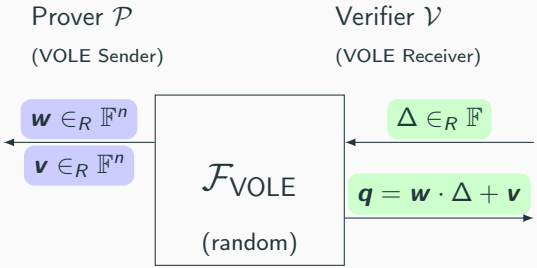

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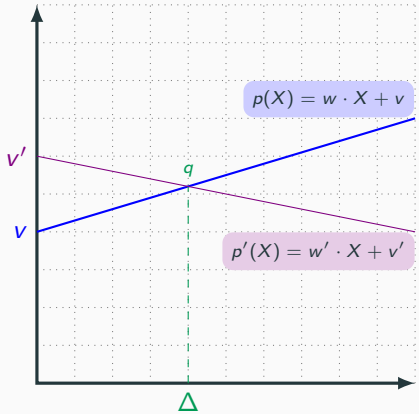

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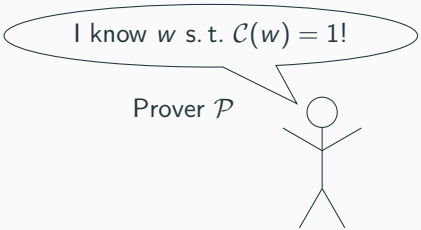

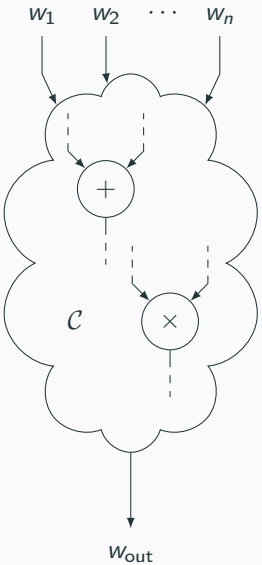

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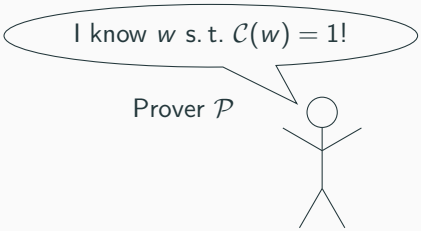
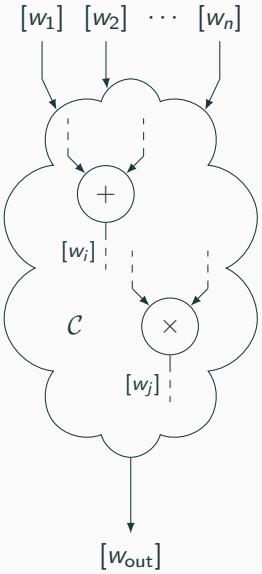


Ingredients:

1. linearly homomorphic commitments $[\cdot]$
   - can compute $[z] \leftarrow a \cdot [x] + [y] + b$ ✅



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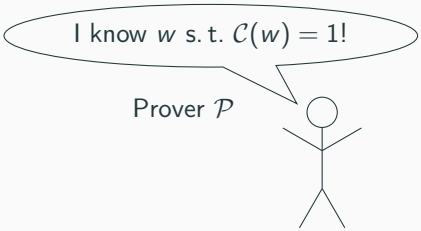

Ingredients:

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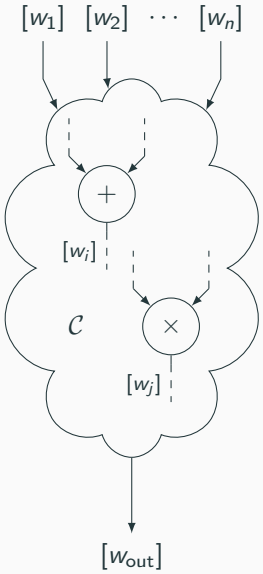
2. multiplication check
   - given $([a], [b], [c])$, verify $a \cdot b \stackrel{?}{=} c$



Verifying Multiplications – LPZK [DIO21], QuickSilver [YSWW21]

Goal: Given $([a], [b], [c])$, verify that $a \cdot b = c$ in $\mathbb{F}$

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use a random linear combination to verify many multiplications

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- $p(\Delta) = 0$, and
- $e_2 := c - a \cdot b \neq 0$

$\implies$ p has degree 2 $\implies$ p has at most 2 roots $\implies$ soundness error $2/|\mathbb{F}|$

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Generalizable to higher-degree constraints:

- $[a + b]^{\max(d_a, d_b)} \leftarrow [a]^{d_a} + [b]^{d_b}$ and $[a \cdot b]^{d_a + d_b} \leftarrow [a]^{d_a} \cdot [b]^{d_b}$
- for degree-d circuit $\mathcal{C}$: $[y]^d \leftarrow \mathcal{C}([\boldsymbol{x}]^1)$

$$[y]^d\text{:} \quad q_y = y \cdot \Delta^d + \sum_{k=0}^{d-1} p_k \cdot \Delta^k \qquad \rightsquigarrow \text{proof size } d \text{ field elements}$$

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Conversions

Emulating $\mathbb{F}_2$ Arithmetic over $\mathbb{F}_p$

Bit decomposition

$$[x]_p = \sum_{k=0}^{m-1} 2^k \cdot [x_i]_p$$

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$$[x]_p = \sum_{k=0}^{m-1} 2^k \cdot [x_i]_p \qquad \wedge \qquad \bigwedge_{k=0}^{m-1} [x_i]_p \cdot (1 - [x_i]_p) = 0$$

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Arithmetizing Boolean operations:

$$x \wedge y = x \cdot y$$
$$x \oplus y = x + y - 2 \cdot x \cdot y \qquad \text{for } x, y \in \{0, 1\}$$

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- ✗ Committing to $[x_0]_p, \ldots, [x_{m-1}]_p$ costs $m^2 = (\log p)^2$ bits of communication
- ✗ XOR not $\mathbb{F}_p$-linear $\rightsquigarrow$ no longer free

Conversions Based on (Extended) Doubly Authenticated Bits

Given $[x]_p, [\boldsymbol{x}]_2$, verify $x = \sum_i 2^i \cdot x_i$ with $m := \log p$

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- need large batches for $B = 3$ ⚠

⇝ VOLE-ZK: A2B/Mystique [B**B**MRS21]; [WYXKW21]

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Can we create daBits with low communication?

daBits from Low-Degree PRGs

Creating daBits with Low Communication – Our Approach

1. Commit to short seed $\boldsymbol{x} \in \{0,1\}^{\ell}$ over both $\mathbb{F}_2$ and $\mathbb{F}_p \rightsquigarrow ([\boldsymbol{x}]_2, [\boldsymbol{x}]_p)$
   - commit to ℓ daBits and prove consistency
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2. Non-interactively apply a low-degree PRG to obtain long $\boldsymbol{y} \in \{0,1\}^{\ell^s}$
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3. Use daBits $([y_i]_2^{d_2}, [y_i]_p^{d_p})$ for conversions and other gadgets 🎉
4. Reuse ℓ bits of the output as seed for next iteration

(reduce degree)

Mixed Low-Degree PRGs

Instantiation from Goldreich-style Random Local PRGs [Gol00]

$$k\text{-local PRG}\colon \{0,1\}^{\ell} \to \{0,1\}^{\ell^s} \text{ with stretch } s$$

$$\mathrm{PRG}(\boldsymbol{x})_i = P(x_{i_1}, \ldots, x_{i_k}) \text{ for } P\colon \{0,1\}^k \to \{0,1\} \text{ and } \{i_1, \ldots, i_k\} \subset_R [\ell]$$

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TSPA

$$P(x_1, \ldots, x_5) = x_1 \oplus x_2 \oplus x_3 \oplus (x_2 \oplus x_4) \wedge (x_3 \oplus x_5) \qquad \begin{cases} \mathbb{F}_2\text{-degree } d_2 = 2 \\ \mathbb{F}_p\text{-degree } d_p = 3 \end{cases}$$

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XOR_4-$\mathrm{Majority}_7$ (better stretch)

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$$[r_i]_2^{d_2} \leftarrow P_2([x_{i_1}]_2, \ldots, [x_{i_k}]_2) \qquad [r_i]_p^{d_p} \leftarrow P_p([x_{i_1}]_p, \ldots, [x_{i_k}]_p)$$

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Performance and Summary

Performance

Rust Implementation: `https://github.com/AarhusCrypto/vole-zk-conversions/`

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Interactive VOLE-ZK with $p = 2^{61} - 1$ – Comparison with A2B [B**B**MRS21] (edaBits)

Improvements	LAN Time	WAN Time	Communication	#VOLEs
2^{10} conversions	2–3×	2–3×	2×	
2^{20} conversions	0.3–0.5×	≈	10×	
fixed-point multiplication	5×	13×	10×	

- amortized ≈ 0 VOLEs per conversion for $\text{XOR}_4\text{-Maj}_7$

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VOLE-in-the-Head [B**B**DKORS23] with $p \approx 2^{128}$

- VOLEs are more expensive ⇝ estimated 20–150× smaller proofs

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VOLE-based Zero-Knowledge

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Low-Bandwidth Conversions from Low-Degree PRGs

- Up to 10× less communication, a bit more computation
- Vastly fewer VOLEs needed $\rightsquigarrow$ great for VOLE-in-the-Head
- Instantiations from Goldreich PRGs and sparse LPN
- More: Fixed-point arithmetic, comparisons, bit extraction, ReLU

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VOLE-based Zero-Knowledge

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Open Question

- How to efficiently commit to even fewer bits over a large field?

Thank you!

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PRG Parameters

Predicate	(d_2, d_p)	ℓ	s	ℓ^s
TSPA	$(2, 3)$	4096	1.19	19 893
XOR_4-MAJ_7	$(4, 11)$	1024	2	2^{20}

High(er)-Degree QuickSilver

Recall: QuickSilver [YSWW21] can be used to prove degree-d constraints

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- $[a + b]^{\max(d_a, d_b)} \leftarrow [a]^{d_a} + [b]^{d_b}$ and $[a \cdot b]^{d_a + d_b} \leftarrow [a]^{d_a} \cdot [b]^{d_b}$
- for degree-d circuit $\mathcal{C}$: $[y]^d \leftarrow \mathcal{C}([x]^1)$

$$[y]^d\text{:} \quad q_y = y \cdot \Delta^d + \sum_{k=0}^{d-1} p_k \cdot \Delta^k \quad \rightsquigarrow \quad \begin{array}{l}\text{proof size } d \text{ field elements} \\ \text{soundness error } d/|\mathbb{F}|\end{array}$$

[YSWW21] Yang et al. (2021), "QuickSilver: Efficient and Affordable Zero-Knowledge Proofs for Circuits and Polynomials over Any Field".

[BBMORRRS24] Baum et al. (2024), "One Tree to Rule Them All: Optimizing GGM Trees and OWFs for Post-Quantum Signatures".

Fixed-Point Arithmetic Even Cheaper

We do not are about $\mathbb{F}_2$! Generate committed bits over $\mathbb{F}_p$ only:

- Commit to seed $[\boldsymbol{x}]_p$ and prove all $x_i \in \{0, 1\}$
- Generate committed bits $[r_i]_p^{d_p} \leftarrow P_p([x_{i_1}]_p, \ldots, [x_{i_k}]_p)$

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Fixed-point Multiplication: Given $([a]_p, [b]_p, [c]_p)$ prove $c = \lfloor (a \cdot b)/2^f \rfloor$:

- Use the $[r_i]_p^{d_p}$ to commit to bit decomposition of $c' := a \cdot b$:

$$([c'_0]_p^{d_p}, \ldots, [c'_{m-1}]_p^{d_p})$$

- Prove

$$\left([a \cdot b]_p^2 = \sum_{k=0}^{m-1} 2^k \cdot [c'_k]_p^{d_p}\right) \quad \bigwedge \quad \left([c]_p = \sum_{k=f}^{m-1} 2^{k-f} \cdot [c'_k]_p^{d_p}\right)$$