Multiple Group Action Dlogs With(out) Precomputation

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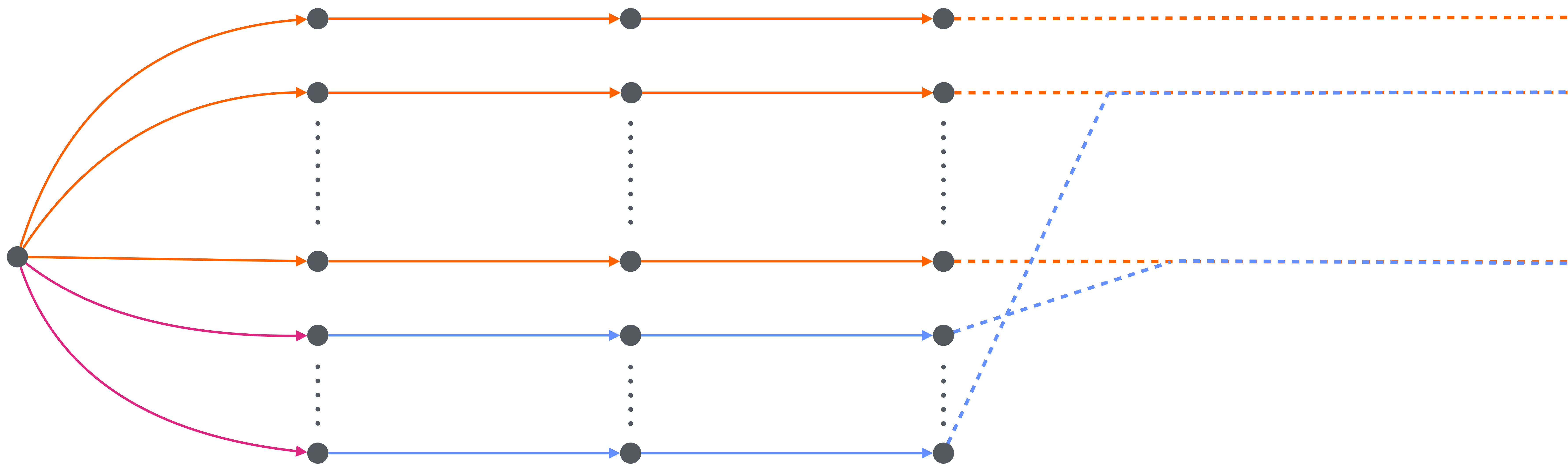

PKC 2025, 14/05/25

Precomputation Attacks

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Company X

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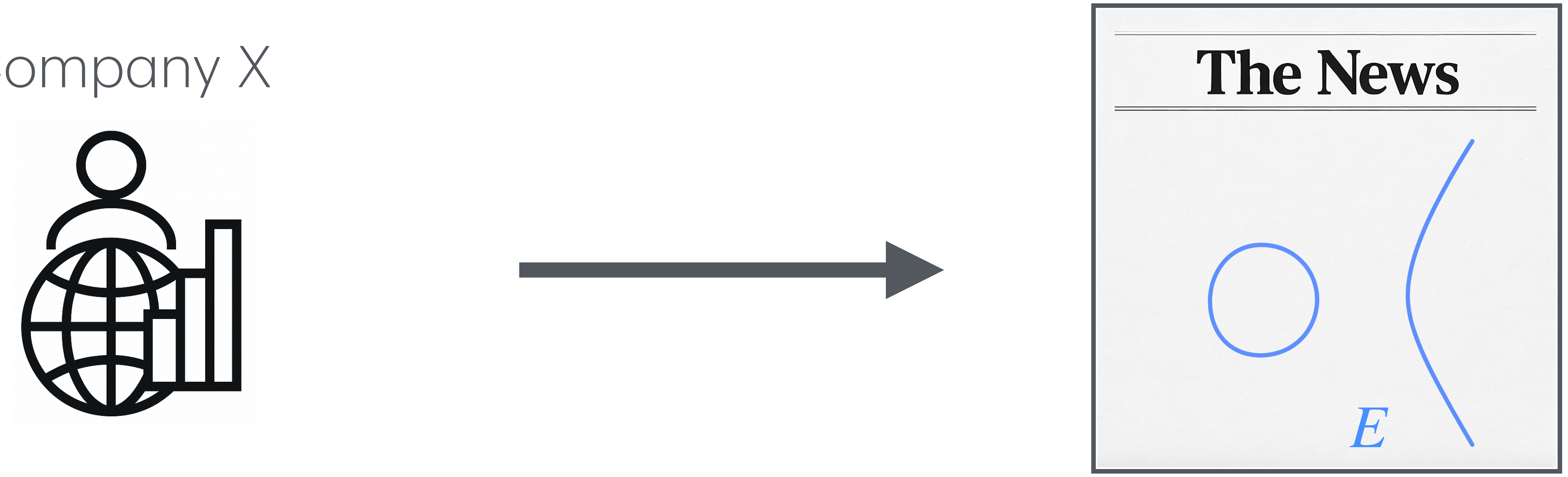

Precomputation Attacks

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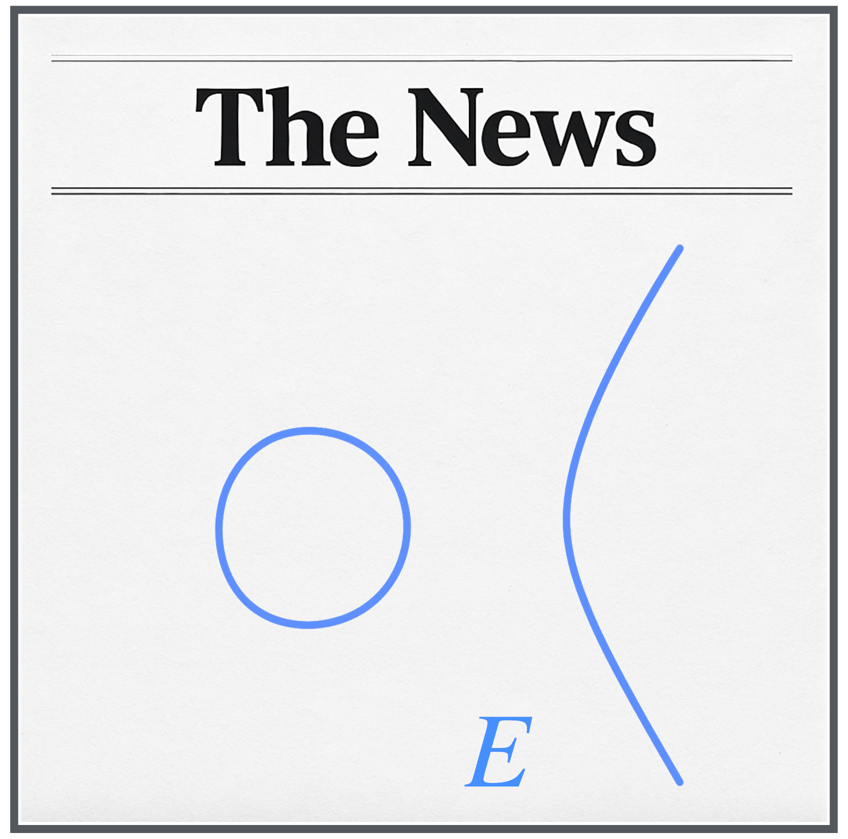

EC97: Shoup

If $|E| = N$, the time *lower bound* to solve one Dlog instance on E is $N^{1/2}$

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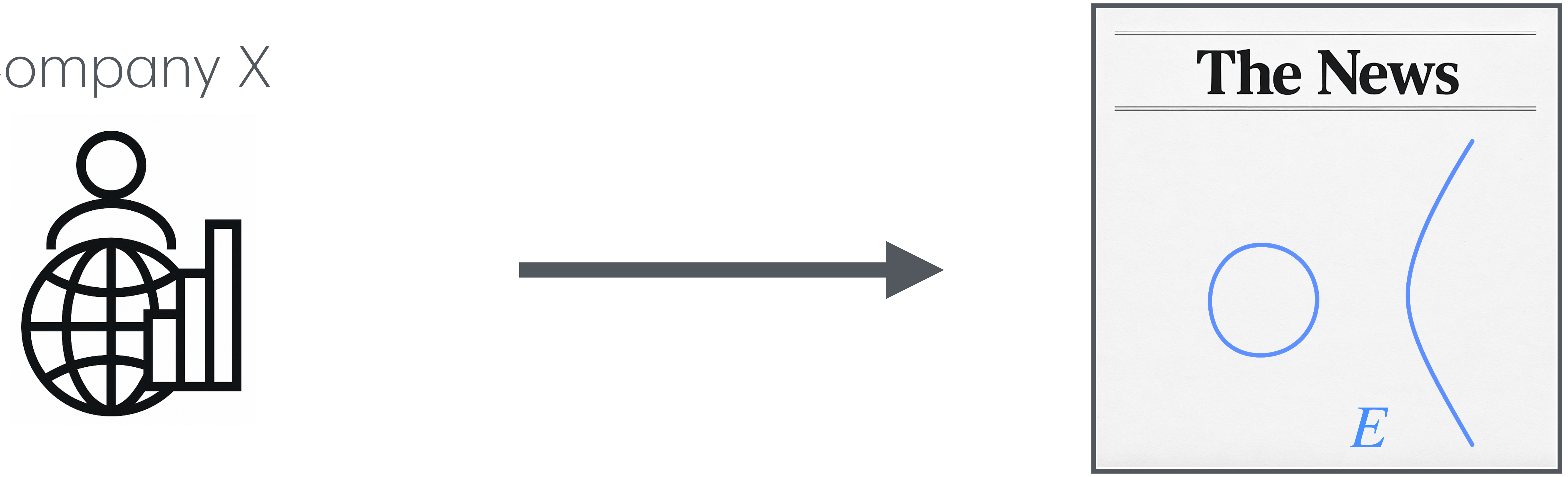

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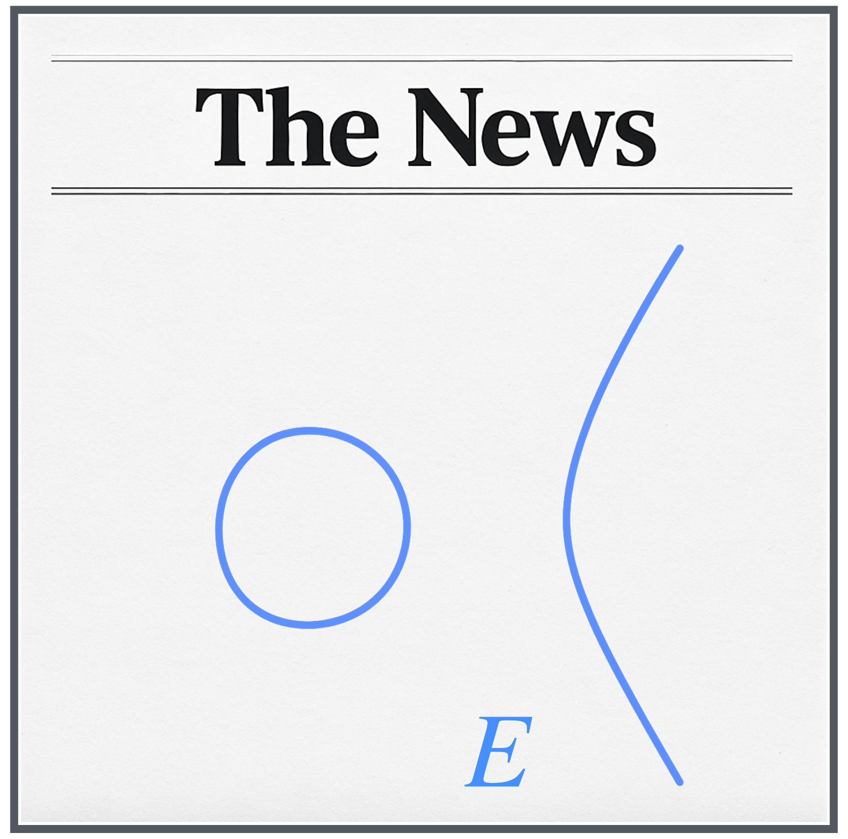
Trivial Precomputation Attack

Compute and store the whole E → Upon receiving an instance, look up the corresponding Dlog

Precomputation Attacks

Company X



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EC18: Corrigan-Gibbs & Kogan

Precomputation of time: $N^{2/3}$ → Online time: $N^{1/3}$ <u>instead of</u> $N^{1/2}$

Precomputation Attacks

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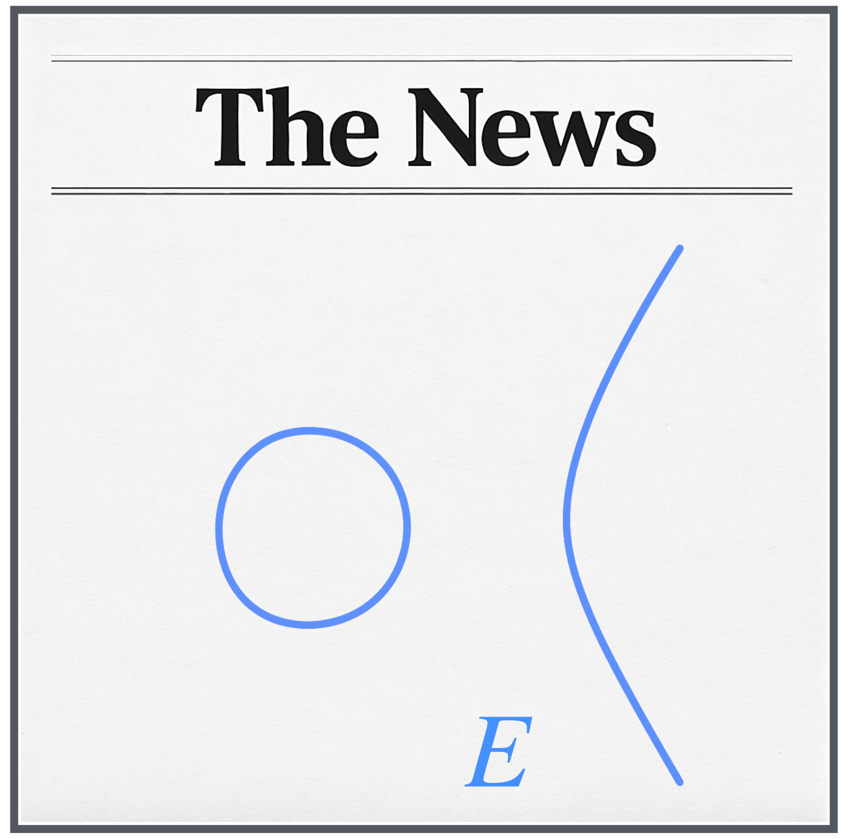


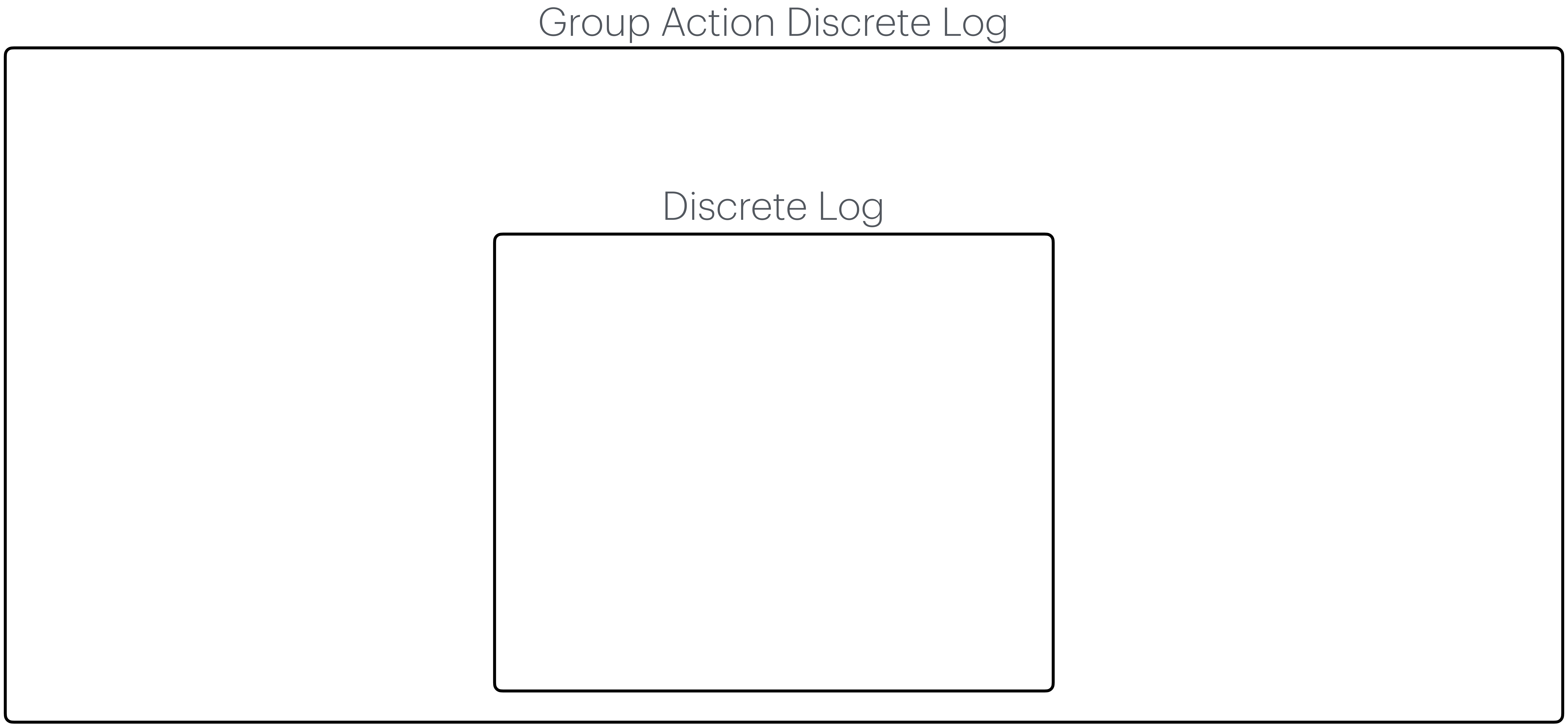
Precomputation Phase

Online Phase

Perform (heavy) *instance-independent* computations to obtain a hint

Upon receiving an instance, leverage the hint to solve faster

Extensions?



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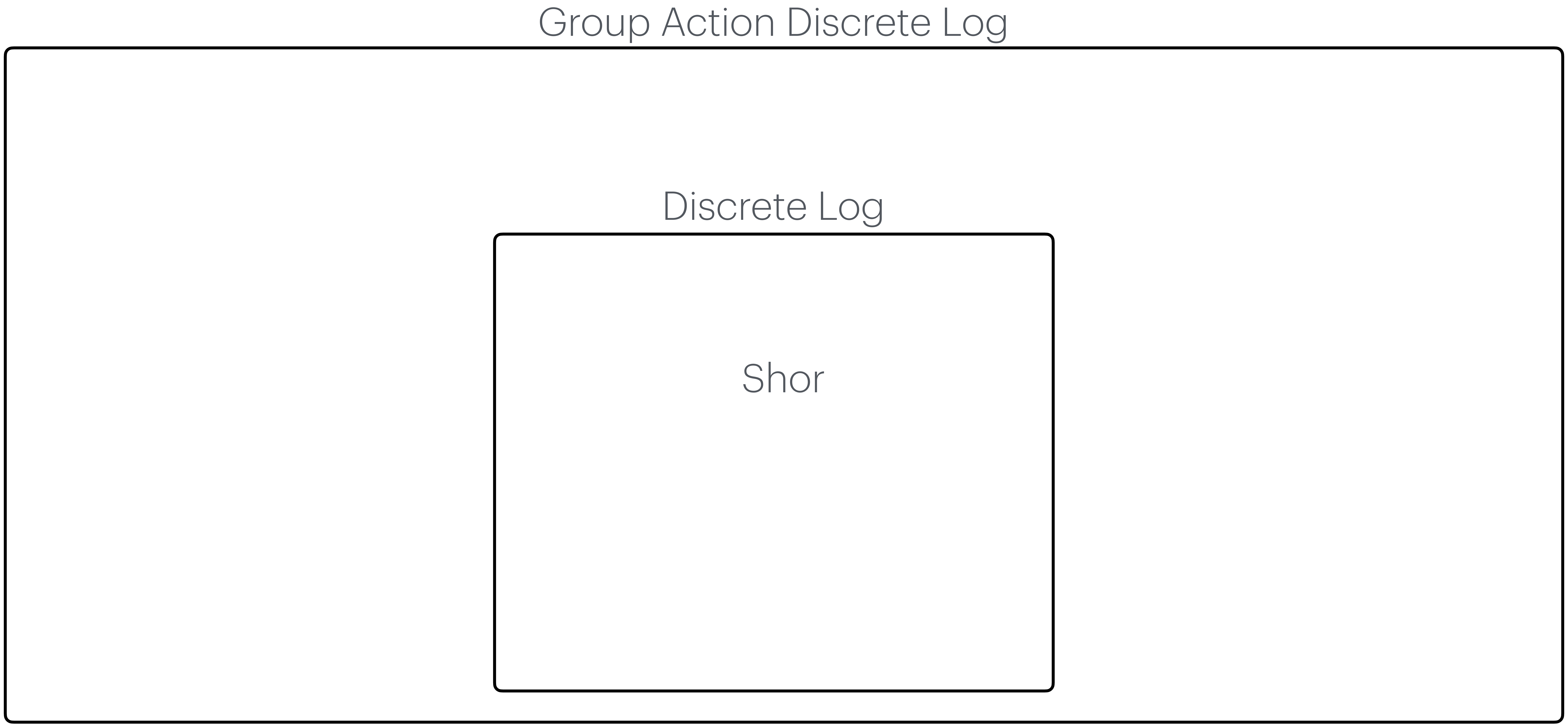

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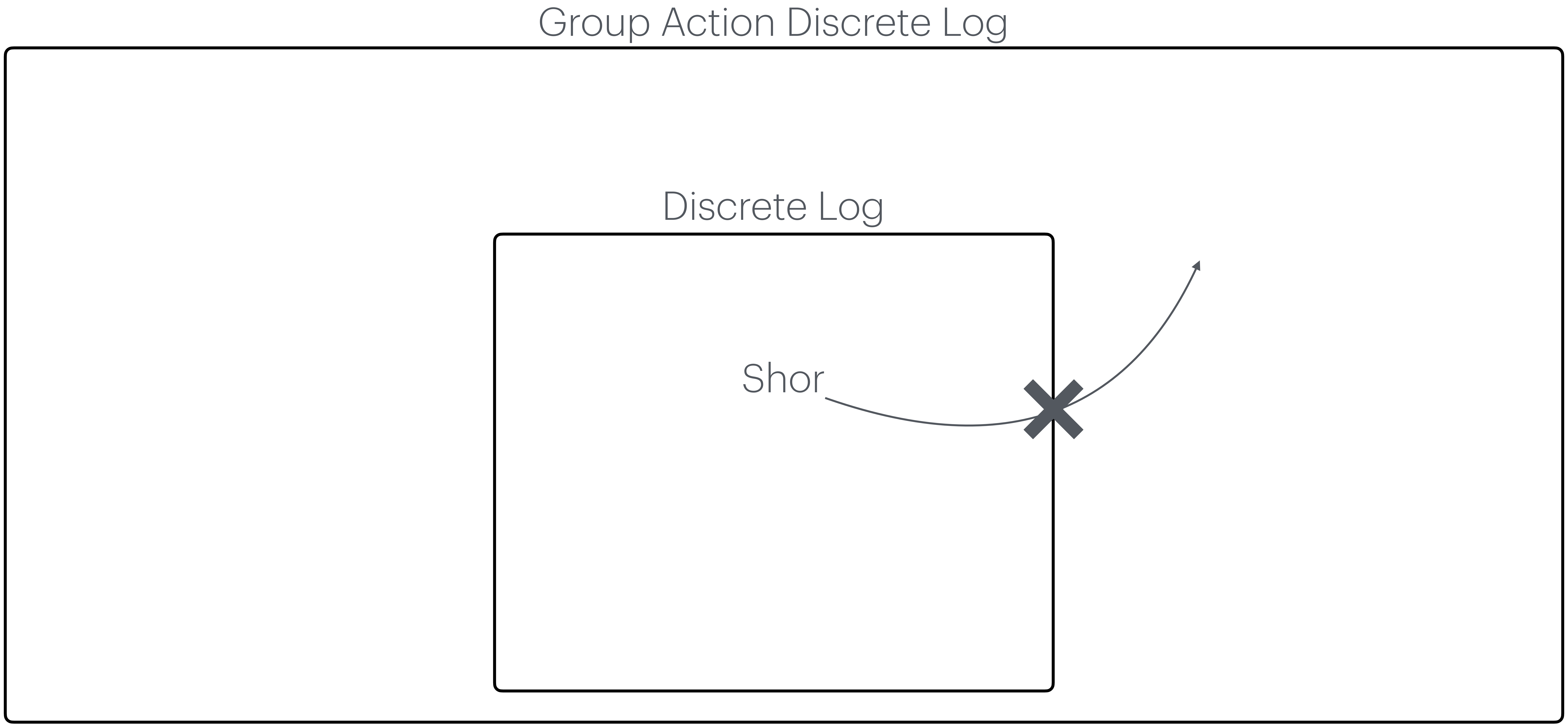

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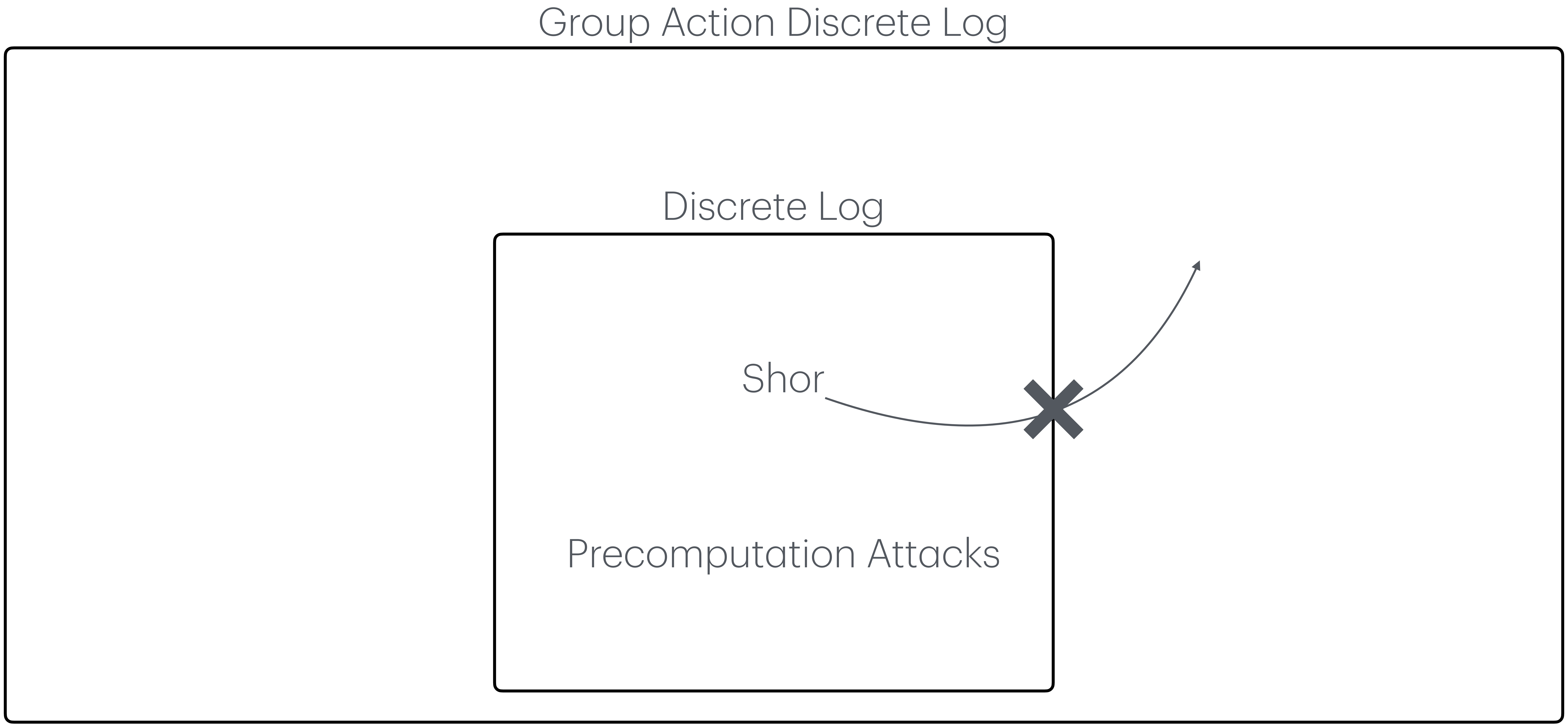
Group Action Discrete Log
Discrete Log
Shor
Precomputation Attacks

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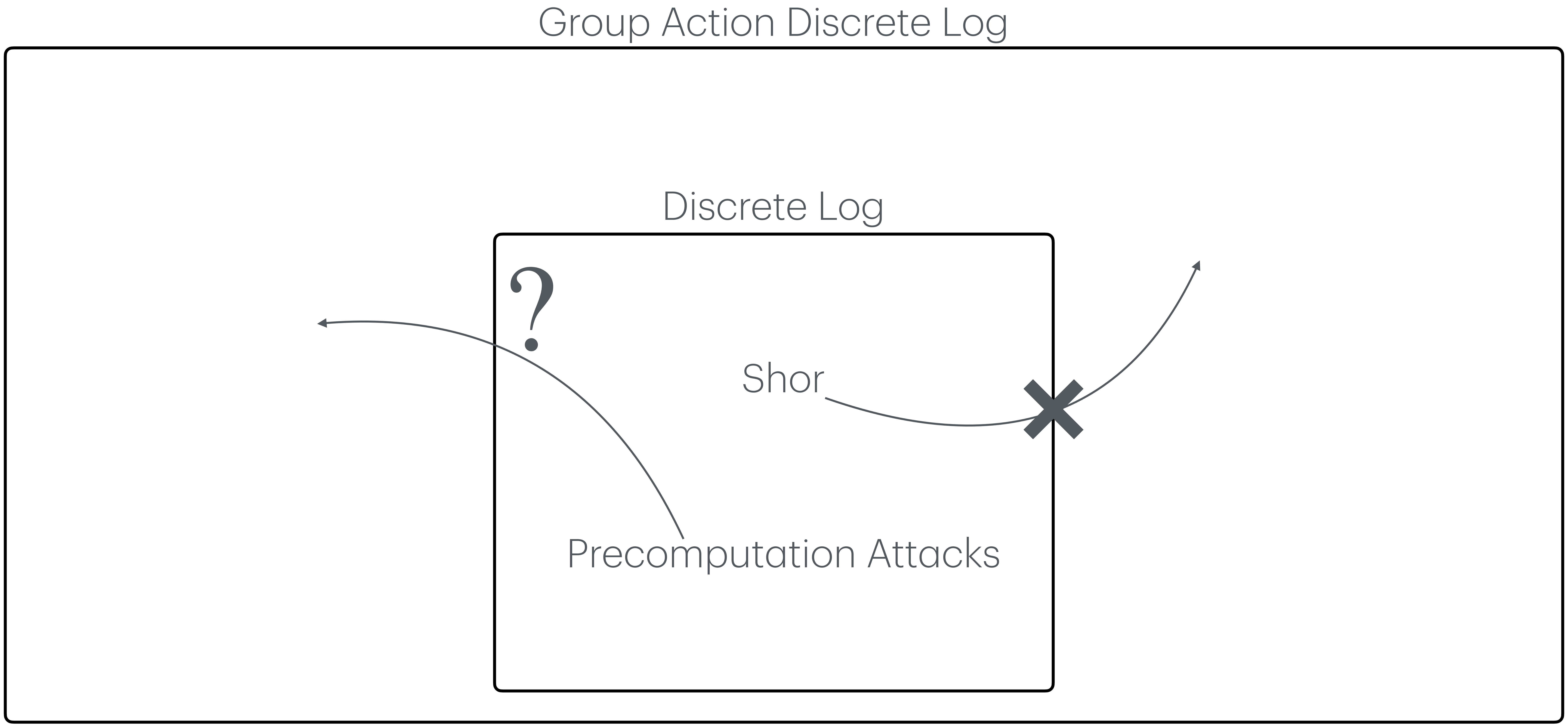
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Precomputation Attacks

Group Actions

Basic Definitions

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Given...

Any set $\mathcal{X}$, with a distinguished element $x \in \mathcal{X}$, called *origin*

A finitely generated abelian group $\mathcal{G} = \langle g_1, \ldots, g_n \rangle$, $|\mathcal{G}| = N$

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Then...

A map $\star : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ is a *group action* if it satisfies:

Identity: $1 \star y = y$ for all $y \in \mathcal{X}$

Compatibility: $g \star (h \star y) = (gh) \star y$ for all $g, h \in \mathcal{G}$ and $y \in \mathcal{X}$

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A special and familiar example...

$\mathcal{X} = H = \langle h \rangle$ a finite cyclic group

Let $1 \in H$ be the origin, $|H| = N$

$\mathcal{G} = \mathbb{Z}_N$

$\star : \mathbb{Z}_N \times H \to H$, $(v, g) \mapsto h^v \cdot g$

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Notation...

For $v \in \mathbb{Z}^n$, write $\mathbf{g}^v = g_1^{v_1} \cdot \cdots \cdot g_n^{v_n}$.

Denote by Λ the kernel of the map $v \mapsto \mathbf{g}^v$

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The GA-Dlog is the usual Dlog

Our Results

For GA-Dlogs

Extend the generic precomputation algorithms to the *Group Action Dlog* setting:

- Single-instance with precomputation
- Multi-instance with precomputation

Multi-instance "without" precomputation algorithm for GA-Dlogs

Multi-instance "without" precomputation algorithm for usual Dlogs

Finding GA-Dlogs

$N = |\mathcal{G}|$

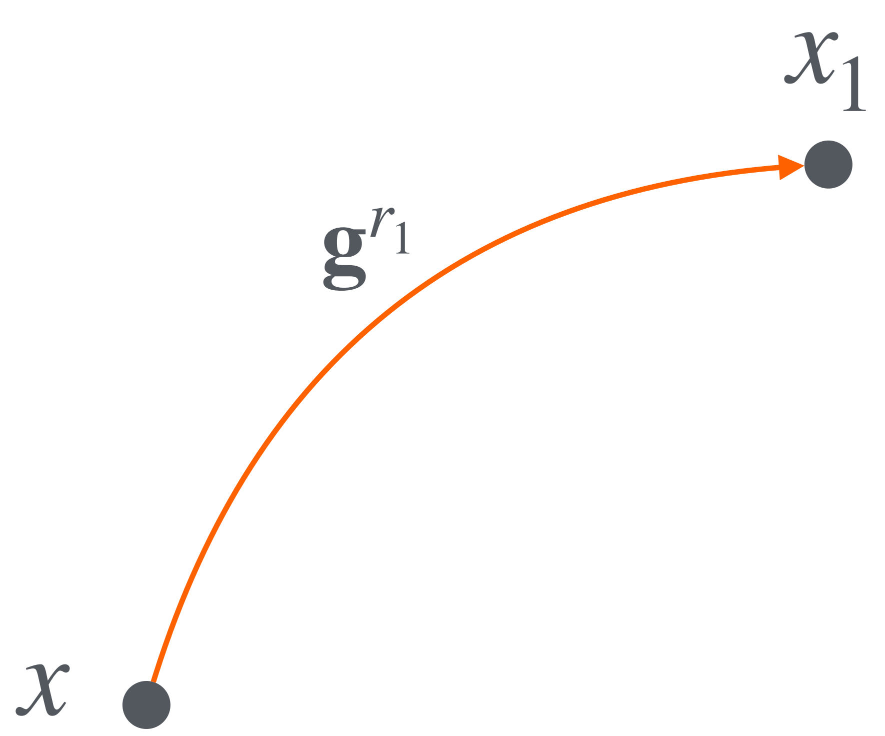
Single: Precomputation Phase

x

Finding GA-Dlogs

$N = |\mathcal{G}|$

Single: Precomputation Phase



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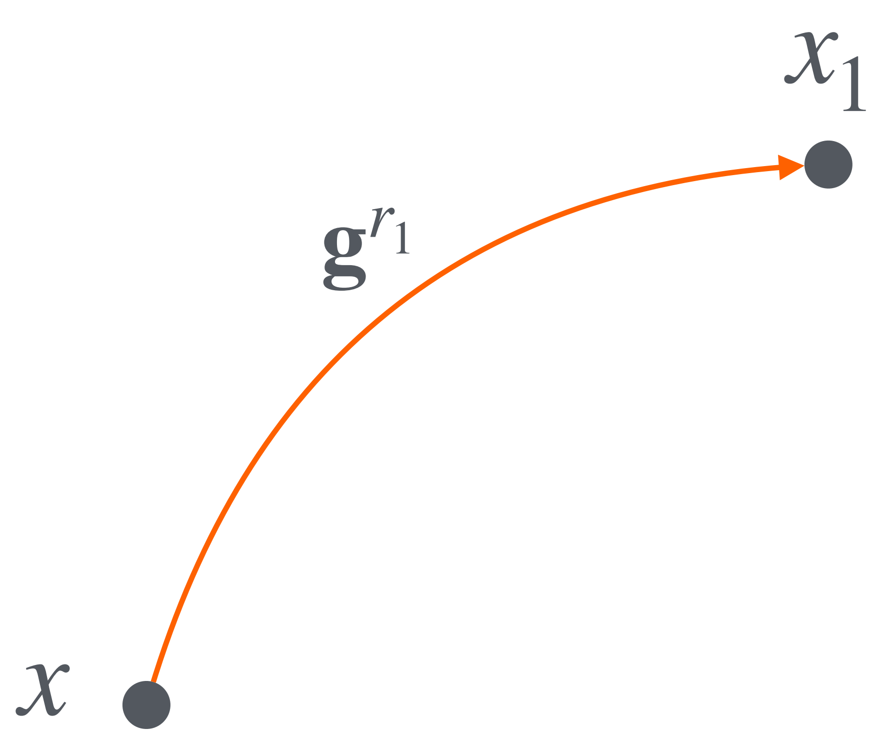

Memoryless Walk

The next step of the walk only depends on the vertex *currently* visited

Finding GA-Dlogs

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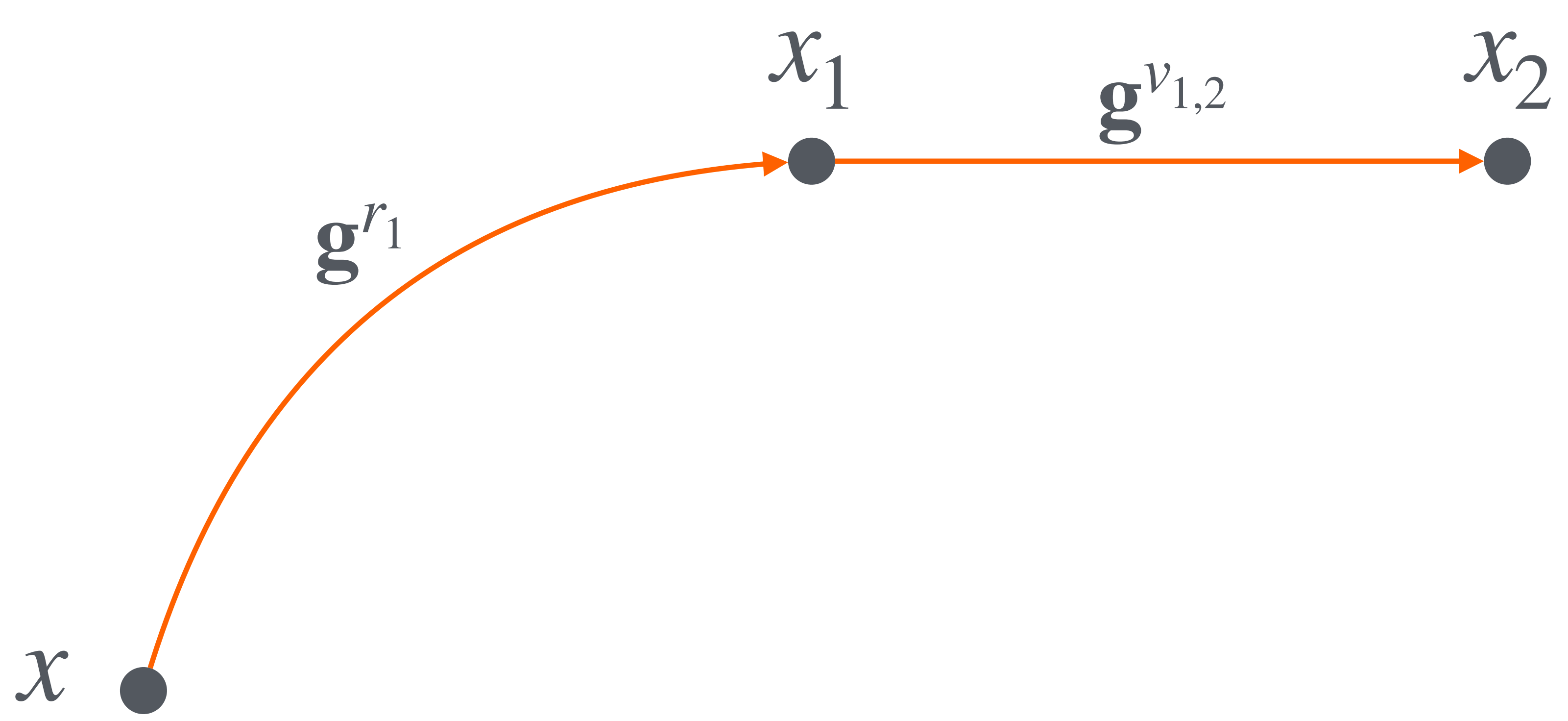

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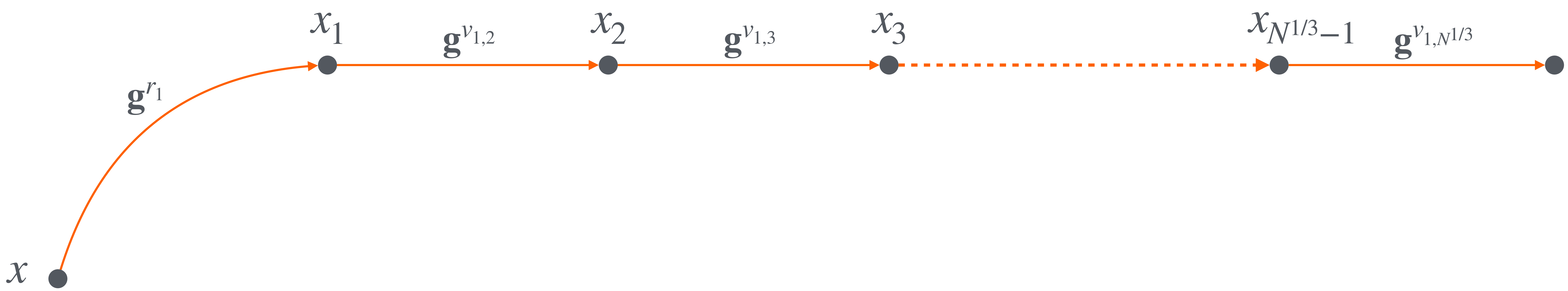

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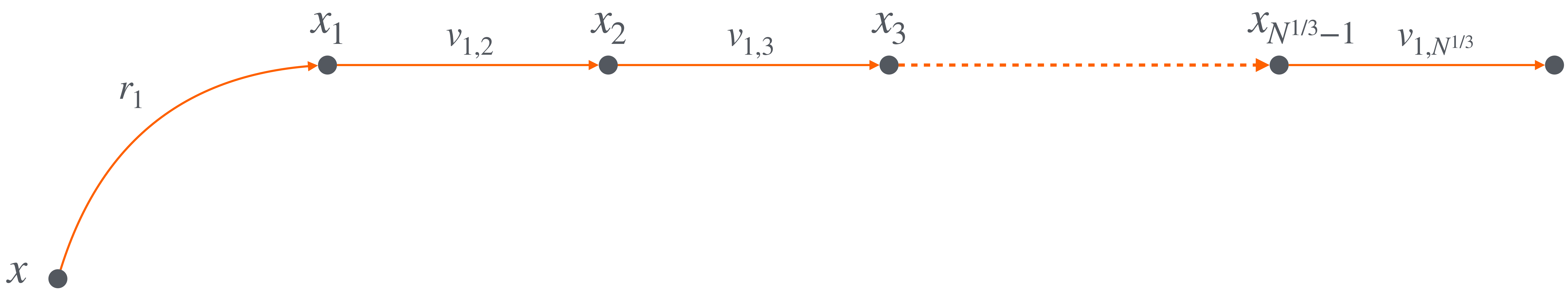

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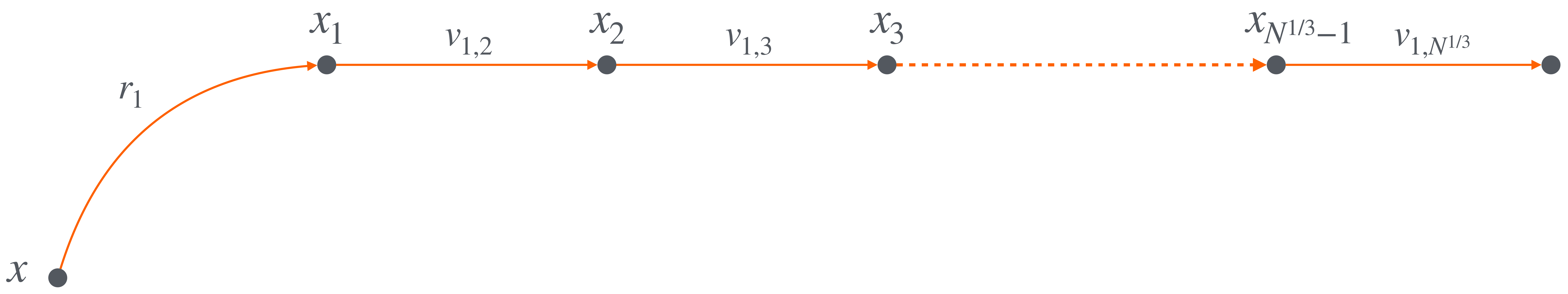

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Notation

$$v^{(1)} := r_1 + v_{1,2} + \cdots + v_{1,N^{1/3}}$$

Finding GA-Dlogs

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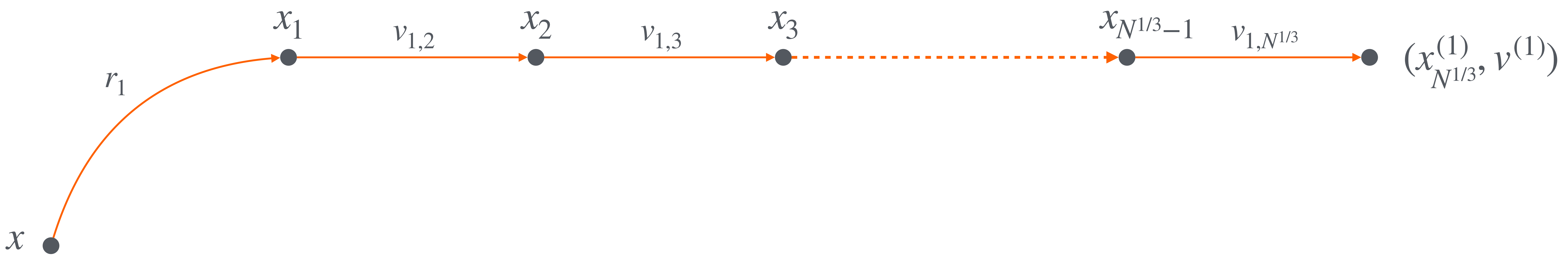

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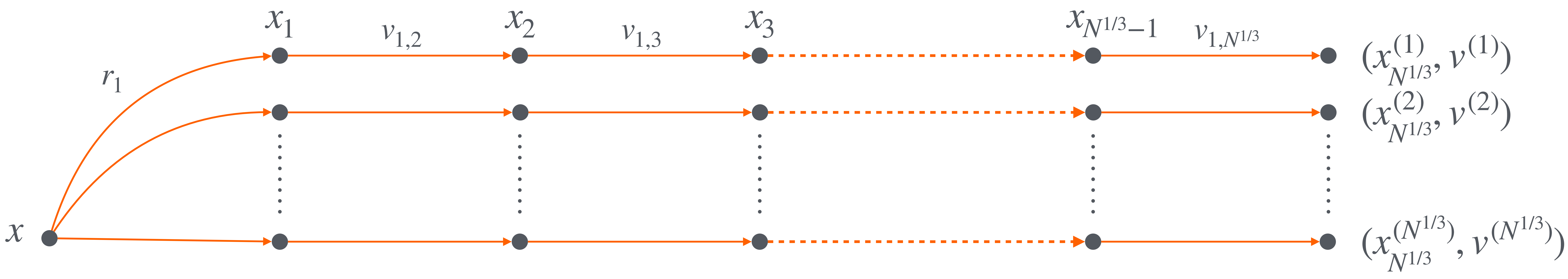

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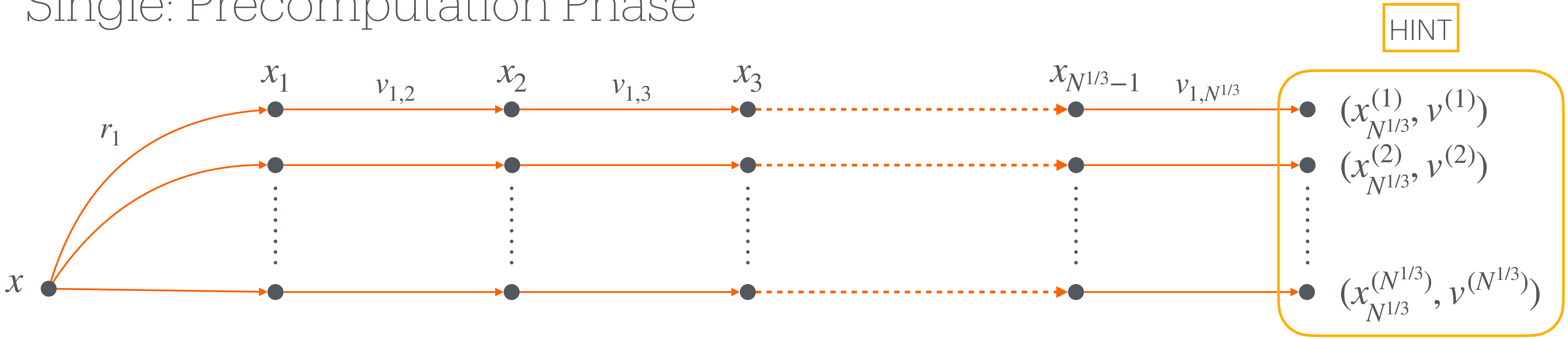

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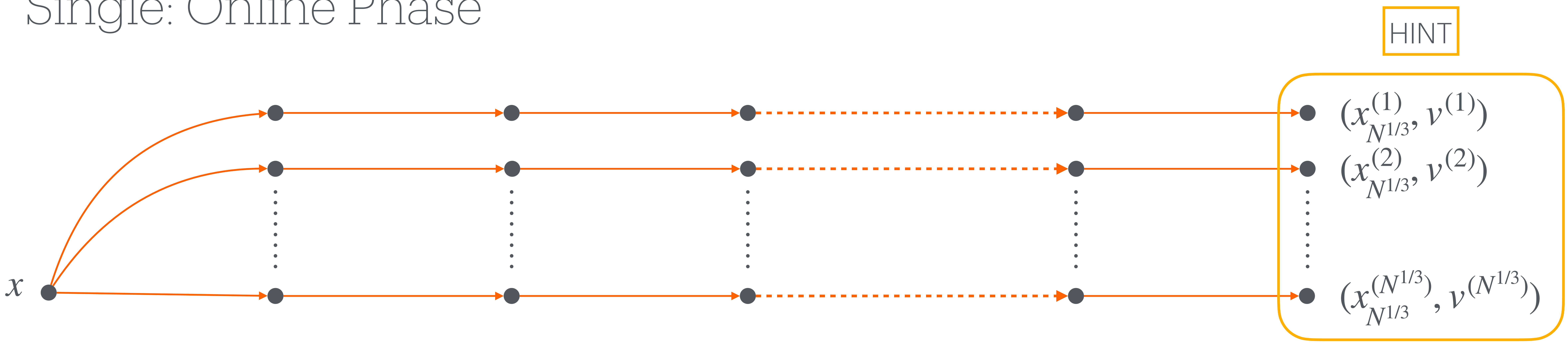

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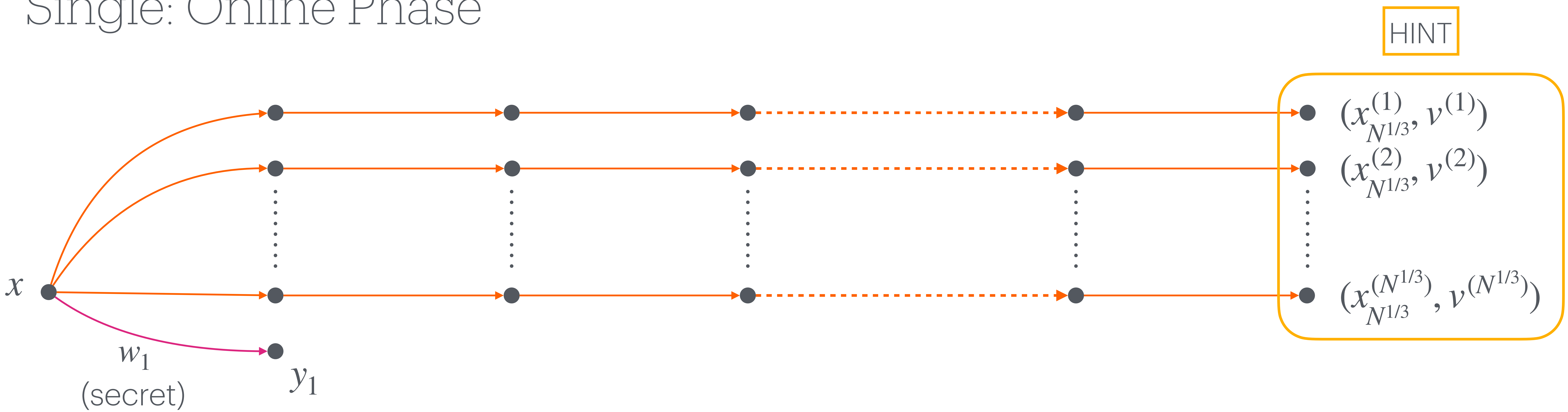

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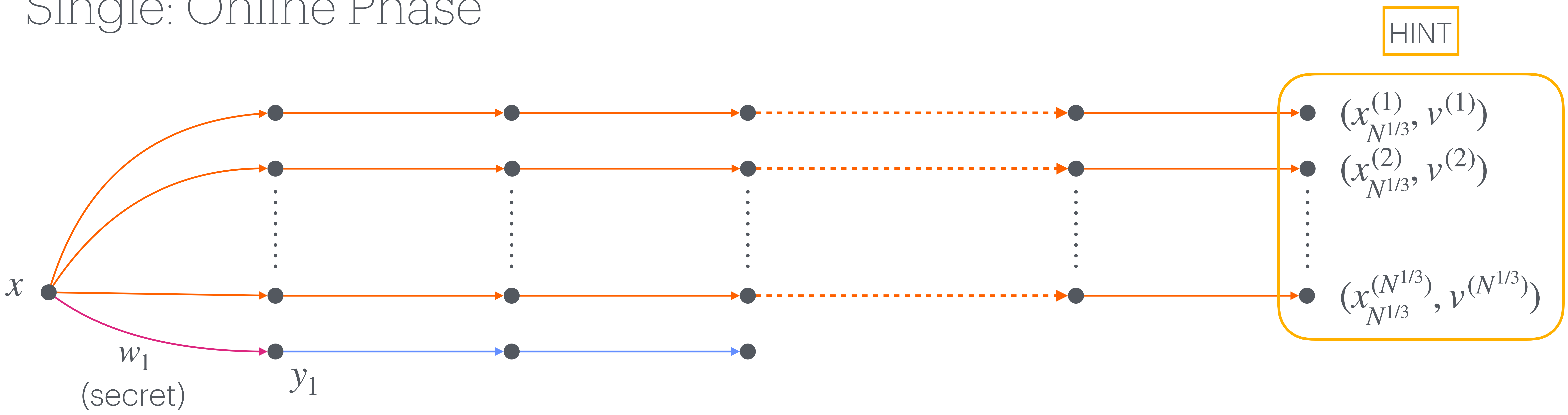

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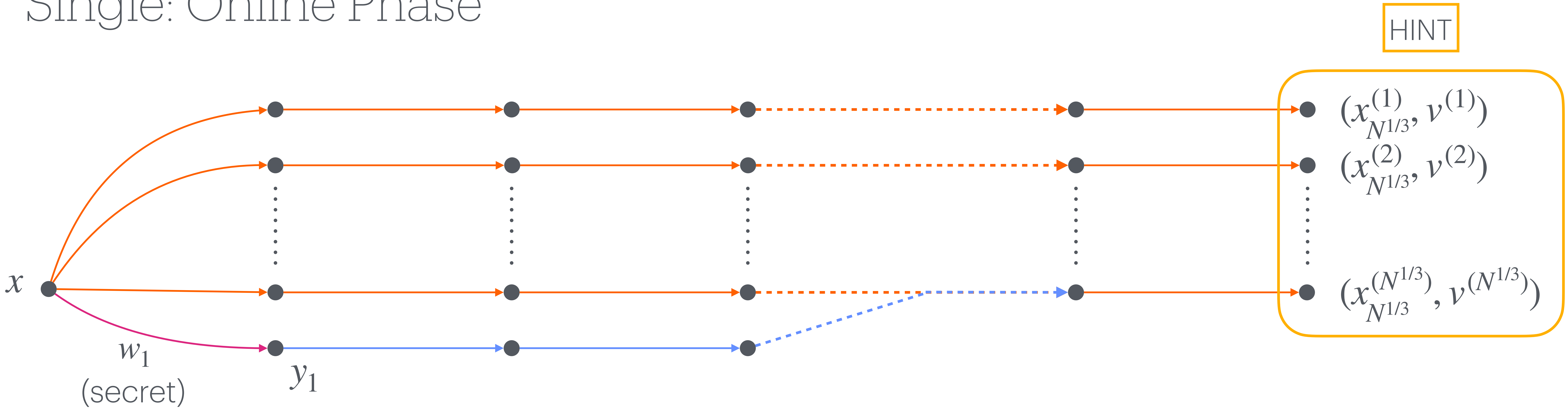

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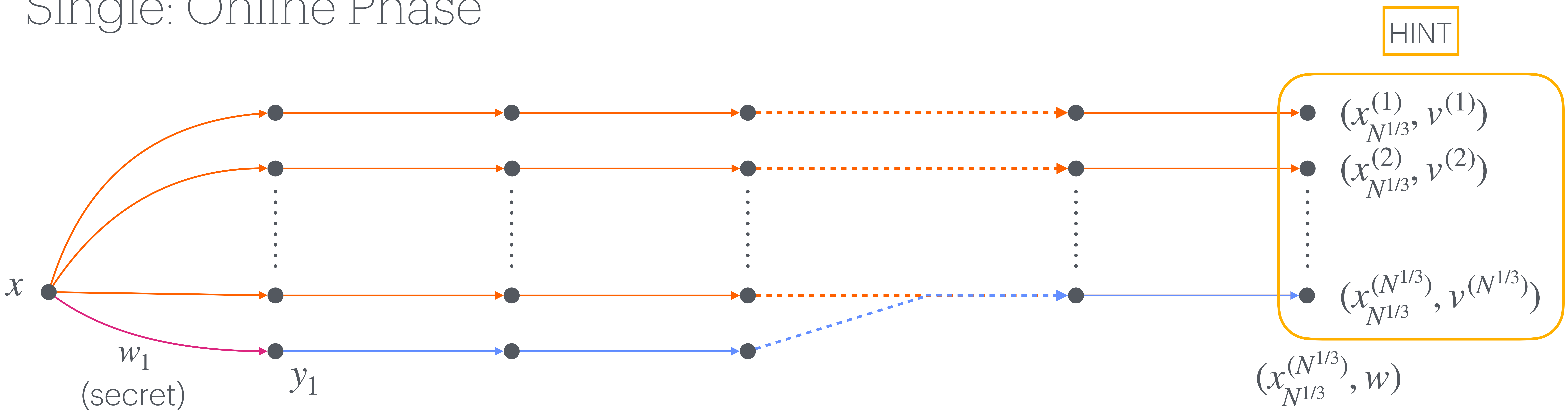

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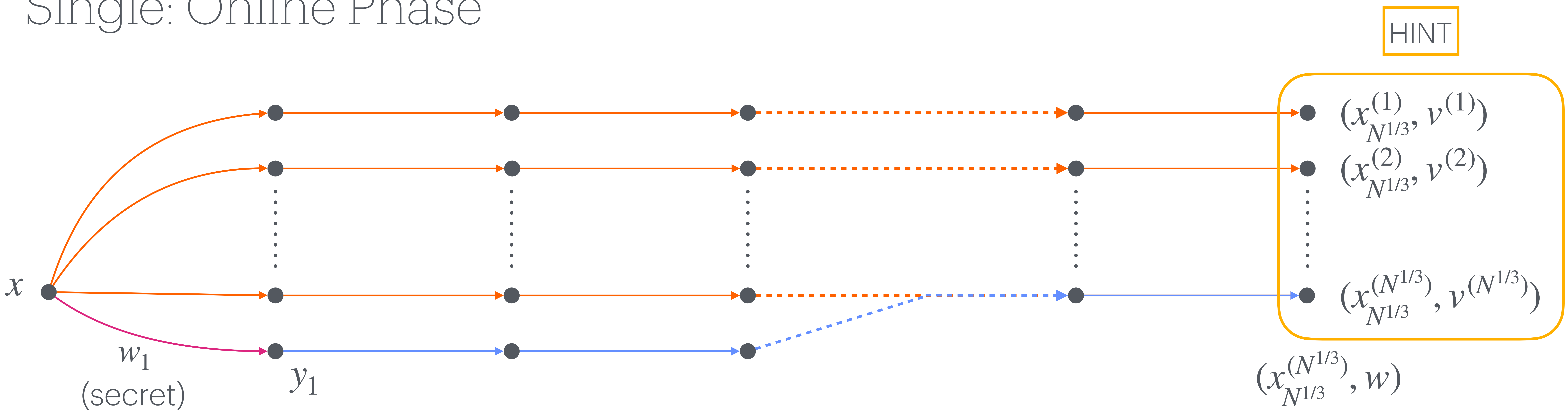

Collision Solves GA-Dlog:

$$x_{(N^{1/3})}^{(N^{1/3})} = \mathbf{g}^{w} \star y_1 \text{ and } x_{(N^{1/3})}^{(N^{1/3})} = \mathbf{g}^{v^{(N^{1/3})}} \star x \longrightarrow \mathbf{g}^{w} \star y_1 = \mathbf{g}^{v^{(N^{1/3})}} \star x \longrightarrow y_1 = \mathbf{g}^{v^{(N^{1/3})}-w} \star x$$

Finding GA-Dlogs

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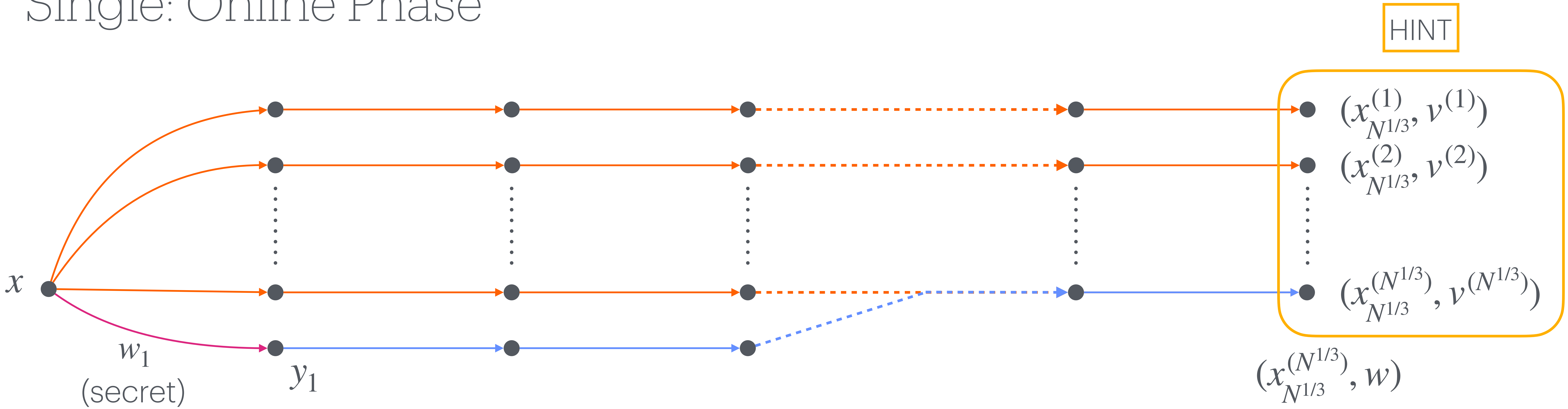


Precomputation: $N^{1/3} \cdot N^{1/3} = N^{2/3}$

Space: $N^{1/3}$

Online: $N^{1/3}$

→ Solve ONE GA-Dlog with Constant Success Probability

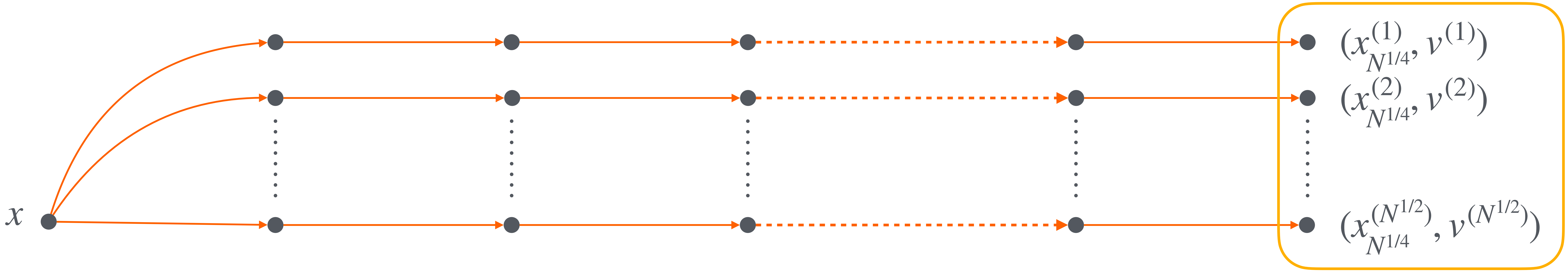
Finding GA-Dlogs

Multiple

$N = |\mathcal{G}|$

$N^{1/4}$ instances

HINT



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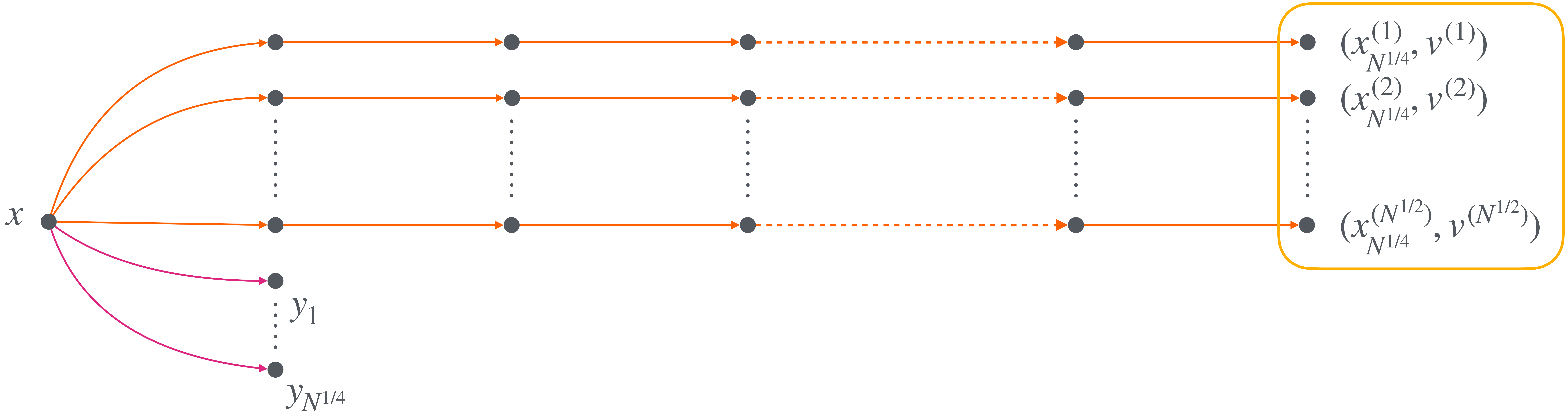

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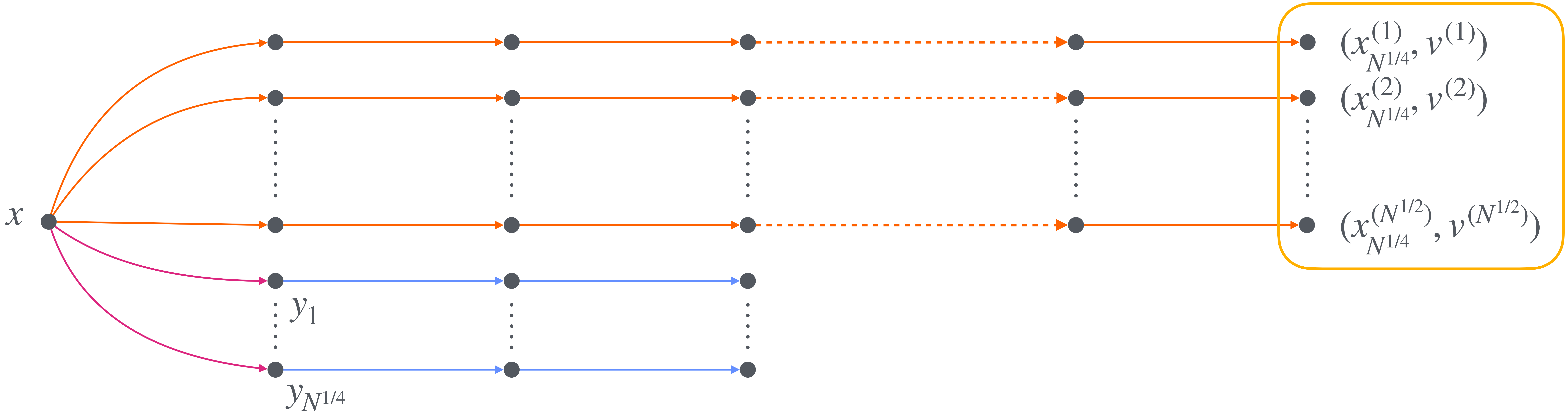

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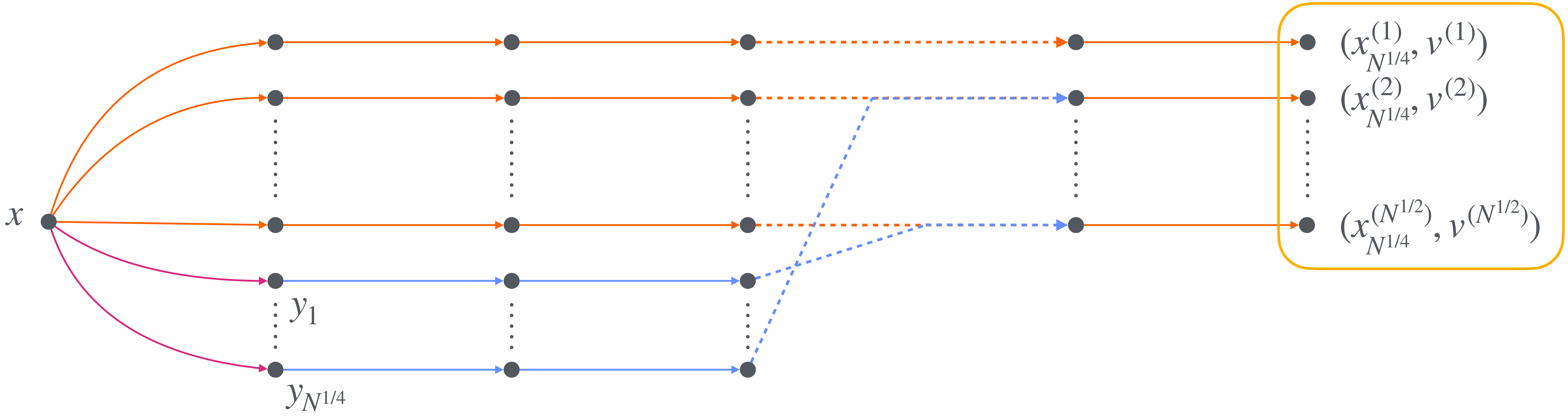

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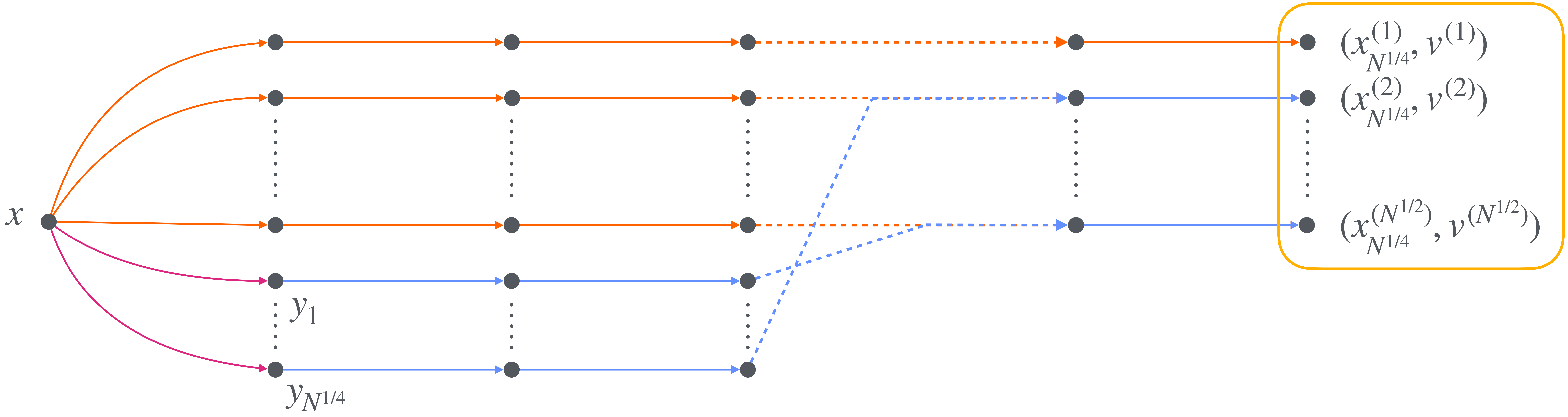

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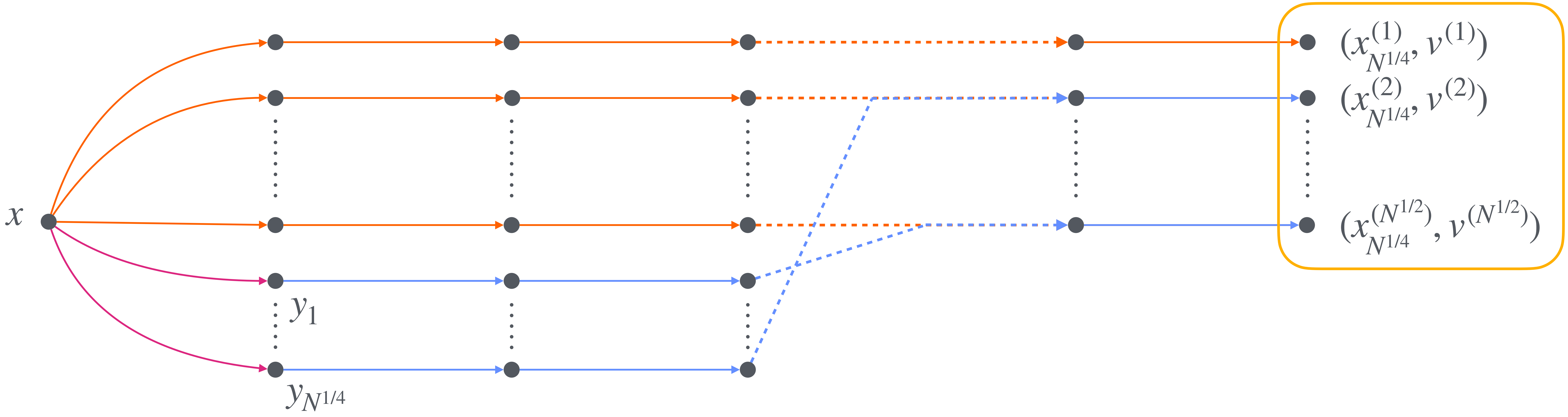

Precomputation: $N^{1/4} \cdot N^{1/2} = N^{3/2}$

Space: $N^{1/2}$

Online: $N^{1/4} \cdot N^{1/4} = N^{1/2}$ (expected)

→ Solve ALL GA-Dlog with Constant Success Probability

Finding GA-Dlogs

Multiple, “without” Precomputation

Finding GA-Dlogs

Multiple, "without" Precomputation

Naïvely

Repeat the $N^{1/2}$ algorithm m times	→	Solve ALL m GA-Dlog in time $m \cdot N^{1/2}$

Finding GA-Dlogs

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Balancing Precomputation and Online times...

Precomputation: $m^{1/2} \cdot N^{1/2}$ Space: m Online: $m^{1/2} \cdot N^{1/2}$	→	Solve ALL m GA-Dlog with runtime $m^{1/2} \cdot N^{1/2}$

Experiments

On CSIDH

From the Theorems...	In practice...

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The probability of success of the online phase is $\geq 1/8$ ↓ On average, online phase needs to be repeated 8 times	

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On CSIDH

From the Theorems...

The probability of success of the online phase is $\geq 1/8$

↓

On average, online phase needs to be repeated 8 times

In practice...

$\log N$	# of runs
5	1.3
8	1.0
10	1.2
12	1.0
15	1.0
18	1.0
21	1.1
24	1.2
27	1.1
29	1.1

Summary

Precomputation Attacks for Dlog can be extended to the GA-Dlog framework

New multi-instance “without” precomputation attack as a corollary

In practice, the technique performs better than in theory

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Questions?

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