Transparent SNARKs over Galois Rings

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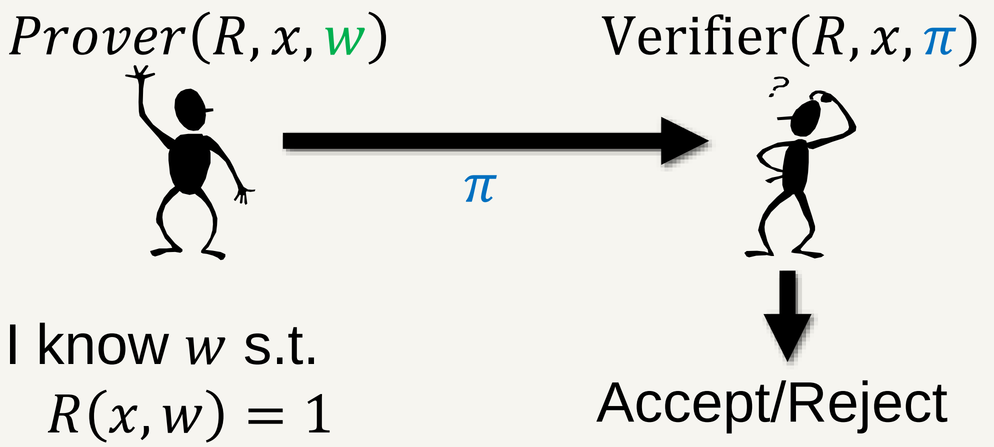
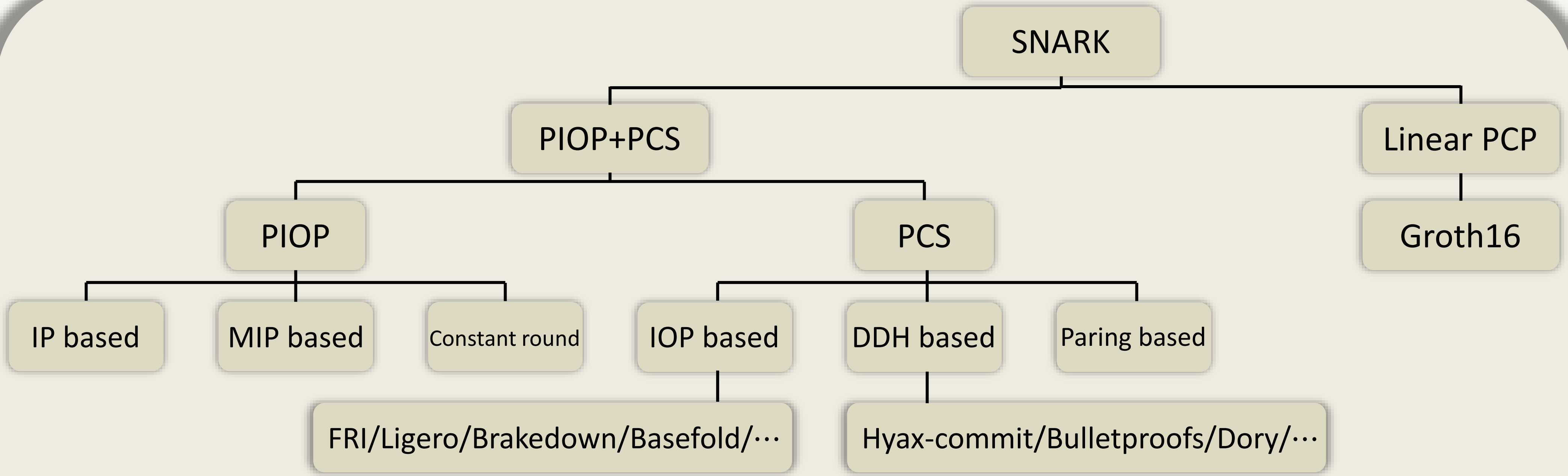
SNARK

- SNARK is a succinct non-interactive argument of knowledge.

- **Non-interactive.**
- **Succinctness:** sublinear proof sizes and sublinear verifier time.
- **Transparent:** It does not require a trusted setup.



Why we need SNARKs over Galois Rings?



Source: [THA22]

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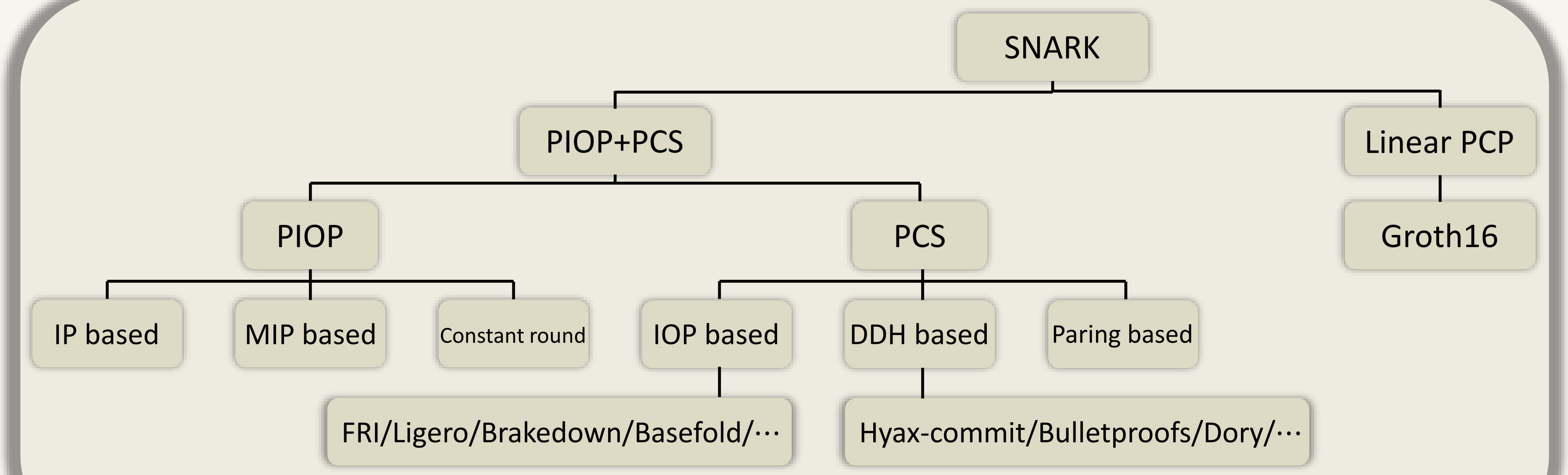

Source: [THA22]

- CPU computation over 2^{32} or 2^{64};
- Floating-point operation over 2^{k};
- FHE ciphertext in integer rings (can be mapped to Galois rings);
- ...

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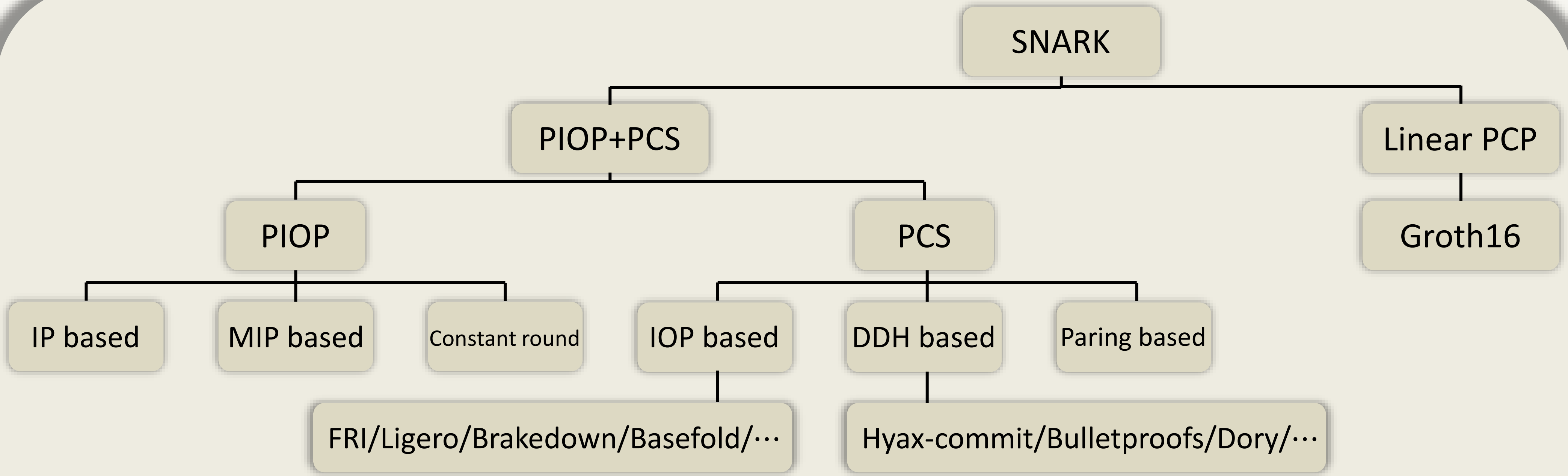

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- …

- DDH based SNARK: Secp256k1 256bit
- Paring based SNARK: BN254 254bit
- FRI based SNARK: Goldilocks 64bit
- Expander code based SNARK: 128bit

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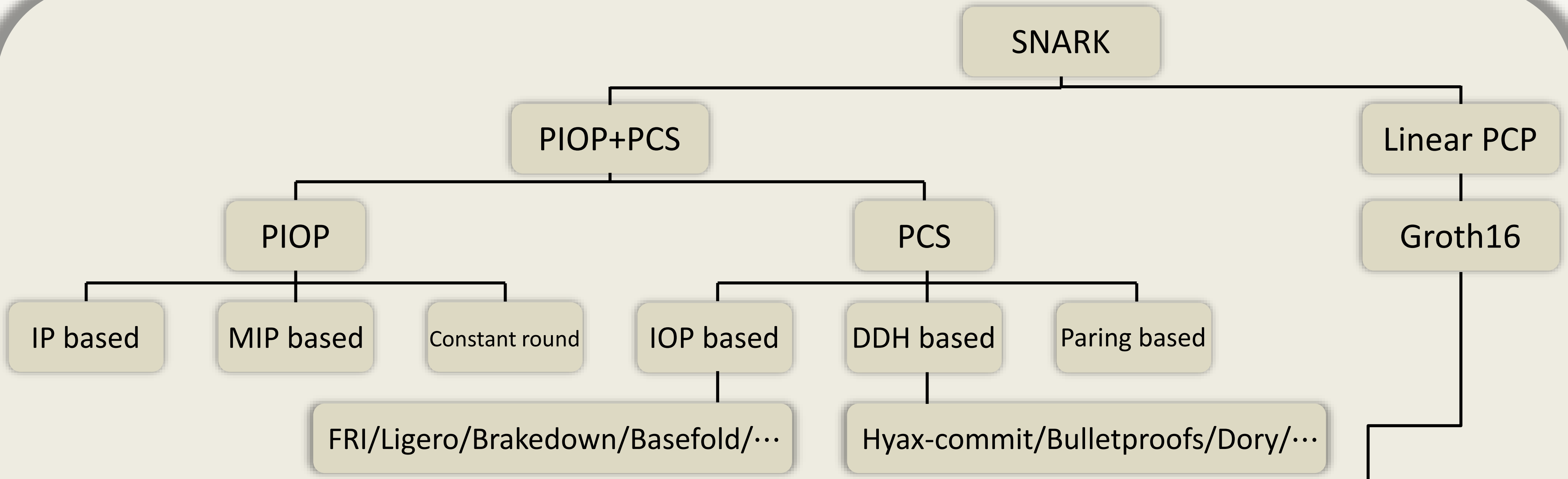


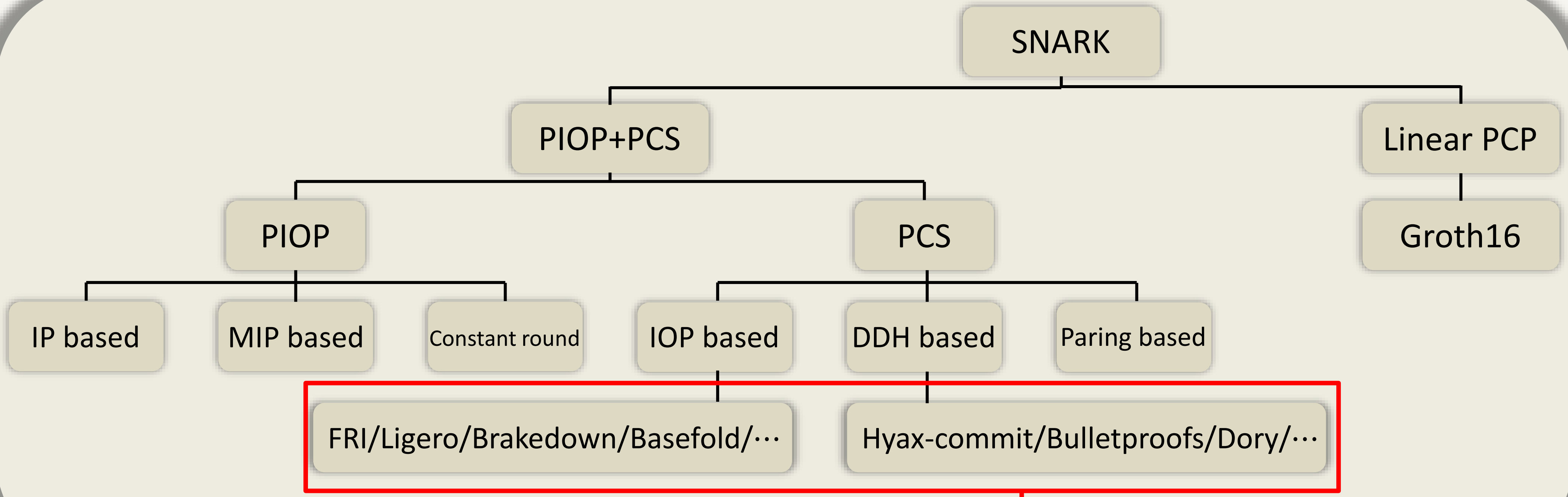
Source: [THA22]

Rinocchio [GNSV23]

- $O(l + m\log m)$ prover time;
- $O(1)$ proof size;
- $O(|x|)$ verifier time, where x is the statement;
- $O(l + m)$ length CRS.

l : wire number
m: multiplication numer
x: statement

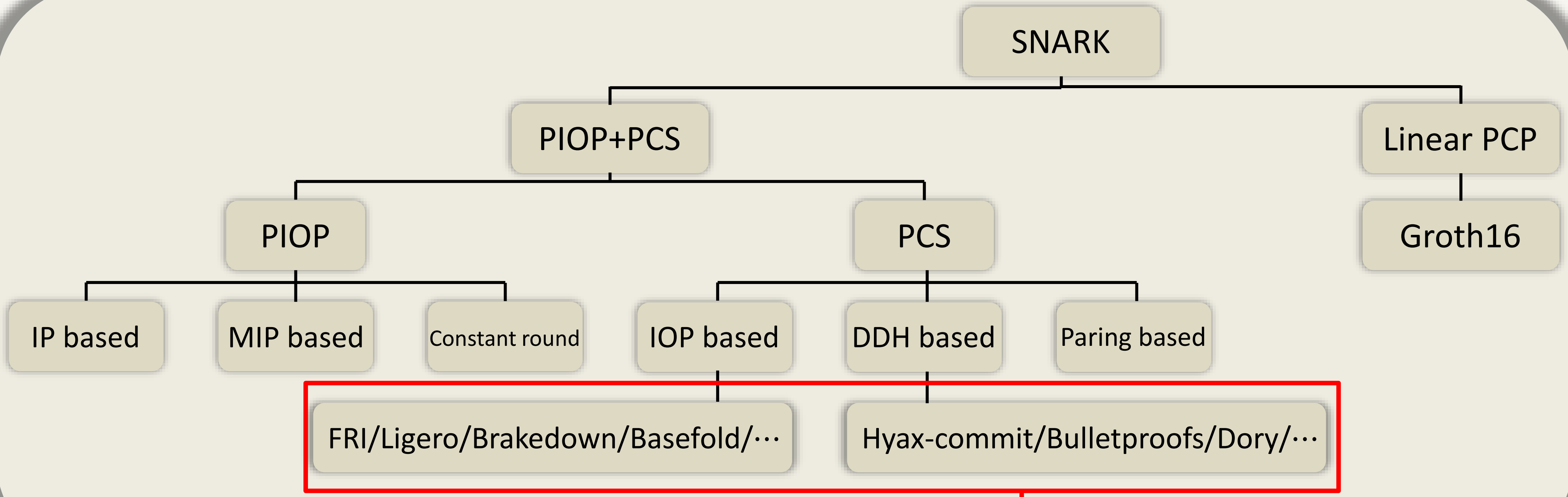
Why we need SNARKs over Galois Rings?



Source: [THA22]

Transparent but rely on finite field algebraic structure.

Why we need SNARKs over Galois Rings?



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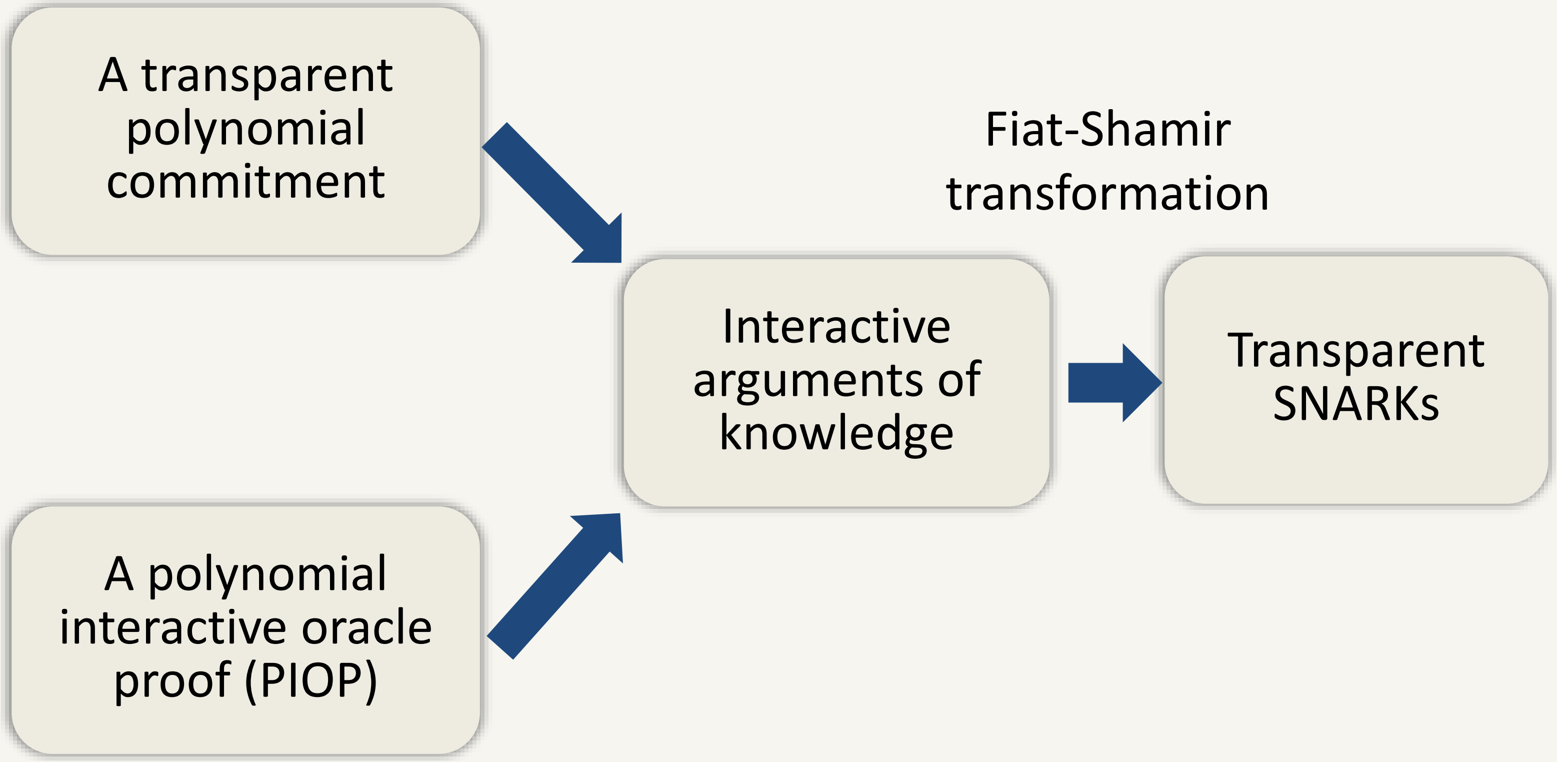
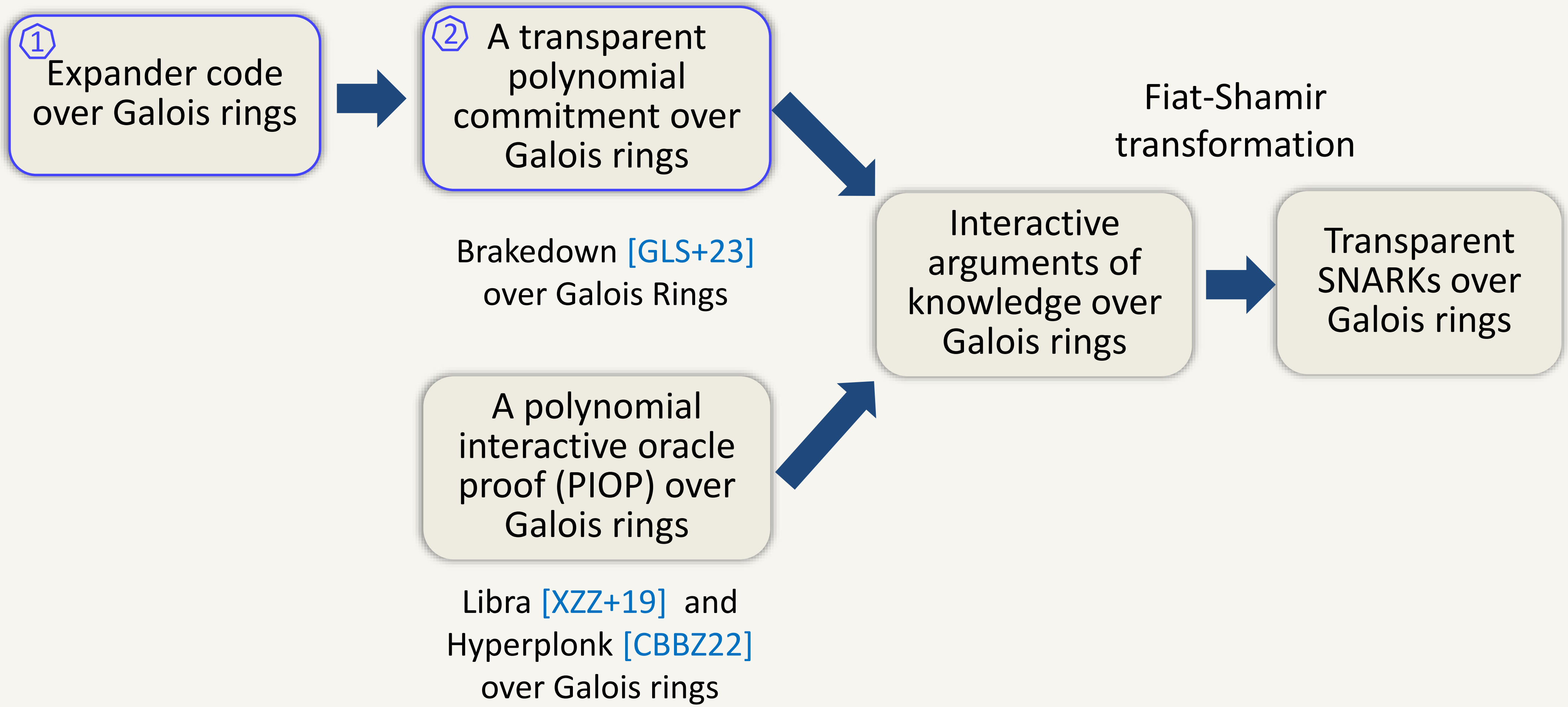
Transparent but rely on finite field algebraic structure.

Are Transparent Polynomial Commitments and SNARKs Possible Over Galois Rings?

PIOP+PCS


A transparent polynomial commitment
A polynomial interactive oracle proof (PIOP)
Fiat-Shamir transformation
Interactive arguments of knowledge
Transparent SNARKs

Our Contributions



Remainder of the talk

- **Expander code over Galois rings construction**
- **Brakedown commitment over Galois rings**
- **PIOP over Galois rings**

Galois Rings

- **Galois Rings :**

  $GR(p^s, r) \cong \mathbb{Z}_{p^s}[x]/f(x)$, where $f(x)$ is a monic polynomial of degree r which is irreducible modulo p^s.

- **Why Are SNARKs over Galois Rings So Challenging?**

  The presence of **zero divisors** in Galois rings invalidates the **Schwartz-Zippel lemma**, which is a fundamental component in proving the soundness of SNARKs.

  Schwartz-Zippel lemma over fields

  Let $P \in \mathbb{F}[x_1, \cdots, x_n]$ be a non-zero polynomial of total degree $d \geq 0$ over the field $\mathbb{F}$ and let $r_1, \cdots, r_n$ be selected at random independently and uniform from $\mathbb{F}$, then

$$\Pr[P(r_1, \cdots, r_n) = 0] \leq \frac{d}{|\mathbb{F}|}$$

Generalized Schwartz-Zippel lemma

- **Exceptional Set** [GNSV23]

  Let $A = \{a_1, \cdots, a_n\} \subset R$. We say that A is an exceptional set if $\forall i \neq j, a_i - a_j \in R^*$, where R^* is the set of all invertible elements in the ring R.

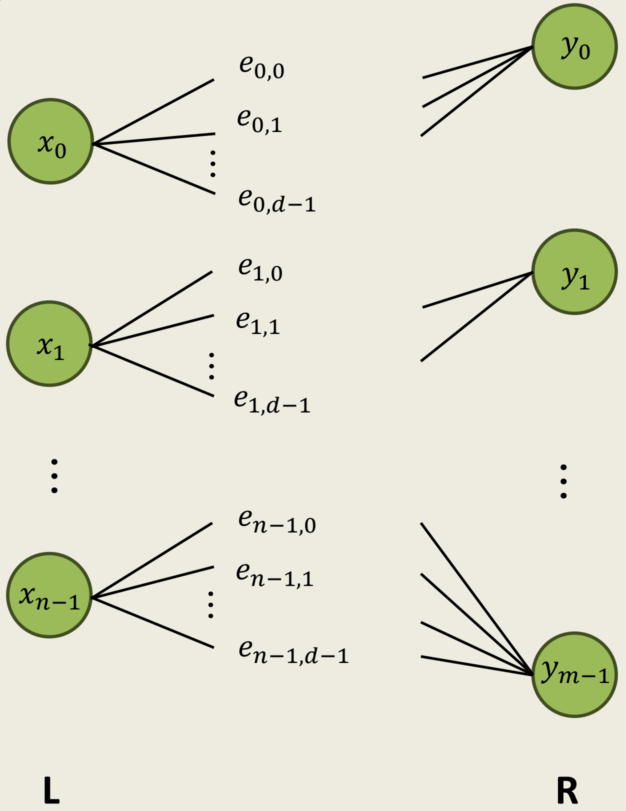
- **Generalized Schwartz-Zippel Lemma** [GNSV23]

  Let $f: R^n \to R$ be an n-variate nonzero polynomial. Let $A \subseteq R$ be a finite exceptional set. Let $\deg(f)$ denote the total degree of f. Then

$$\Pr_{a \in A^n}[f(a) = 0] \leq \frac{\deg(f)}{|A|}$$

The exceptional set of $GR(p^s, r)$ is $GF(p, r)$

Core Procedure in Expander Code



A bipartite graph $G = (L, R, E)$

$(x_0, \cdots, x_{n-1}) \rightarrow (y_0, \cdots, y_{m-1})$

$y_i = \sum_{x_j \in N(y_i)} e_{j,i} \cdot x_j$, $N(y_i)$ denote the neighbors of y_i.

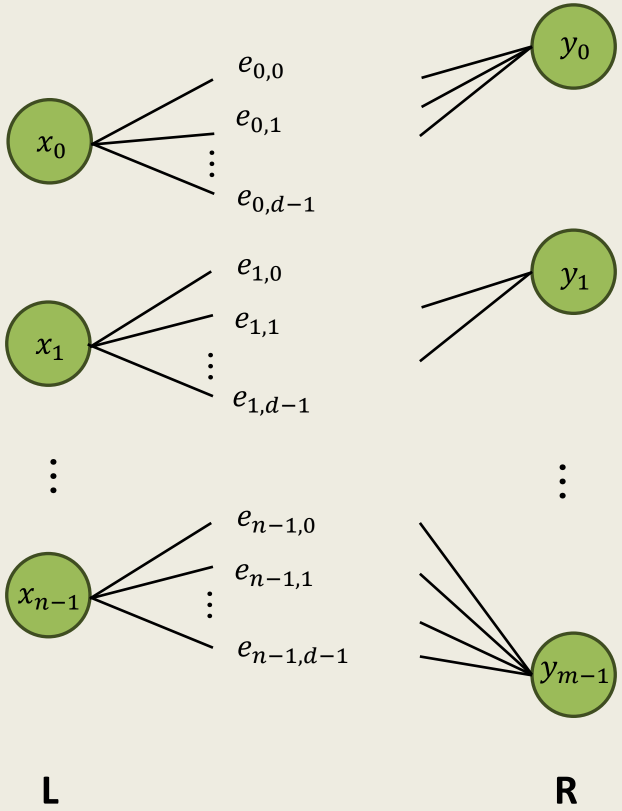
- **Expansion:**

  For every subset $S \subseteq L$ with $|S| = k$,$|N(S)| \geq b(k)$, where $b(k) = \max(k + 4, 1.28k)$

- **Nonzero:**

  For every subset $S \subseteq L$ satisfying the expansion and there is at least one nonzero element in S, the neighborhood $N(S)$ contains at least one non-zero element.

Core Procedure in Expander Code



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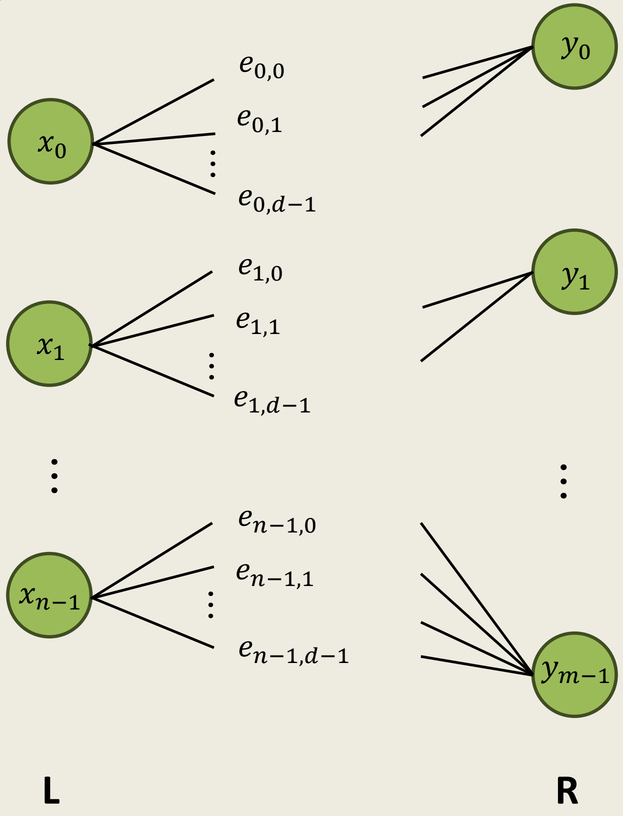

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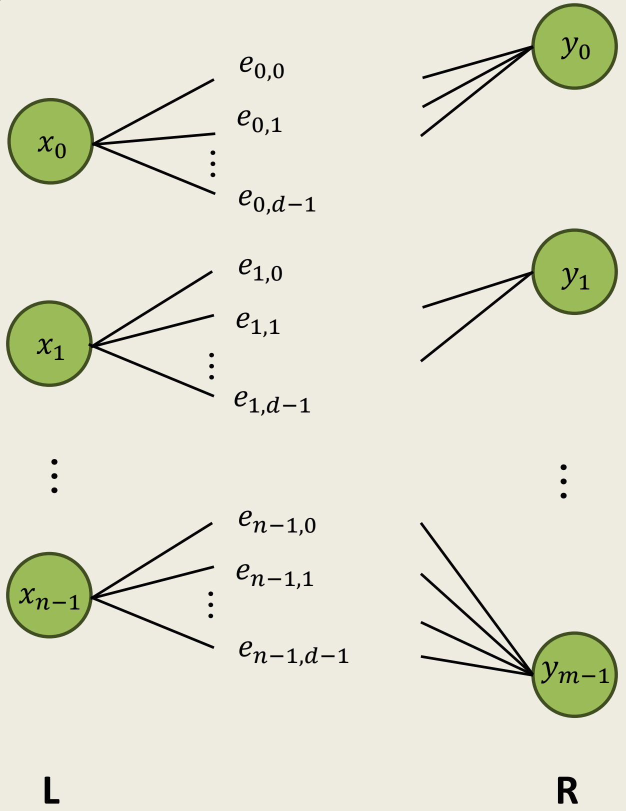

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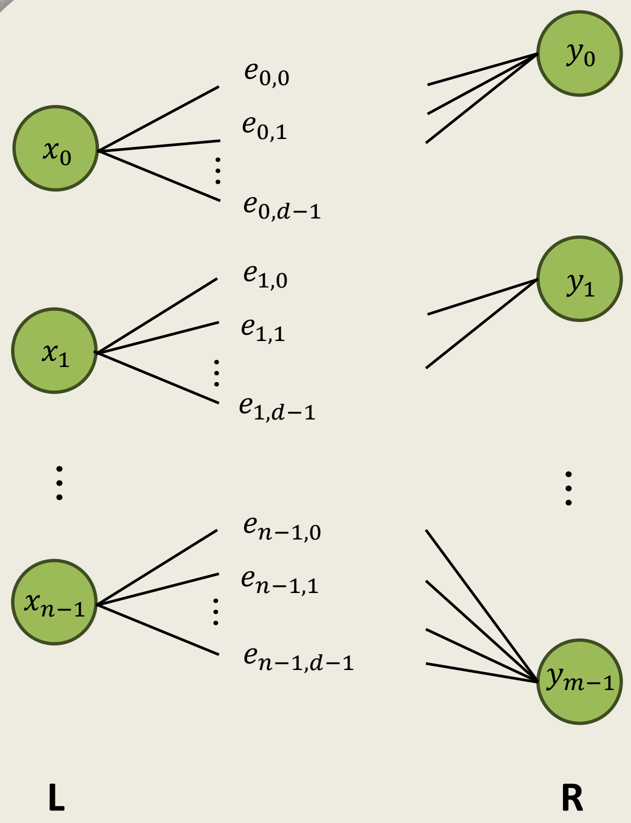
$$\downarrow$$

$$\Pr_{e_{i,j} \in GR(p^s, r)} [y_i(e_{j,i}) = 0]$$

Core Procedure in Expander Code



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$$\downarrow$$

$$\Pr_{e_{i,j} \in GR(p^s, r)} [y_i(e_{j,i}) = 0]$$

Generalized Schwartz-Zippel lemma fails to provide tight enough probability bounds.

Our Solution: Refined Parameter Analysis

- **Tightening the Bounds: Beyond Generalized Schwartz-Zippel**

  GCD over Galois rings:

  a is an element of ring $GR(p^s, r)$ and n is an integer. We define $\mathrm{GCD}(a, n)$ as $\mathrm{GCD}(a_0, \cdots, a_{r-1}, n)$. Where a is represented by $a_0 + a_1 x + \cdots + a_{r-1} x^{r-1}$.

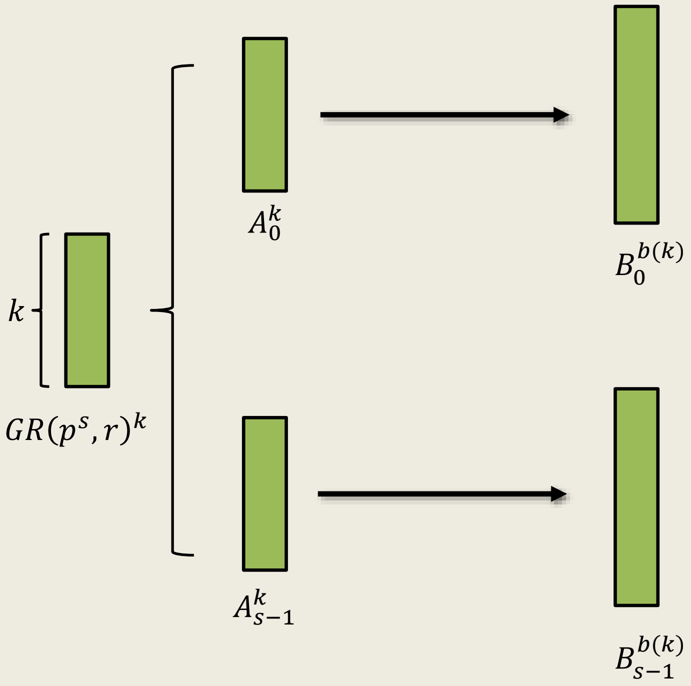
- **A key observation:**

  Consider elements $a, b \in GR(p^s, r)$. Let $d = \mathrm{GCD}(a, p^s)$. The linear equation $ax = b$ has at most d^r solutions.

$$\frac{d^r}{p^{sr}} \leq \frac{1}{p^r}$$

Equality is achieved when d attains its maximal value of p^{s-1}.

Our Solution: Refined Parameter Analysis



Classification by GCD with p^s

$$GR(p^s,r) = A_0 \cup \cdots \cup A_{s-1} \cup \{0\}$$

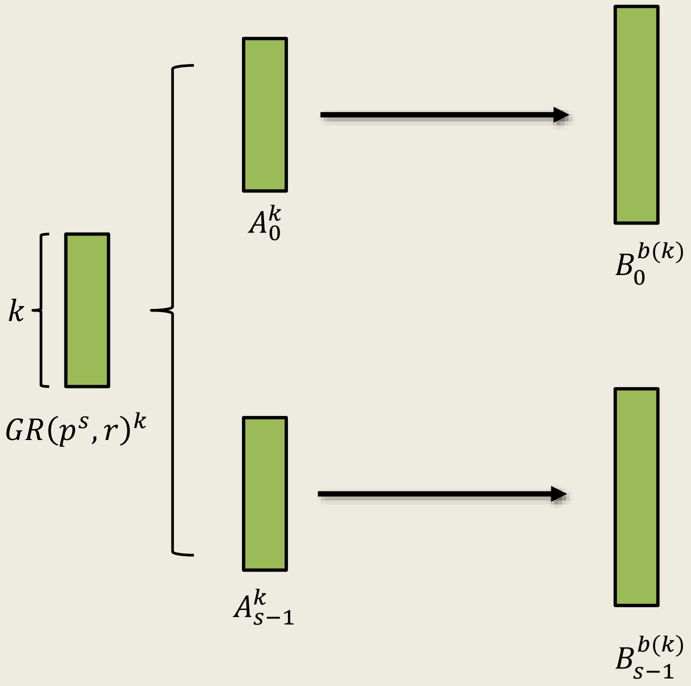
$$A_i = \{a | a \in GR(p^s,r) \cap \gcd(a,p^s) = p^i\},\ |A_i| = \left(\frac{p^s}{p^i}\right)^r$$

$$B_i = \{a | a \in GR(p^s,r) \cap \gcd(a,p^s) \geq p^i\}$$

Define the event E_i as A_i^k transformed to get $0^{b(k)}$:

$$\Pr[E_i] \leq |A_i|^k \frac{(p^i)^r}{p^{sr}} = \left(\frac{p^s}{p^i}\right)^{rk} \left(\frac{p^i}{p^r}\right)^{rb(k)} = \left(\left(\frac{p^i}{p^s}\right)^r\right)^{b(k)-k}$$

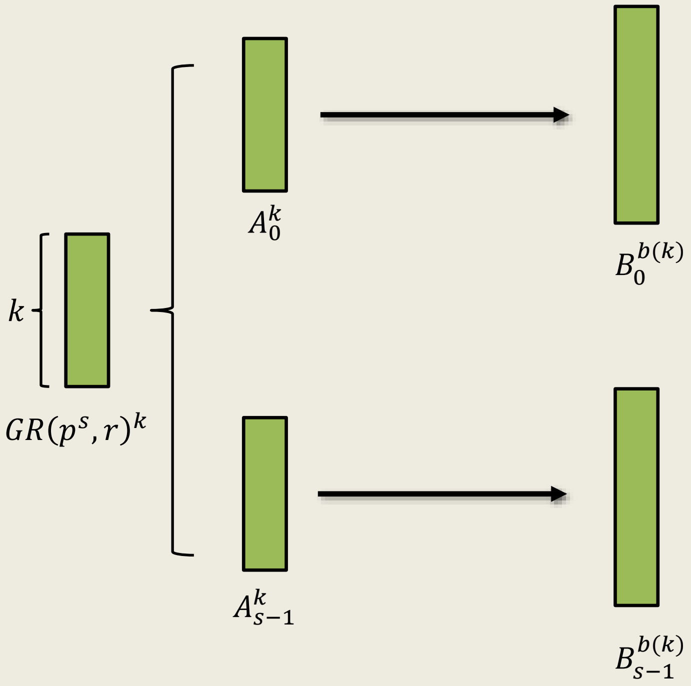
Our Solution: Refined Parameter Analysis



Classification by GCD with p^s

$$\begin{aligned}
\Pr[nonzero] &\le \Pr[E_0] + \Pr[E_1] + \cdots + \Pr[E_{s-1}] \\
&\le \sum_{i\in[0,s-1]} \left(\left(\frac{p^i}{p^s}\right)^r\right)^{b(k)-k} \\
&\le s\left(\left(\frac{p^{s-1}}{p^s}\right)^r\right)^{b(k)-k} \\
&= s\left(\frac{1}{p^r}\right)^{b(k)-k}
\end{aligned}$$

Our Solution: Refined Parameter Analysis



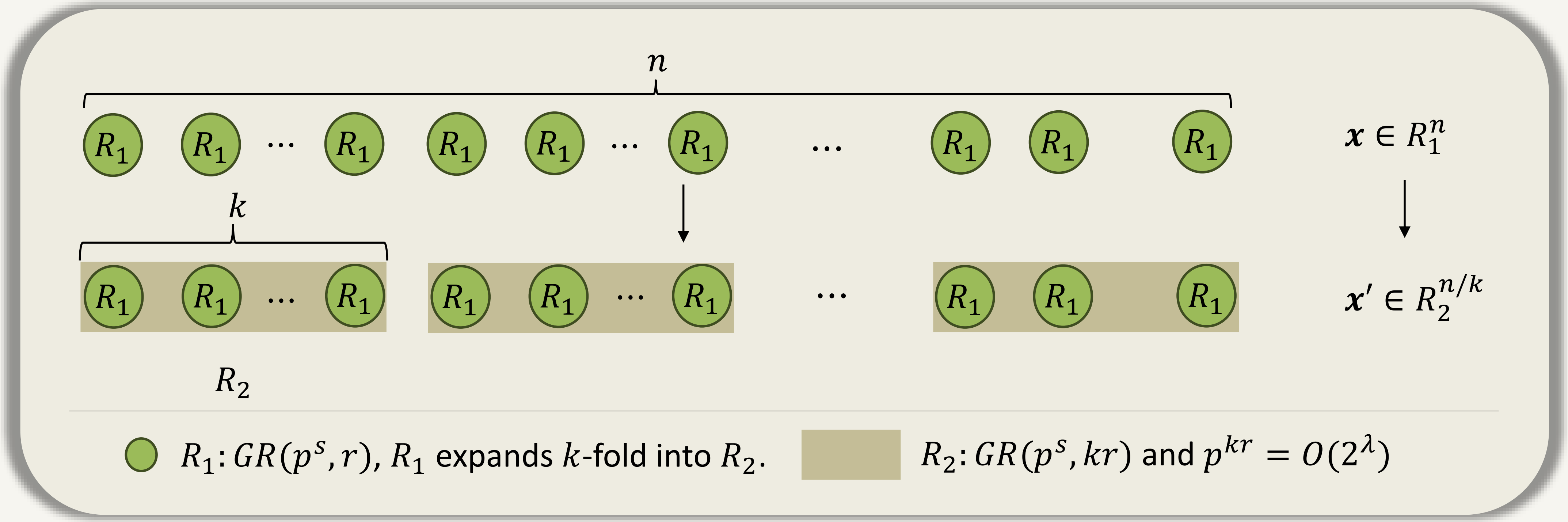
Classification by GCD with p^s

$$\Pr[nonzero] \le \Pr[E_0] + \Pr[E_1] + \cdots + \Pr[E_{s-1}]$$
$$\le \sum_{i\in[0,s-1]} \left(\left(\frac{p^i}{p^s}\right)^r\right)^{b(k)-k}$$
$$\le s\left(\left(\frac{p^{s-1}}{p^s}\right)^r\right)^{b(k)-k}$$
$$= s\left(\frac{1}{p^r}\right)^{b(k)-k}$$

We need $\left(\frac{1}{p^r}\right) = negl(\lambda)$, but What if p^r is not large enough?

Our Solution: Refined Parameter Analysis

- **Extend Binius [DP23] Block-level encoding to Galois rings**



Let Enc' is for linear encoding on R_2, then $Enc(\boldsymbol{x}) = Enc'(\boldsymbol{x}')$.

$\forall\, a \in R_1, a \cdot Enc(\boldsymbol{x}) = a \cdot Enc'(\boldsymbol{x}') = Enc'(a \cdot \boldsymbol{x}') = Enc(a \cdot \boldsymbol{x})$

Brakedown over Galois Rings

$$f(x_0, \cdots, x_{l-1}) = \sum_{b \in \{0,1\}^l} \prod_{i \in [0, l-1]} \big((1 - x_i)(1 - b_i) + x_i b_i\big) f(b)$$

$$U = \begin{bmatrix} f(0, \cdots, 0,0, \cdots 0) & f(0, \cdots, 0,0, \cdots 1) & \cdots & f(0, \cdots, 0,1, \cdots 1) \\ f(0, \cdots, 1,0, \cdots 0) & f(0, \cdots, 1,0, \cdots 1) & \cdots & f(0, \cdots, 1,1, \cdots 1) \\ \vdots & \vdots & \vdots & \vdots \\ f(1, \cdots, 1,0, \cdots 0) & f(1, \cdots, 1,0, \cdots 1) & \cdots & f(1, \cdots, 1,1, \cdots 1) \end{bmatrix}$$

$$\boldsymbol{s_1} = \big((1 - r_0, r_0) \otimes \cdots \otimes (1 - r_{l/2-1}, r_{l/2-1})\big)$$
$$\boldsymbol{s_2} = \big((1 - r_{l/2}, r_{l/2}) \otimes \cdots \otimes (1 - r_{l-1}, r_{l-1})\big)$$

$$f(r_0, \cdots, r_{l-1}) = \boldsymbol{s}_1^\top U \boldsymbol{s}_2$$

Brakedown over Galois Rings

$$f(x_0,\cdots,x_{l-1}) = \sum_{b\in\{0,1\}^l} \prod_{i\in[0,l-1]} \big((1-x_i)(1-b_i)+x_ib_i\big)f(b)$$

$$U = \begin{bmatrix} f(0,\cdots,0,0,\cdots 0) & f(0,\cdots,0,0,\cdots 1) & \cdots & f(0,\cdots,0,1,\cdots 1) \\ f(0,\cdots,1,0,\cdots 0) & f(0,\cdots,1,0,\cdots 1) & \cdots & f(0,\cdots,1,1,\cdots 1) \\ \vdots & \vdots & \vdots & \vdots \\ f(1,\cdots,1,0,\cdots 0) & f(1,\cdots,1,0,\cdots 1) & \cdots & f(1,\cdots,1,1,\cdots 1) \end{bmatrix}$$

$$\boldsymbol{s_1} = \Big((1-r_0,r_0)\otimes\cdots\otimes\big(1-r_{l/2-1},r_{l/2-1}\big)\Big)$$

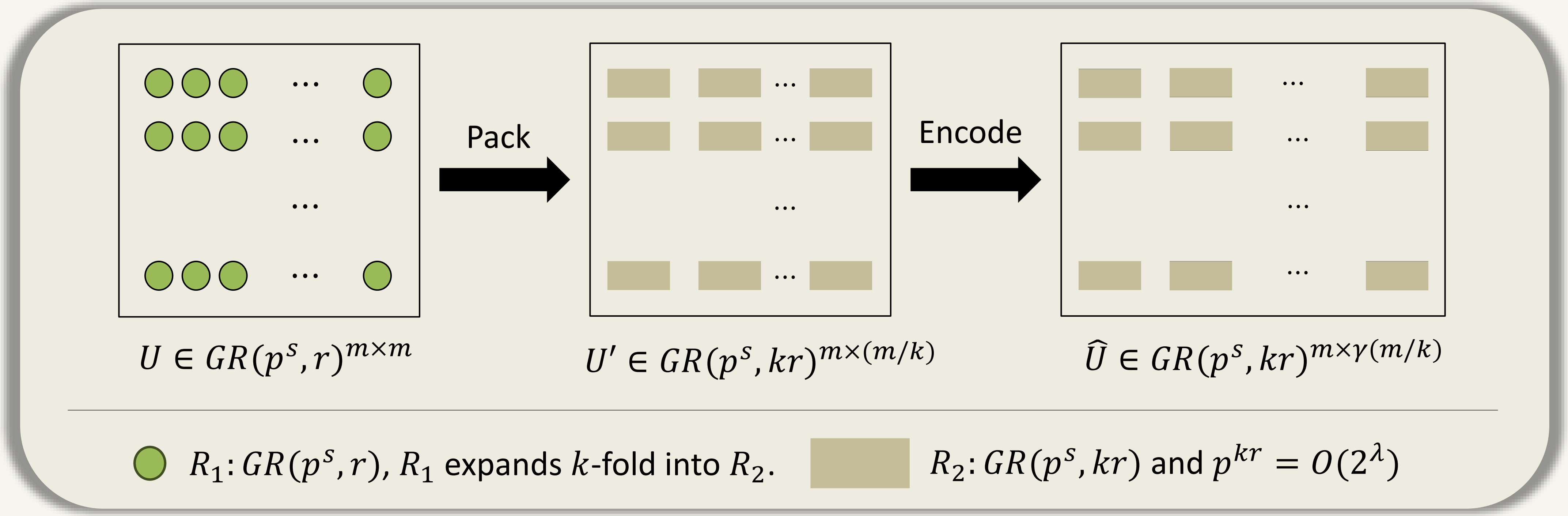
$$\boldsymbol{s_2} = \Big(\big(1-r_{l/2},r_{l/2}\big)\otimes\cdots\otimes(1-r_{l-1},r_{l-1})\Big)$$

$$f(r_0,\cdots,r_{l-1}) = \boxed{\boldsymbol{s}_1^{\top}U}\boldsymbol{s}_2$$

Brakedown over Galois Rings : Commit Phase

Let $\boldsymbol{U} \in GR(p^s, r)^{m \times m}$ be the coefficient matrix of the l-variable multilinear polynomial f to be committed, where $m = 2^{l/2}$ and $\boldsymbol{U} = (\boldsymbol{u}_0, \cdots, \boldsymbol{u}_{m-1})$.

Prover:

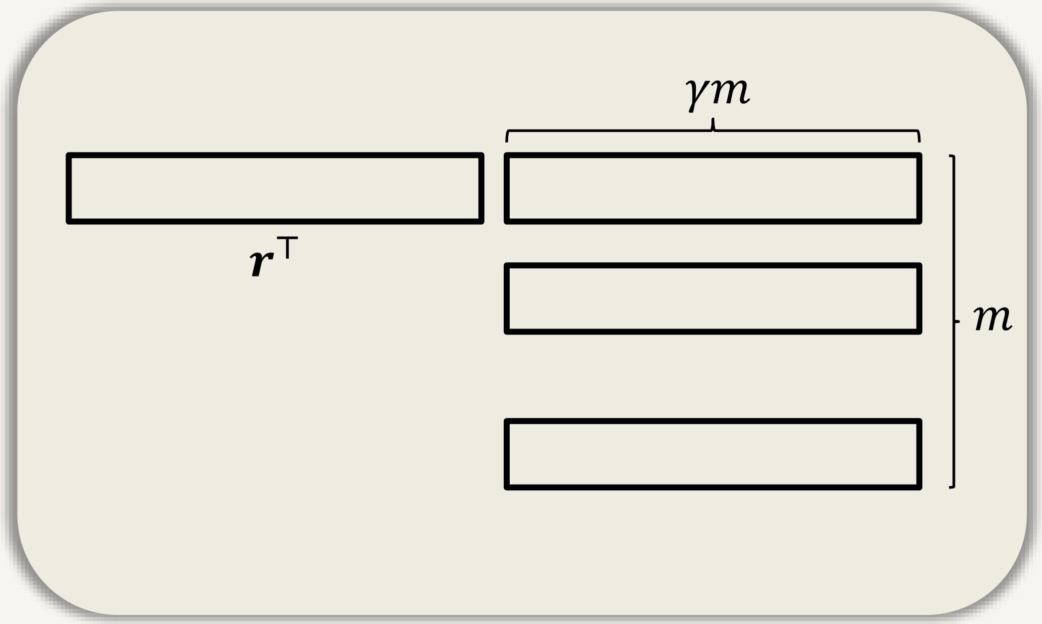


Where $\frac{1}{\gamma}$ is the code rate. The prover then constructs a Merkle tree from the $\hat{U}$ and sends its root hash to the verifier as the commitment.

Brakedown over Galois Rings: Testing Phase

Soundness Weakness

The verifier randomly checks a linear combination of matrix $\boldsymbol{U}$ by sampling an m-length vector $\boldsymbol{r}_1$. By the Schwartz-Zippel Lemma, the soundness error is at most $\frac{1}{p^r}$. However, the required soundness error must be $\frac{1}{p^{kr}}$ to meet security guarantees.

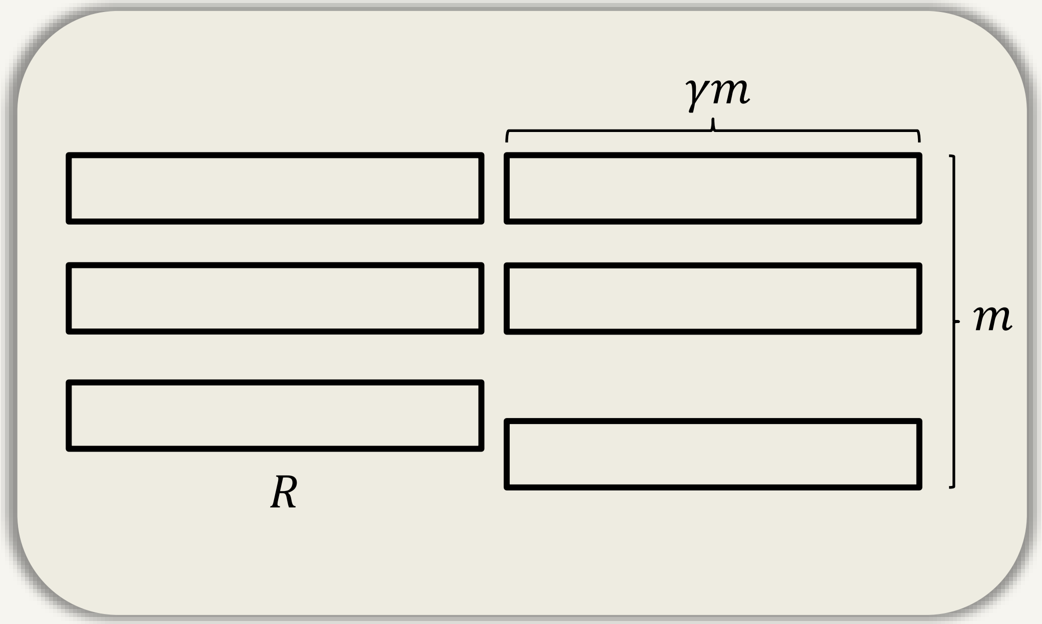


Our Solution: Repetition

[AHIV17] Repetition Version

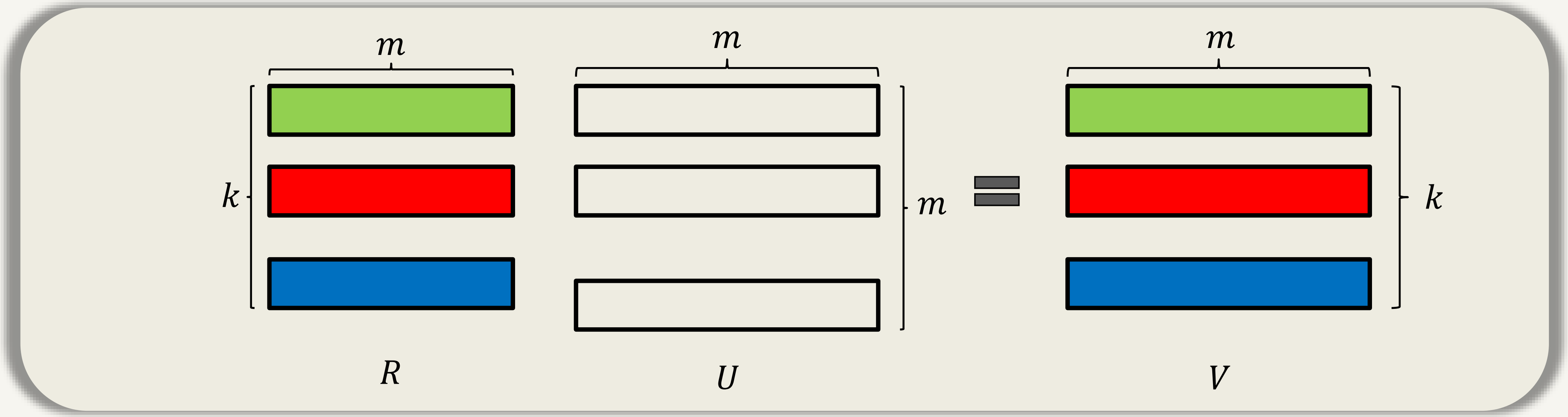
Fixed any $[l, n, d]$ code $C \subset R_2^l$ and a proximity parameter $e \in \{0, \cdots, \left\lfloor \frac{d-1}{3} \right\rfloor\}$. For a matrix $\widehat{U} \in R_2^{m \times l}$ with $d\left(\widehat{U}, C^m\right) > e$, and a matrix $R \in R_1^{k \times m}$ where each element of R is randomly chosen from R_1, let $W = RU$. Then:

$$\Pr\left[d\left(W, C^k\right) \leq e\right] \leq \frac{e+1}{p^{kr}}$$



where $d\left(W, C^k\right) := \frac{|\{j \in [l] | \exists i \; s.t. W_i[j] \neq c_i[j]\}|}{n}$ and c_i denote the closet codeword with row U_i in C.

Brakedown over Galois Rings: Testing Phase

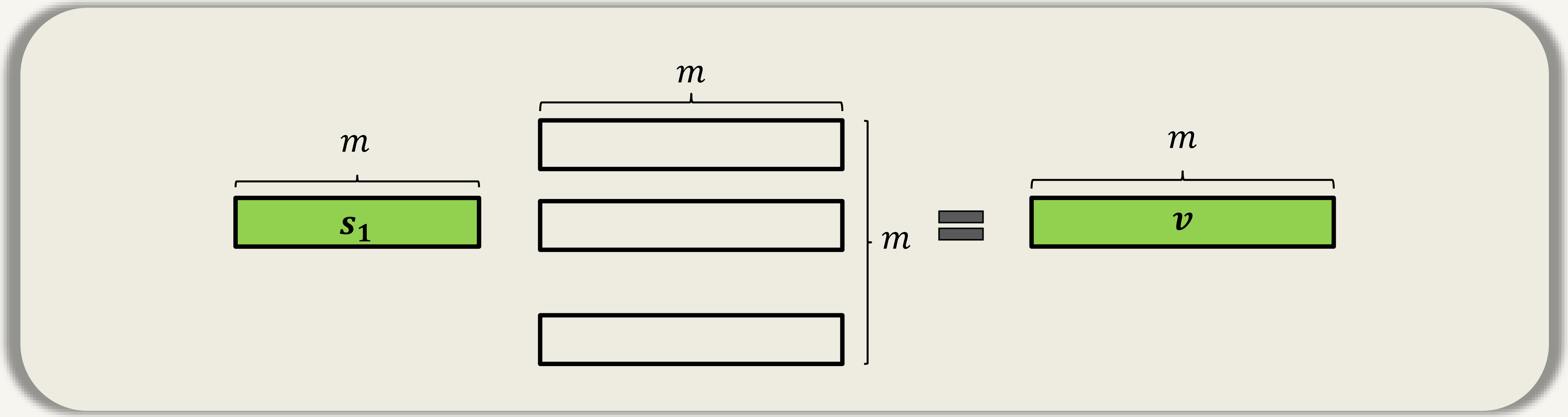


The procedure:

- V sends a random matrix $R \in \left(GR(p^s, r)\right)^{k\times m}$,
- P compute $V = RU$ and sends V,
- V picks $\Theta(\lambda)$ column indices and check $\forall\, i \in [0, k-1]$: $Enc(V_i)[j] = \sum_{s\in[0,m-1]} R_i[s] \cdot \widehat{U}_s[j]$

PS: The fundamental verification unit after encoding is the $GR(p^s, kr)$.

Brakedown over Galois Rings: Evaluation Phase



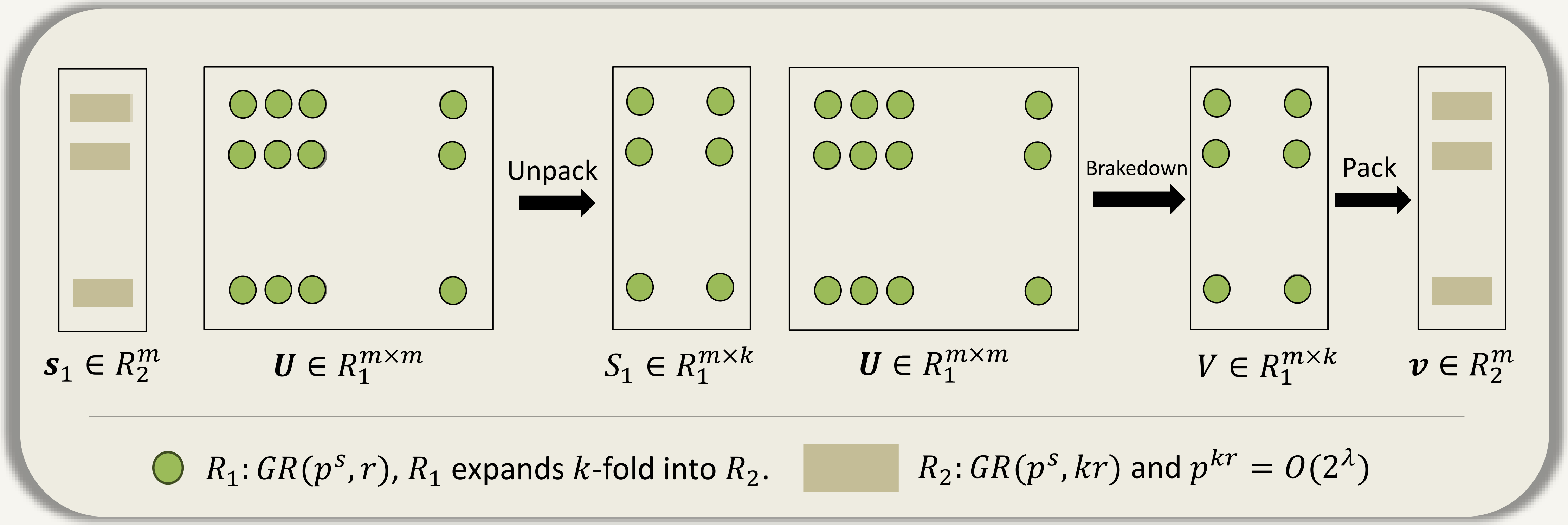
The procedure follows the testing phase exactly, except:

- The verifier substitutes the matrix R with the vector $\boldsymbol{s}_1^\top$, and
- Computes the evaluation $\boldsymbol{v}\top\boldsymbol{s}_2$ upon receiving vector v from the prover and successfully verifying it.

Efficient Computation in Galois Ring Extensions

Polynomial Commitments with Coefficients over R_1 and Evaluations over R_2

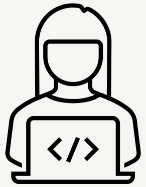
The arithmetic circuit C performs all computations over R_1, ensuring the polynomial f's coefficients lie in R_1, while the evaluation of f is opened over R_2 for verifiable safety.



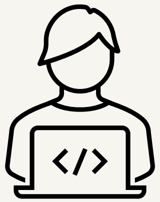
Sumcheck over Galois Rings

statement $\sum_{x_1}\sum_{x}\cdots\sum_{x} p(x_0,\cdots,x_{l-1}) = H$ Let $H_0 = H$, and $r_{<0} \coloneqq \emptyset$

Prover



Verifier



Round i, $i \in [0, l-2]$

- Compute $p_i(X_i) = \sum_{(x_{i+1},\cdots,x_{l-1})} p(r_0,\cdots,r_{i-1},X_i,x_{i+1},\cdots,x_{l-1})$

$p_i(X_i)$ →

← r

- Check if $p_i(0) + p_i(1) = H_i$
- Randomly chose r from the **exceptional set** from R
- Compute $H_{i+1} = p_i(r)$

(The size of exceptional set is $O(2^\lambda)$)

Round l

- Compute $p_{l-1}(X_{l-1}) = p(r_0,\cdots,r_{l-2},X_{l-1})$

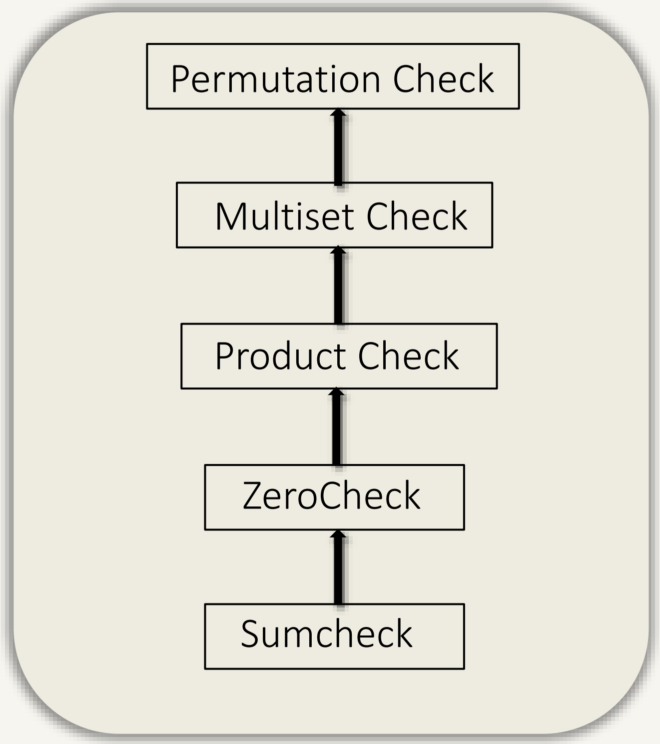
$p_{l-1}(X_{l-1})$ →

- Check if $p_{l-1}(0) + p_{l-1}(1) = H_{l-1}$
- Randomly chose r from the **exceptional set** from R
- Compute $p(r_0,\cdots,r_{l-1}) = p_{l-1}(r)$

HyperPlonk over Galois Rings

HyperPlonk **[CBBZ22]**

- **Gate Constraints** SumCheck
- **Permutation Constraints**



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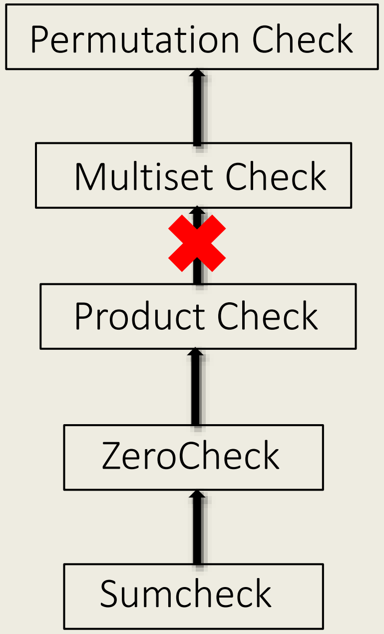


Finite fields:

$S \subset \mathbb{F}_p, \phi: S \to \mathbb{F}[x], \quad \phi(S) = f_s: f_s(x) = \prod_{a \in S}(x - a)$

By the Schwartz-Zippel lemma, the function ϕ is guaranteed to be injective.

Galois rings:

The zero divisors interfere with the Schwartz-Zippel Lemma, causing ϕ not to be injective, e.g., under mod 8, both sets $\{3,5\}, \{1,7\}$ get the polynomial $x^2 - 1$.

HyperPlonk over Galois Rings

HyperPlonk [CBBZ22]

- **Gate Constraints** ZeroCheck
- **Permutation Constraints**

$$\boxed{\text{Permutation Check}} \uparrow \boxed{\text{ZeroCheck}} \uparrow \boxed{\text{Sumcheck}}$$

Permutation $\sigma: \{0,1\}^l \to \{0,1\}^l$, $\tilde{\sigma} = \left(\sigma_0(\boldsymbol{x}), \cdots, \sigma_{l-1}(\boldsymbol{x})\right)$, where σ_i denotes the i-th bit of the permutation.

$$f\left(\tilde{\sigma}(\boldsymbol{x})\right) - g(\boldsymbol{x}) = 0, \forall\, \boldsymbol{x} \in \{0,1\}^l$$

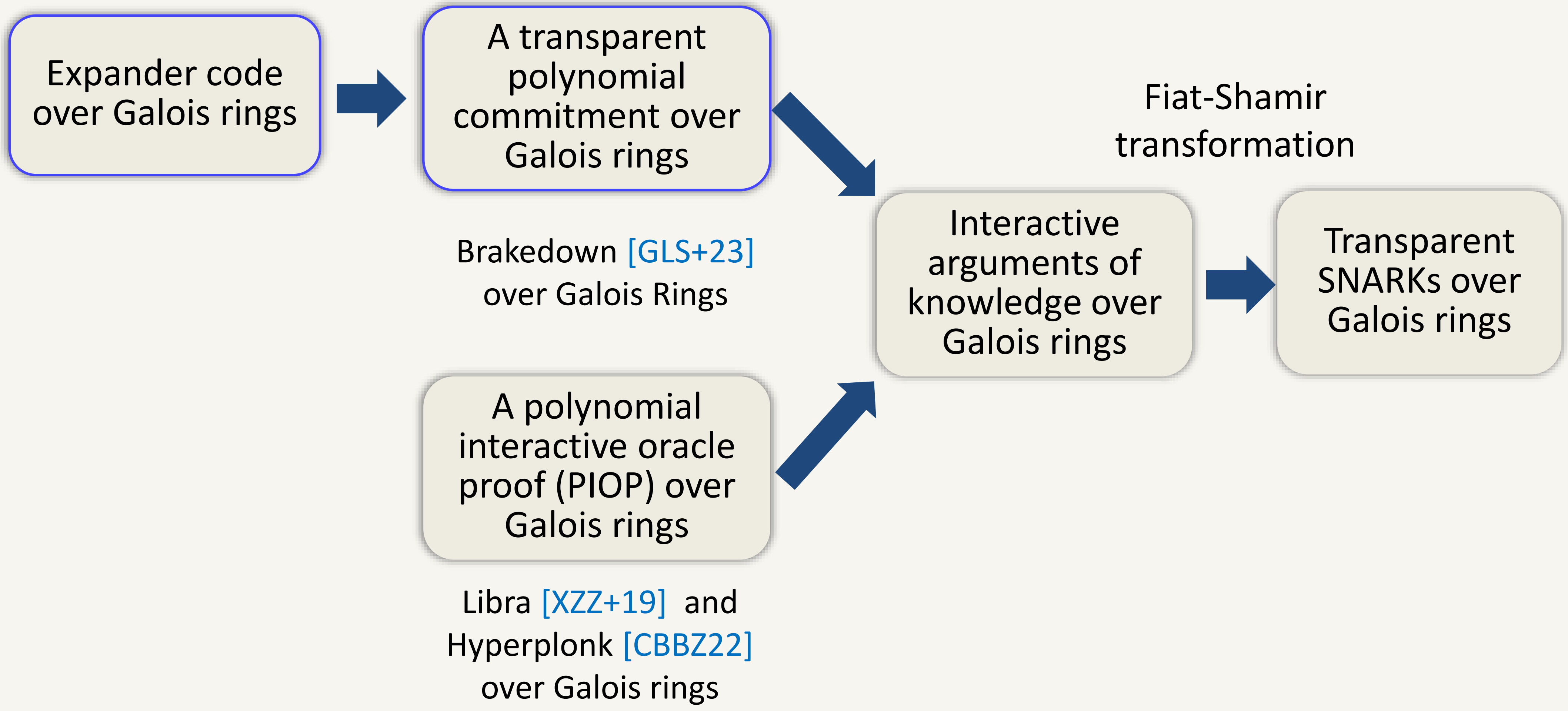
$$\Downarrow$$

$$\sum_{y \in \{0,1\}^l} \left(f(\boldsymbol{y}) \cdot eq(\tilde{\sigma}(\boldsymbol{x}), \boldsymbol{y}) - g(\boldsymbol{y}) \cdot eq(\boldsymbol{x}, \boldsymbol{y})\right) = 0\,, \forall \boldsymbol{x} \in \{0,1\}^l$$

$$\Downarrow$$

$$\sum_{x \in \{0,1\}^l} eq(\boldsymbol{x}, \boldsymbol{y}) \sum_{y \in \{0,1\}^l} \left(f(\boldsymbol{y}) \cdot eq(\tilde{\sigma}(\boldsymbol{x}), \boldsymbol{y}) - g(\boldsymbol{y}) \cdot eq(\boldsymbol{x}, \boldsymbol{y})\right) = 0$$

Transparent SNARK over Galois Rings



Thank you for your attention

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