Unconditional foundations for supersingular isogeny-based cryptography

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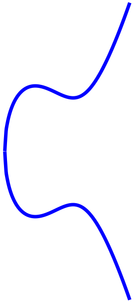
The Isogeny Problem



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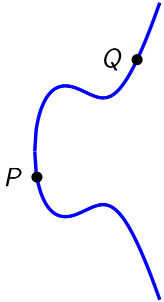

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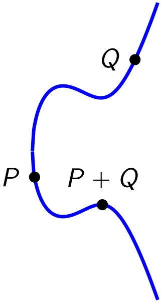

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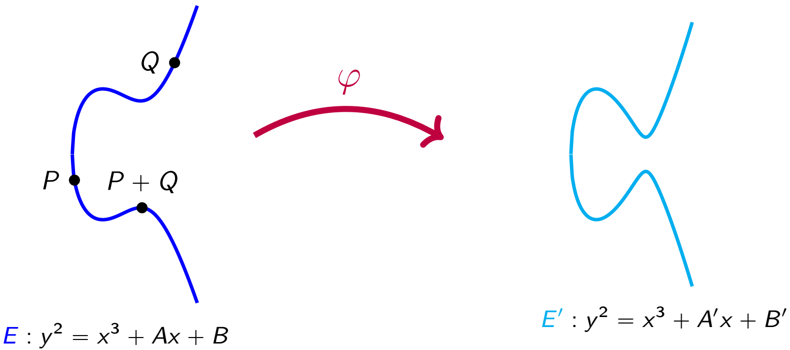

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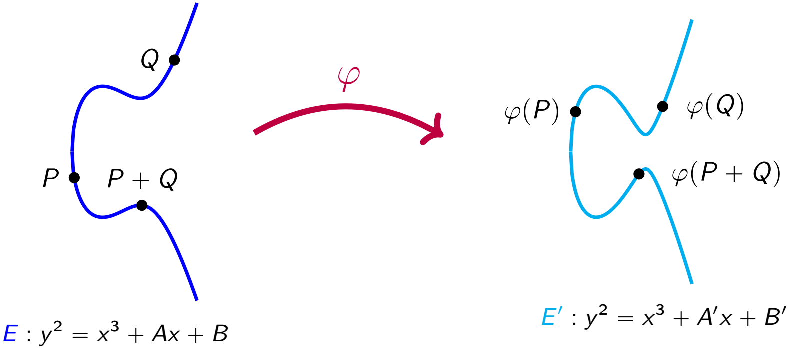

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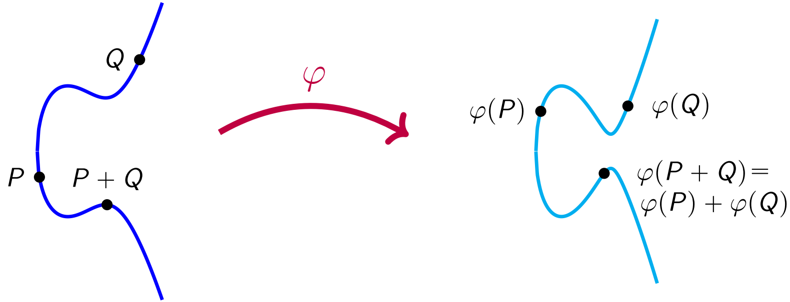

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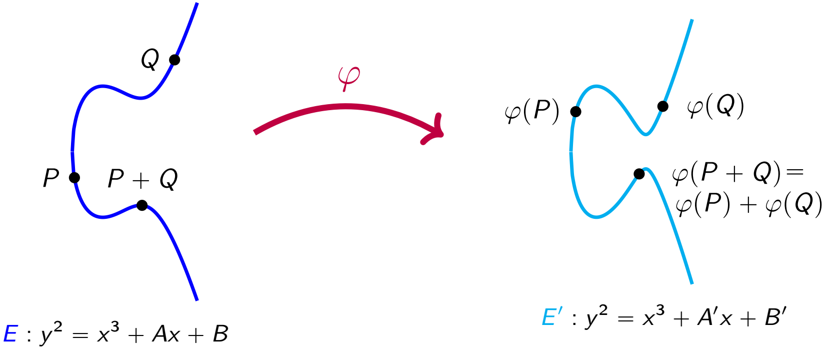

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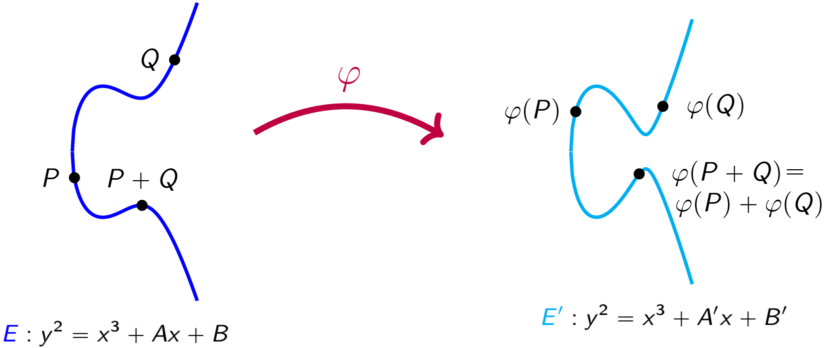

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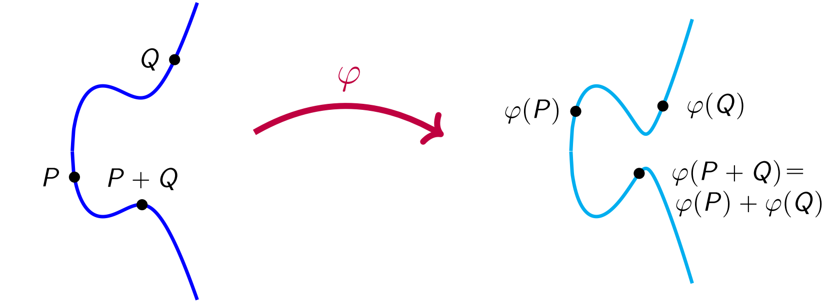

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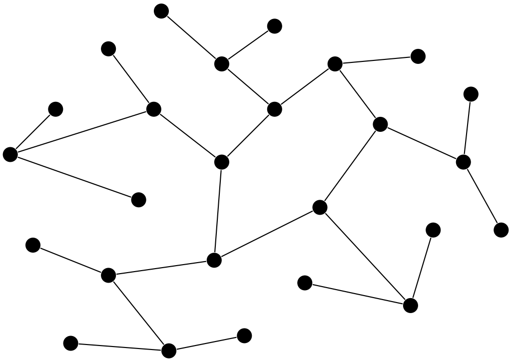

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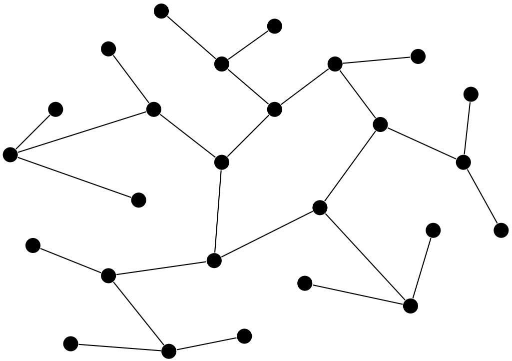

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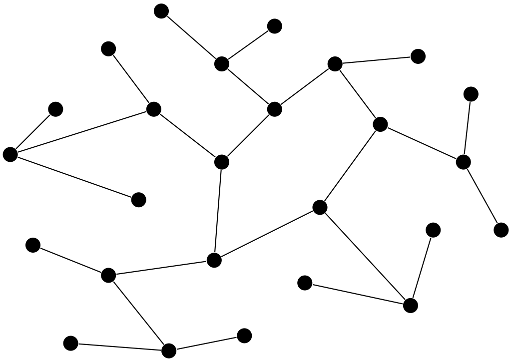

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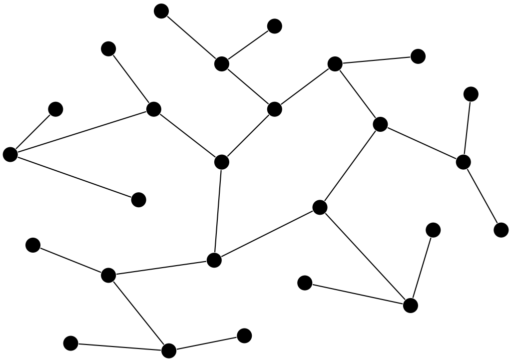

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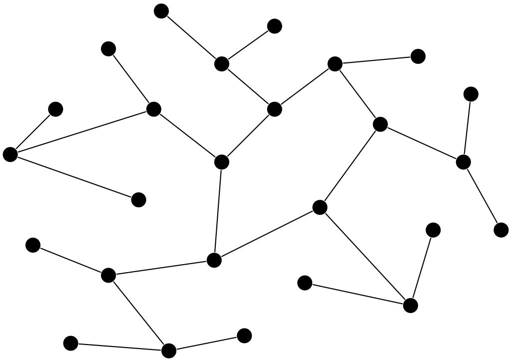

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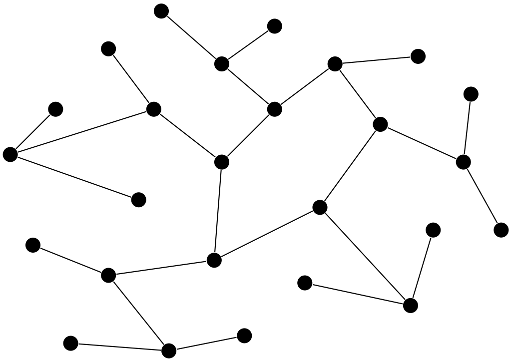

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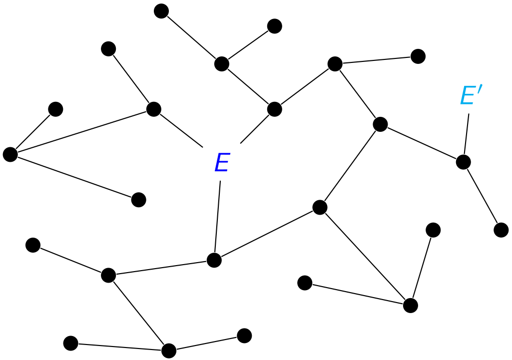

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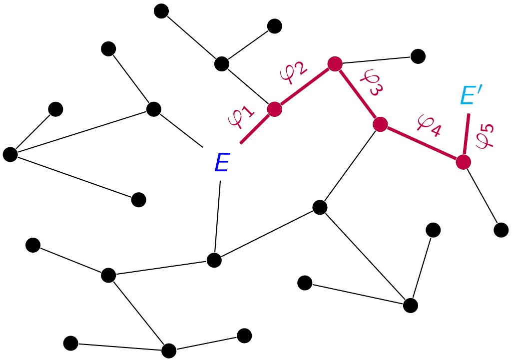

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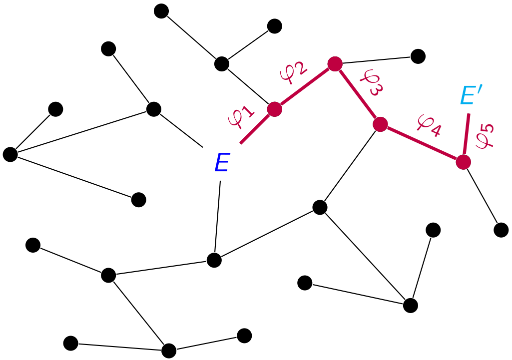

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$$\text{End}(E) = \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z} + \alpha_4\mathbb{Z}.$$

The Maximal Order Problem (MaxOrder)

Given $E/\mathbb{F}_{p^2}$, find **four quaternions** $\alpha_1, \ldots, \alpha_4$ in $\left(\frac{-p,-q_p}{\mathbb{Q}}\right)^*$ such that

$$\text{End}(E) \simeq \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z} + \alpha_4\mathbb{Z}.$$

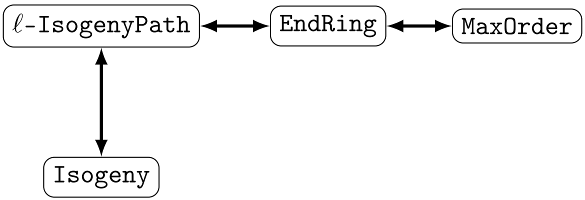
$\text{End}(E)$	a maximal order $\mathcal{O}$ in $\left(\frac{-p,-q_p}{\mathbb{Q}}\right)$
$\varphi : E_1 \to E_2$	I_φ left $\mathcal{O}_1$-ideal and right $\mathcal{O}_2$-ideal
$\deg \varphi$	norm of I
$\varphi \circ \psi$	$I_{\varphi\circ\psi} = I_\psi \cdot I_\varphi$

Hard to compute → (left column: $\varphi : E_1 \to E_2$) ← **Easy to compute** (right column: I_φ)

Deuring correspondence in a nutshell

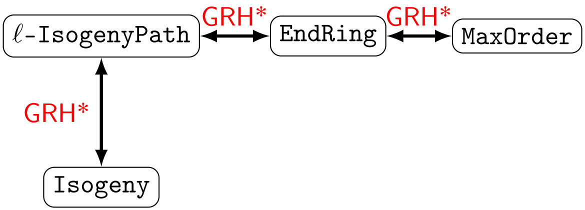
$* : \left(\frac{-p,-q_p}{\mathbb{Q}}\right) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + ij\mathbb{Q}$ with $i^2 = -p$, $j^2 = -q_p$ and $ij = -ji$.

Previous state-of-the-art



Polynomial reductions between isogeny-based problems

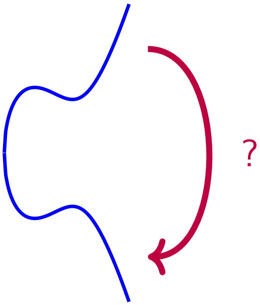
Previous state-of-the-art

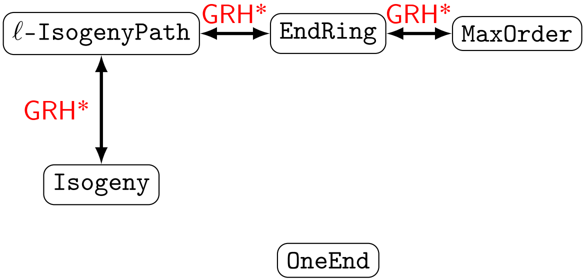


Polynomial reductions between isogeny-based problems

*: Proven under heuristics [Eis+18] then under the Generalised Riemann Hypothesis [Wes22].

One Endomorphism problem





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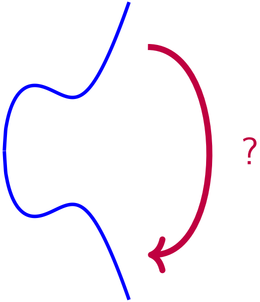

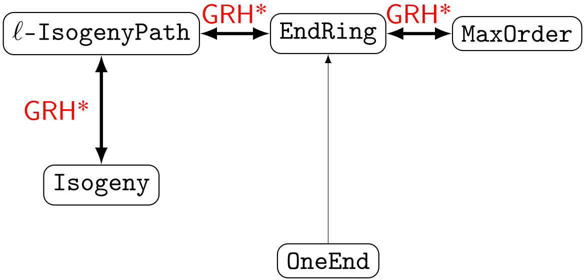

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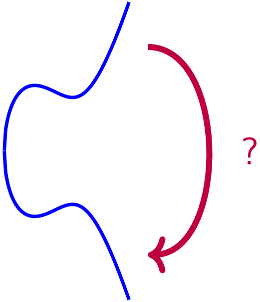

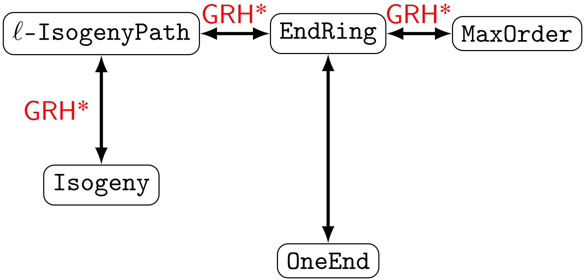

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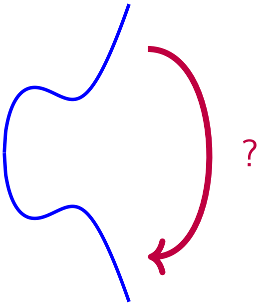

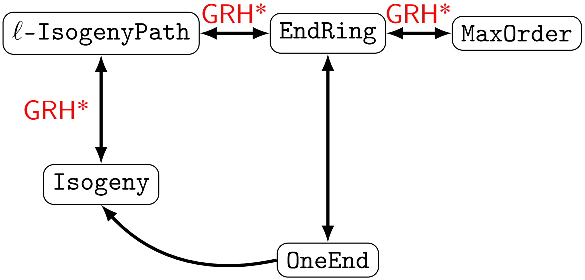

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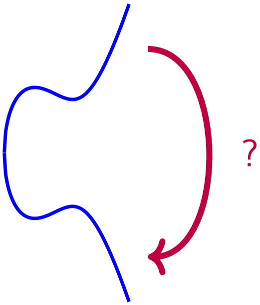

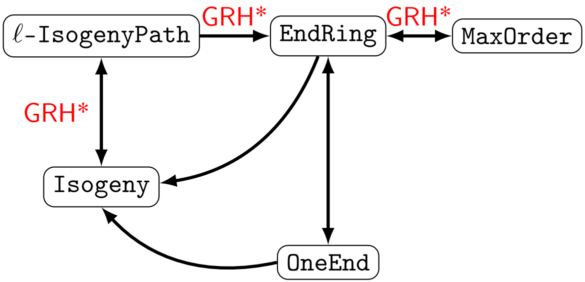

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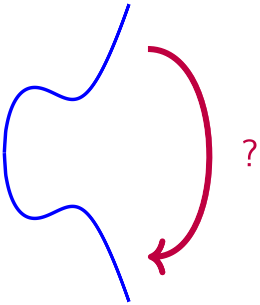

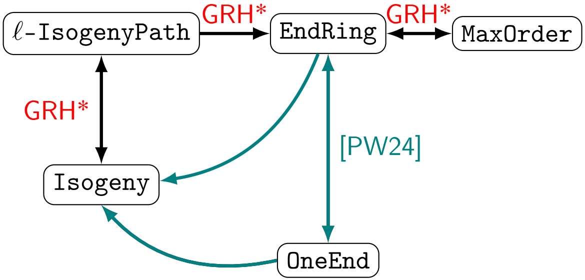

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Previous state-of-the-art

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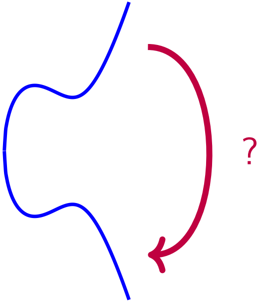

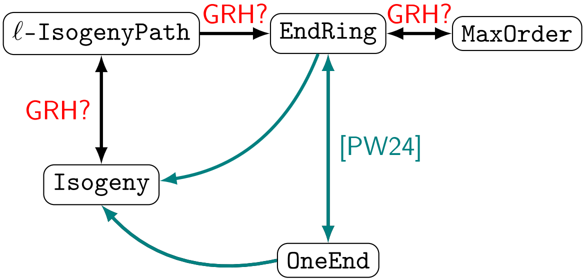

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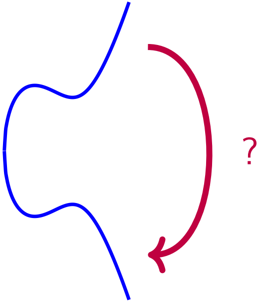

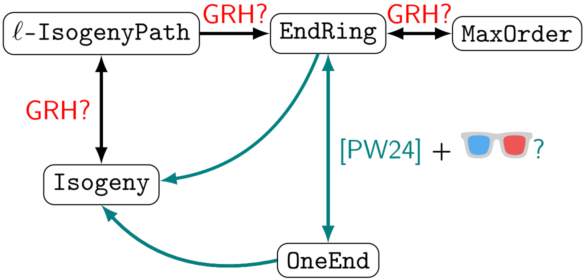

Polynomial reductions between isogeny-based problems

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: higher dimensional results following SIDH's attacks [CD23; Mai+23; Rob23]

Why consider such a variety of problems?

OneEnd

EndRing

Why consider such a variety of problems?

Good for security proofs

OneEnd

EndRing

Why consider such a variety of problems?

Good for security proofs

`OneEnd`

Good for attacks

`EndRing`

Why consider such a variety of problems?

Good for security proofs

`OneEnd`

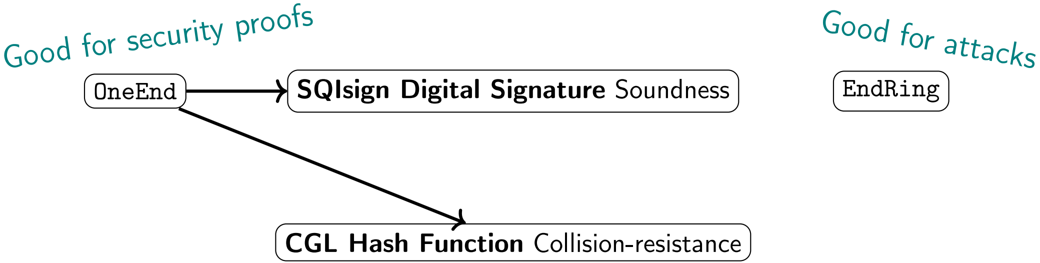
SQIsign Digital Signature Soundness

Good for attacks

`EndRing`

CGL Hash Function Collision-resistance

Why consider such a variety of problems?



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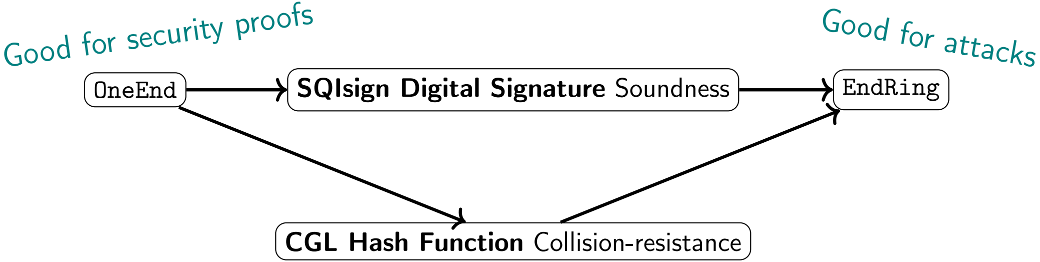

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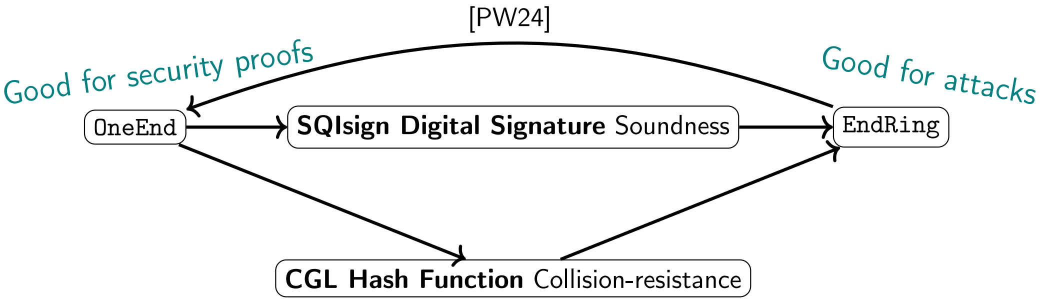
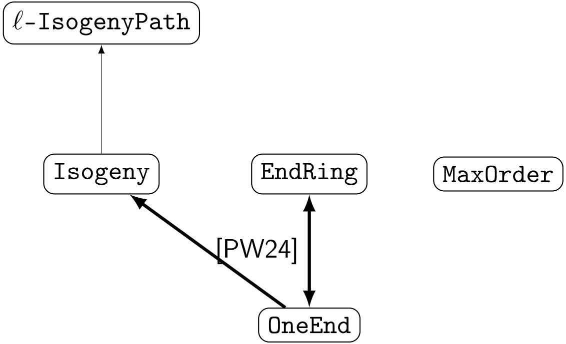
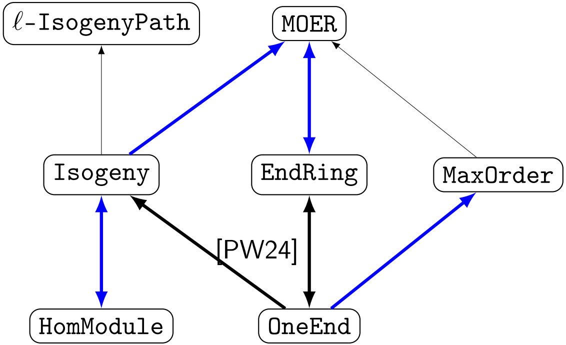


Equivalences **without GRH**



Polynomial reductions between isogeny-based problems **without GRH**

Equivalences **without GRH**



Polynomial reductions between isogeny-based problems **without GRH**

The Maximal Order Problem (MaxOrder)

Given $E/\mathbb{F}_{p^2}$, find **four quaternions** $\alpha_1, \dots, \alpha_4$ in $\left(\frac{-p,-q_p}{\mathbb{Q}}\right)$ such that

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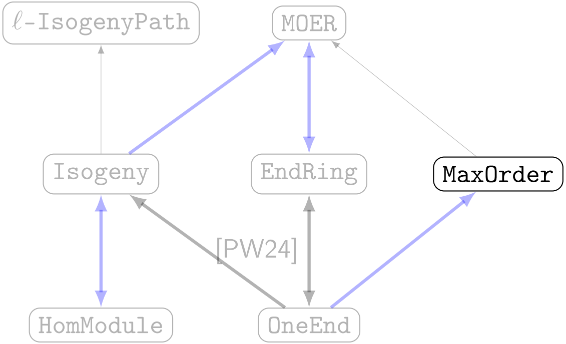

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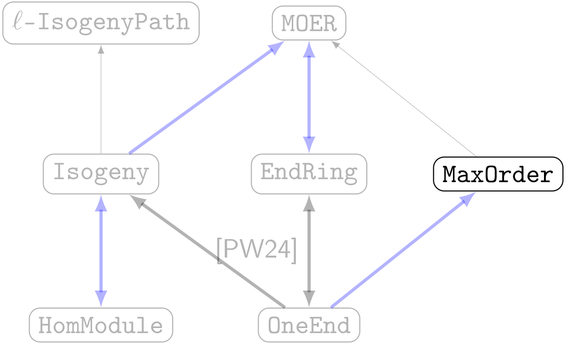


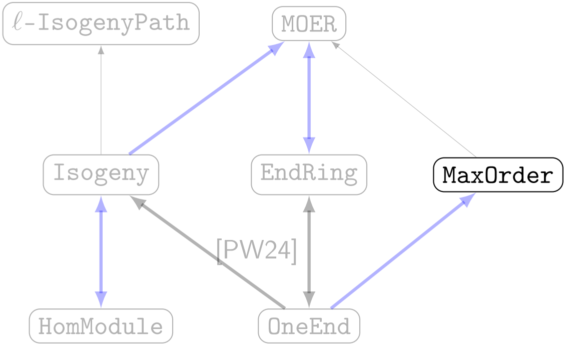
Polynomial reductions between isogeny-based problems **without GRH**

Equivalences **without GRH**

The MaxOrder Problem (MaxOrder)

Given $E/\mathbb{F}_{p^2}$, compute
two integers $a, b \in \mathbb{Z}_{>0}$ and
four quaternions $\alpha_1, \ldots, \alpha_4$ in $\left(\frac{-a,-b}{\mathbb{Q}}\right)$
such that

$$\mathrm{End}(E) \simeq \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z} + \alpha_4\mathbb{Z}.$$



Polynomial reductions between isogeny-based problems **without GRH**

Equivalences **without GRH**

The MaxOrder + EndRing Problem (MOER)

Given $E/\mathbb{F}_{p^2}$, compute
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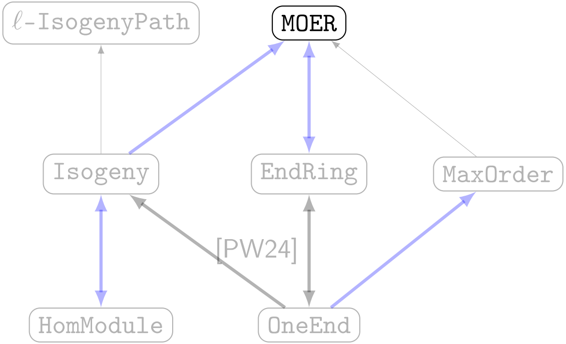
and **an isomorphism**

$$\varepsilon : \mathrm{End}(E) \xrightarrow{\sim} \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z} + \alpha_4\mathbb{Z}.$$



Polynomial reductions between isogeny-based problems **without GRH**

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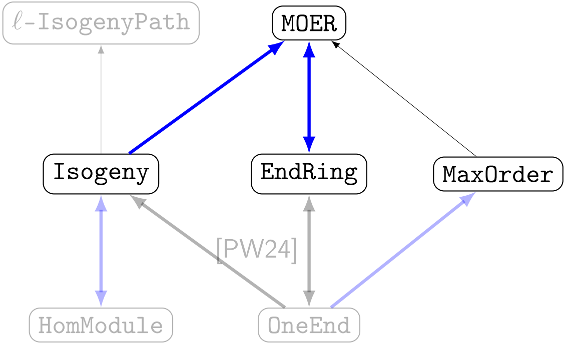
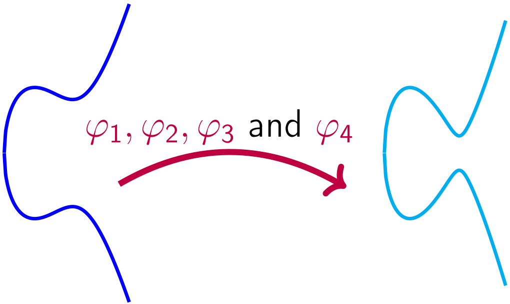
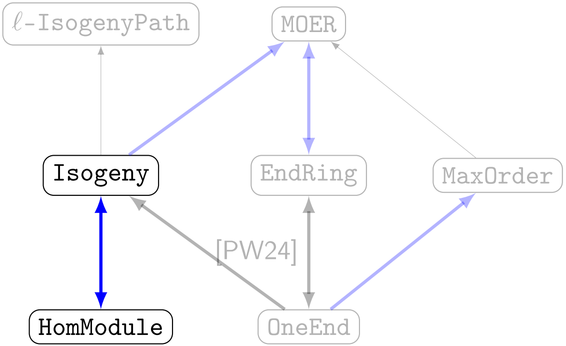


Polynomial reductions between isogeny-based problems **without GRH**

Adding a new problem

Homomorphism Module Problem





$$\mathrm{Hom}(\ ,\ ) = \varphi_1\mathbb{Z} + \varphi_2\mathbb{Z} + \varphi_3\mathbb{Z} + \varphi_4\mathbb{Z}$$

Polynomial reductions between isogeny-based problems **without GRH**

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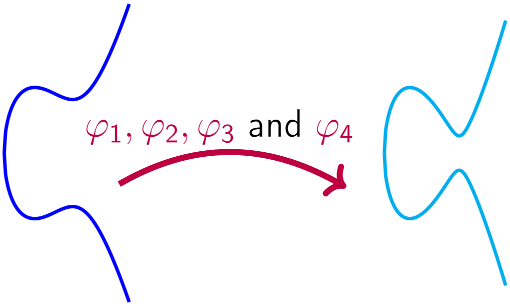

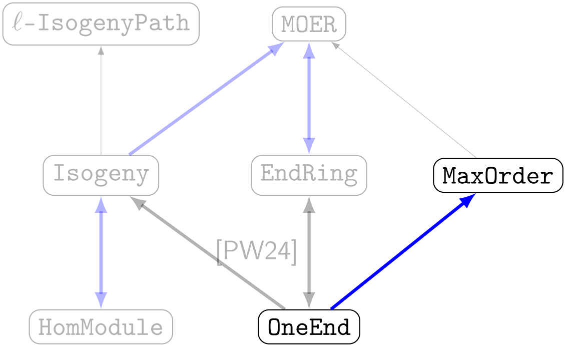

$$\mathrm{Hom}(\;,\;) = \varphi_1\mathbb{Z} + \varphi_2\mathbb{Z} + \varphi_3\mathbb{Z} + \varphi_4\mathbb{Z}$$

Polynomial reductions between isogeny-based problems **without GRH**

Reducing `OneEnd` to `MaxOrder`

Goal: Compute a non-scalar endomorphism $\alpha \in \text{End}(E) \backslash \mathbb{Z}$ given a `MaxOrder` oracle.

Reducing OneEnd to MaxOrder

Goal: Compute a non-scalar endomorphism $\alpha \in \mathrm{End}(E)\backslash\mathbb{Z}$ given a MaxOrder oracle.

Idea: $\mathcal{O} \overset{\varepsilon}{\simeq} \mathrm{End}(E)$

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Step 1: Compute a "local" correspondence between isogenies and ideals for a small prime ℓ.

E

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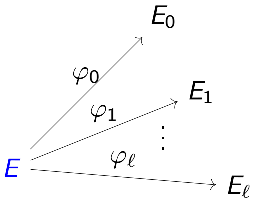

Compute all the isogenies of degree ℓ from E

Reducing `OneEnd` to `MaxOrder`

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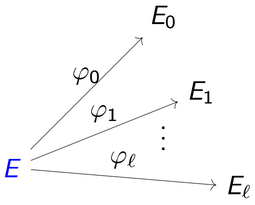

Compute all the isogenies of degree ℓ from E

$\mathcal{O} \simeq \mathrm{End}(E)$

$\mathcal{O}_0 \simeq \mathrm{End}(E_0)$

$\mathcal{O}_1 \simeq \mathrm{End}(E_1)$

$\vdots$

$\mathcal{O}_\ell \simeq \mathrm{End}(E_\ell)$

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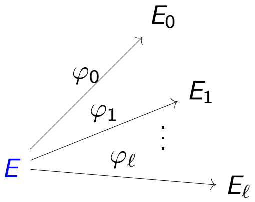

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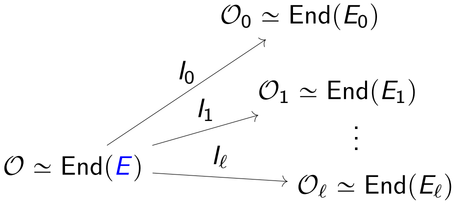

Connect all the maximal orders with ideals of norm ℓ

Goal: Compute a non-scalar endomorphism $\alpha \in \mathrm{End}(E) \backslash \mathbb{Z}$ given a MaxOrder oracle.

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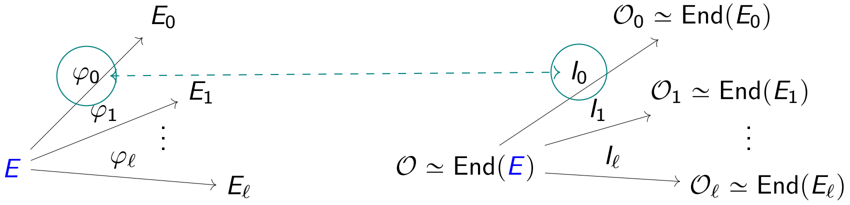

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Connect all the maximal orders with ideals of norm ℓ

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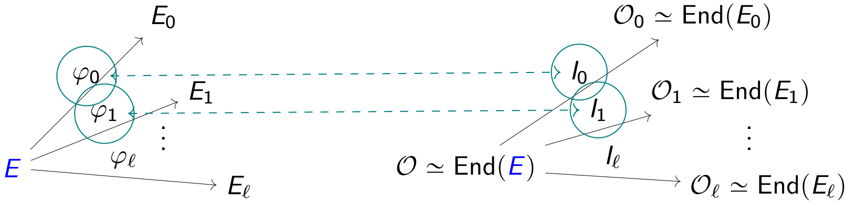

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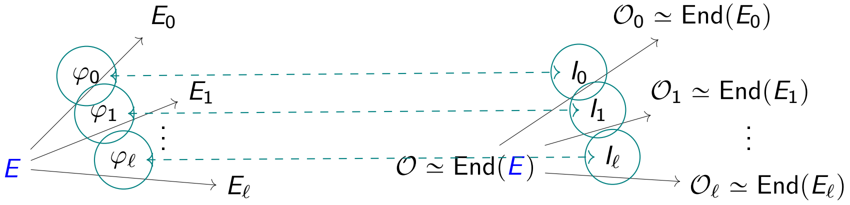

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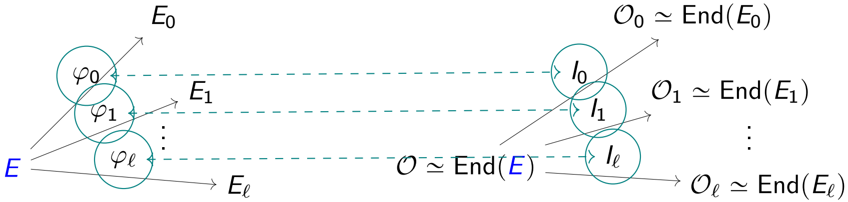

Compute all the isogenies of degree ℓ from E

Connect all the maximal orders with ideals of norm ℓ

Step 2: Compute a "local" isomorphism $\varepsilon_\ell : \mathcal{O}/\ell\mathcal{O} \xrightarrow{\sim} \mathrm{End}(E[\ell])$.

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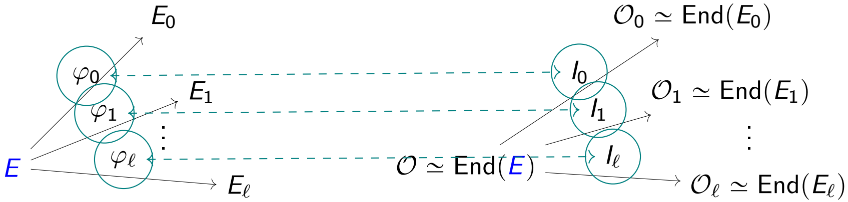

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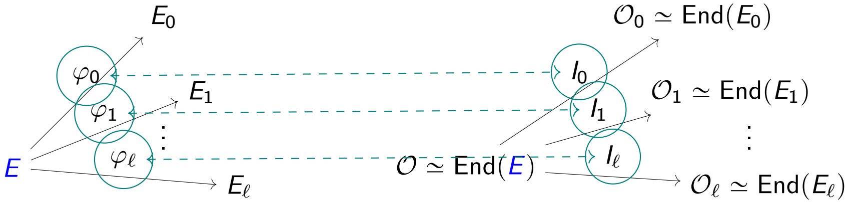
$$\forall \gamma \in \mathcal{O}, \forall P \in E \text{ of order } \ell, \text{we have that } \varepsilon_\ell(\gamma)(P) = \varepsilon(\gamma)(P).$$

Reducing OneEnd to MaxOrder

Goal: Compute a non-scalar endomorphism $\alpha \in \mathrm{End}(E)\backslash\mathbb{Z}$ given a MaxOrder oracle.

Idea: Choose a non-scalar $\beta \in \mathcal{O} \overset{\varepsilon}{\simeq} \mathrm{End}(E)$ and compute $\alpha = \varepsilon(\beta)$. We don't know ε...

Step 1: Compute a "local" correspondence between isogenies and ideals for a small prime ℓ.



Compute all the isogenies of degree ℓ from E

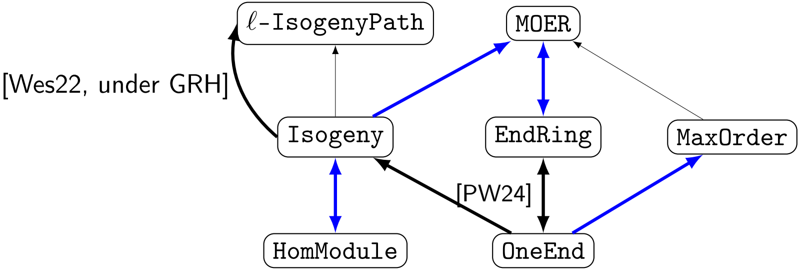
Connect all the maximal orders with ideals of norm ℓ

Step 2: Compute a "local" isomorphism $\varepsilon_\ell : \mathcal{O}/\ell\mathcal{O} \xrightarrow{\sim} \mathrm{End}(E[\ell])$.

$$\forall \gamma \in \mathcal{O}, \forall P \in E \text{ of order } \ell, \text{we have that } \varepsilon_\ell(\gamma)(P) = \varepsilon(\gamma)(P).$$

Step 3: Interpolate $\alpha = \varepsilon(\beta)$ from its evaluations on many small points.

New state-of-the-art



Polynomial reductions between isogeny-based problems.

Theorem (This paper)

The **Isogeny**, **EndRing**, **MaxOrder**, **OneEnd**, **MOER** *and* **HomModule** *problems are equivalent under classical probabilistic polynomial reductions.*

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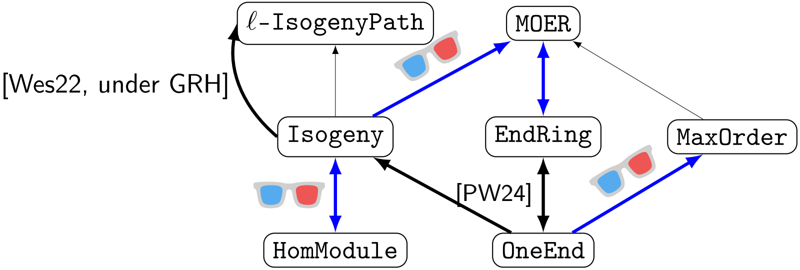

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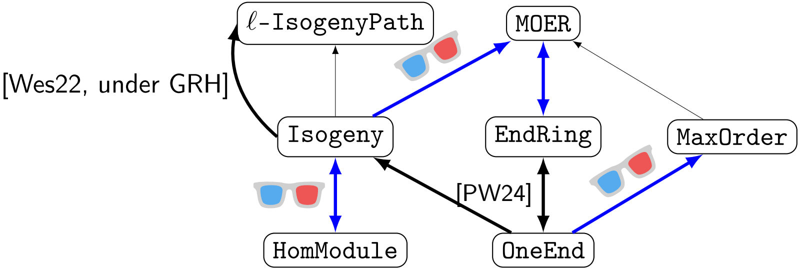

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: HD results following SIDH's attacks [CD23; Mai+23; Rob23] including IsogenyInterpolation [Rob24], IdealToIsogeny [PR23] and IsogenyDivision[Rob22; HW25] algorithms.

Worst-case to average-case reductions

Theorem (This paper)

For any pair of problems (P, Q) chosen from the problems

$$\mathtt{Isogeny}, \ell\text{-}\mathtt{IsogenyPath}, \mathtt{EndRing}, \mathtt{OneEnd}, \mathtt{MaxOrder}, \mathtt{MOER} \textit{ and } \mathtt{HomModule},$$

there exists an unconditional probabilistic polynomial time reduction

$$P \textit{ worst-case} \longrightarrow Q \textit{ average-case},$$

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$$P \text{ worst-case} \longrightarrow Q \text{ average-case},$$

$$\text{except if } \begin{cases} & P = \ell\text{-}\mathtt{IsogenyPath} \\ \text{or} & Q = \mathtt{MaxOrder} \text{ and } p \equiv 1 \bmod 8 \end{cases} \text{ then one needs to assume GRH.}$$

Conclusion

Corollary

Isogeny *worst-case hardness* $\Longrightarrow$ **Isogeny**, ℓ-**IsogenyPath**, **MOER**, **OneEnd**, **EndRing**, **HomModule**, (**MaxOrder** *if* $p \not\equiv 1 \bmod 8$) *average-case hardness.*

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Open questions:

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Corollary

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Open questions:

- Can we reduce ℓ-`IsogenyPath` to another problem?
- Can we reduce a problem to `MaxOrder` in the average case when $p \equiv 1 \bmod 8$?

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Isogeny *worst-case hardness* $\Longrightarrow$ **Isogeny**, **ℓ-IsogenyPath**, **MOER**, **OneEnd**, **EndRing**, **HomModule**, (**MaxOrder** *if* $p \not\equiv 1 \bmod 8$) *average-case hardness.*

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Thank you for your attention!

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